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Entropy Functions for Symmetric Systems of Conservation Laws

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Using a simple symmetrizability criterion, we show that symmetric systems of conservation laws are equipped with a one-parameter family of entropy functions.

1. INTRODUCTION

An entropy function associated with a system of N conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(f(u))=0, \tag{1.1}$$

is a convex function, U, augmented by an entropy flux function, F, both taking values from R^N smoothly into R, such that for any smooth $u \equiv u(x, t)$ satisfying (1.1) we have

$$\frac{\partial}{\partial t}(U(u))+\frac{\partial}{\partial x}(F(u))=0. \tag{1.2}$$

Carrying out the differentation in (1.2) we find, because of (1.1), that the above requirement amounts to the following *integrability condition*

$$U_u^T(u)\, f_u(u)=F_u^T(u). \tag{1.3}$$

Entropy functions play a significant role in the theory of systems of conservation laws. As observed by Friedrichs and Lax [1], if U is an entropy function for system (1.1), then its Hessian, U_{uu}, symmetrizes that system,

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i.e., symmetrizes f_u. It is fairly easy to see that the converse is also true; for future reference we can therefore state

THEOREM 1. *A convex U serves as an entropy function for system* (1.1), *if and only if, its Hessian, U_{uu}, symmetrizes f_u,*

$$U_{uu} f_u = (U_{uu} f_u)^T. \tag{1.4}$$

For the sake of completeness we include the proof. If U is an entropy in the sense that (1.3) holds, further differentation gives

$$U_{uu} f_u + U_u^T f_{uu} = F_{uu}.$$

The Hessian on the right is symmetric and so is the second matrix on the left, being the product of a vector and a 3-tensor; hence, their difference, $U_{uu} f_u$, is symmetric. Conversely, if $U_{uu} f_u$ is symmetric, so is $(U_u^T f_u)_u = U_{uu} f_u + U_u^T f_{uu}$. Hence $U_u f_u$ has a primitive

$$F(u) = \int^u U_w^T(w)\, f_w(w) \cdot dw \tag{1.5}$$

such that (1.3) holds; in other words, the symmetry of $U_{uu} f_u$—or what amounts to the same thing, of $(U_u^T f_u)_u$—is required as a compatibility condition for $F(u)$ to be well-defined, i.e., for the integral on the RHS of (1.5) to be path independent.

We remark that the convexity of U did not enter into the proof, and was assumed just for the sake of complying with the definition of an entropy function being convex. Apart from it, the "if" part of the above theorem, (1.4), provides us—unlike the integrability condition (1.3)—with a *self-contained* criterion for U being an entropy function. The "only if" part of the theorem on the other hand, reveals the hyperbolic nature of systems equipped with entropy functions; indeed, multiplication of (1.1) by U_{uu} on the left, puts the system in symmetric hyperbolic form (in the sense of Friedrichs), for which the local well-posedness theory of smooth solutions, prevails, see [1].

It is well known that solutions for (1.1) may fail to be smooth at a finite time, after which one must admit these solutions in the weak sense. For the latter, the following *entropy inequality* is imposed as an admissibility criterion [3, 4]

$$\frac{\partial}{\partial t}(U(u)) + \frac{\partial}{\partial x}(F(u)) \leqslant 0 \qquad \text{(weakly)}; \tag{1.6}$$

the inequality (1.6) follows from considerations of the *regularized problem*

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(f(u))=\varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad (1.7_\varepsilon)$$

letting ε goes to zero, $\varepsilon \downarrow 0$. Thus, the nonpositive LHS of (1.6) indicates the existence of vanishing viscosity in an admissible weak solution. In [4], Lax postulated a uniqueness criterion to single out the so called "physically relevant" solution of (1.1), requiring the entropy inequality (1.6) to hold for *all* entropy functions associated with (1.1). This brings us to the question of *how rich is the family of such entropies.*

In the scalar case, $N=1$, this family consists of *all* smooth convex functions; in his penetrating paper [3], Kruzkov has shown, that having the entropy inequality (1.6) for the *one-parameter* family of convex functions $U(u;\lambda)=|u-\lambda|$, $\lambda\in R$—which is in the convex hull of the former—indeed single out the unique, physically relevant, stable solution in L^1. The situation with the general nonscalar case, is however less favorable: the integrability condition is overdetermined unless $N=2$, e.g., [4].

2. Symmetric Systems of Conservation Laws

In this section we restrict our attention to *symmetric* systems of conservation laws, i.e., systems of the form (1.1) with symmetric Jacobians, $f_u=f_u^T$. We will show that such systems are equipped with *one-parameter* family of entropy functions.

To this end we are making use of the symmetrizability criterion of Theorem 1, looking for Hessians which symmetrize f_u,

$$U_{uu}f_u=f_uU_{uu}. \qquad (2.1)$$

An obvious first choice for such a Hessian will be the identity matrix, $U_{uu}=I_N$. This coincides with Godunov's observation, [2], (see also [1]), that for symmetric systems, $U(u)=\frac{1}{2}u^T\cdot u$ serves as entropy function, augmented by an entropy flux, see (1.5), $F(u)=\int w^Tf_w(w)\cdot dw=u^Tf(u)-\int^u f(w)\cdot dw$. Our next choice for symmetrizing Hessian will be f_u: the assumed symmetry of f_u implies, as argued before, that it is indeed a Hessian, $f_u=U_{uu}$ with $U(u)=\int^u f(w)\cdot dw$, augmented by an entropy flux, see (1.5), $F(u)\equiv F(f(u))=\frac{1}{2}f^T(u)\cdot f(u)$; furthermore, (2.1) is trivially satisfied with this choice (we note the identity $U_u^T(u)=F_f^T(f)$ in this case, from which (1.3) follows upon multiplication by f_u on the right). However, the function $U(u)$ so constructed is not convex since its Hessian, f_u, is not necessarily positive definite. This can be easily overcome by considering a sufficiently small neighborhood of the first convex choice of an entropy function. Thus we have shown

THEOREM 2. *Any symmetric system of conservation laws,* (1.1), *is equipped with the following one-parameter family of entropy functions*

$$U(u;\lambda)=\tfrac{1}{2}u^T\cdot u-\lambda\int^u f(w)\cdot dw, \quad \{\lambda\in R; \lambda f_u<I\} \tag{2.2a}$$

with corresponding entropy fluxes

$$F(u;\lambda)=u^T\cdot f(u)-\int^u f(w)\cdot dw-\frac{\lambda}{2}f^T(u)\cdot f(u). \tag{2.2b}$$

Let $u_l(u_r)$ denote the state on the left (respectively, right) of a discontinuity moving with speed s and governed by system (1.1). The entropy inequality (1.2) across such discontinuity amounts to

$$s[U(u_l;\lambda)-U(u_r;\lambda)]-[F(u_l;\lambda)-F(u_r;\lambda)]\leqslant 0. \tag{2.3}$$

Invoking the Rankine–Hugoniot (R–H) relation, $s(u_l-u_r)=f(u_l)-f(u_r)$, the inequality (2.3) reads, after little rearrangement,

$$(1-\lambda s)\left[\tfrac{1}{2}(f(u_l)+f(u_r))\cdot(u_r-u_l)+\int_{u_r}^{u_l} f(w)\cdot dw\right]\leqslant 0.$$

Since λ was chosen so that $\lambda f_u<I$, the R–H relation implies that the first term on the left is positive; hence, the entropy inequality (2.3) for each of the λ-parameter members $U(u;\lambda)$ in (2.2), is consistent with that of $U(u;0)$. Thus, the one-parameter entropies' family, provides us with stability criteria which coincide with that derived from Godunov's original choice, $U(u)=\frac{1}{2}u^T\cdot u$.

3. A NOTE ON THE REGULARIZED PROBLEM

We have mentioned before the *parabolic* regularized problem (1.7_ε), in connection with the entropy inequality (1.6). The key of studying system (1.7_ε) in the large, via standard energy methods, depends on obtaining *á priori* information in the maximum norm $|u(\cdot,t)|_{L^\infty}$. Here we note that such information can be easily obtained when the symmetric system (1.1) is regularized via *disspersive* term

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(f(u))=\frac{\partial^3 u}{\partial x^3}. \tag{3.1}$$

The proof is intimately related to the conserved entropies constructed in Section 2. Let $(\cdot,\cdot)_0$ denote the spatial L_2- inner product of compactly supported functions, $|\cdot|_0^2=(\cdot,\cdot)_0$. Multiplying (3.1) by u^T on the left and integrating we find, that $|u|_0^2$ is conserved in time. Next, differentiate (3.1), multiply by u_x^T and integrate to arrive at

$$\frac{1}{2}\frac{d}{dt}|u_x(\cdot, t)|_0^2+(u_x, f_{xx})_0=0;$$

multiplying (3.1) by f^T on the left, integrating and adding to the above we find that $\frac{1}{2}|u_x(\cdot, t)|_0^2+\int_x \int^u f(w)\cdot dw$ is also conserved. We remark that the last two conserved functionals are in exact agreement with the corresponding first two associated with *KdV* equation. Under appropriate growth assumptions on the flux, f, they yield the required maximum norm bound.

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