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ENTROPY CONSERVATIVE FINITE ELEMENT SCHEMES

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ABSTRACT

We study the question of entropy stability for discrete approximations to hyperbolic systems of conservation laws. We quantify the amount of numerical viscosity present in such schemes, and relate it to their entropy stability by means of comparison. To this end, two main ingredients are used: the entropy variables and the construction of certain entropy conservative schemes in terms of piecewise-linear finite element approximations. We then show that conservative schemes are entropy stable, if and--for three-point schemes--only if, they contain more numerical viscosity than the above mentioned entropy conservative ones.

1. THE ENTROPY VARIABLES

We consider semi-discrete schemes of the form

$$\frac{d}{dt} u_\nu(t) = -\frac{1}{\Delta x_\nu} [f_{\nu+1/2} - f_{\nu-1/2}], \tag{1}$$

which are consistent with the system of conservation laws

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} [f(u)] = 0, \qquad (x,t) \in R\times[0,\infty) \tag{2}$$

Here, $f \equiv f(u) = (f_1,\dots,f_N)^T$ is a smooth flux function of the conservative variables $u \equiv u(x,t) = (u_1,\dots,u_N)^T$, $u_\nu(t)$ denote the discrete solution along the gridline (x_ν,t) with $\Delta x_\nu \equiv \frac{1}{2}(x_{\nu+1} - x_{\nu-1})$ being the variable meshsize, and $f_{\nu+1/2}$[1] is the Lipschitz continuous numerical flux consistent with differential one

$$f_{\nu+1/2} = f(u_{\nu-p+1},\dots,u_{\nu+p}), \qquad f(u,u,\dots,u) = f(u). \tag{3}$$

We are concerned here with the <u>entropy stability</u> question of such schemes. To this end, let $(U \equiv U(u),\ F \equiv F(u))$ be an entropy pair associated with the system (2), such that

$$U_u^T f_u = F_u^T, \qquad U_{uu} > 0. \tag{4}$$

We ask whether the scheme (1) is <u>entropy stable</u> w.r.t. such pair, in the sense that it satisfies a discrete entropy inequality of the form

$$\frac{d}{dt} U(u_\nu(t)) + \frac{1}{\Delta x_\nu}\,[F_{\nu+1/2} - F_{\nu-1/2}] \leq 0, \tag{5}$$

with $F_{\nu+1/2}$ being a consistent numerical entropy flux

$$F_{\nu+1/2} = F(u_{\nu-p+1},\dots,u_{\nu+p}), \qquad F(u,u,\dots,u) = F(u). \tag{6}$$

If, in particular, equality takes place in (5), we say that the scheme (1) is <u>entropy conservative.</u> We note in passing that if it holds for a large enough class of entropy pairs, such a discrete entropy inequality is intimately

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[1] The same notations are used for differential and discrete fluxes; the distinction between the two is by the number of their arguments.

related to both questions of convergence toward a limit solution as well as this limit solution being the unique physically relevant one, e.g. [1], [2], [3].

Making use of the entropy pair (4), Mock [4] (see also [5]), has suggested the following procedure to symmetrize the system (2).

Define the entropy variables

$$v = v(u) = \frac{\partial U}{\partial u}(u). \tag{7}$$

Thanks to the convexity of $U(u)$ the mapping $u \to v$ is one-to-one. Hence, one can make the change of variables $u = u(v)$ which puts the system (2) into its equivalent symmetric form

$$\frac{\partial}{\partial t}[u(v)] + \frac{\partial}{\partial x}[g(v)] = 0, \qquad g(v) \equiv f(u(v)). \tag{8}$$

The system (8) is symmetric in the sense that the Jacobians of its temporal and spatial fluxes are

$$H \equiv H(v) = \frac{\partial}{\partial v}[u(v)] > 0, \qquad B \equiv B(v) = \frac{\partial}{\partial v}[g(v)]. \tag{9}$$

Indeed, if we introduce the so-called potential functions

$$\begin{aligned} \phi &\equiv \phi(v) = v^T u(v) - U(u(v)) \\ \psi &\equiv \psi(v) = v^T g(v) - F(u(v)), \end{aligned} \tag{10}$$

then making use of (4) we find

$$u(v) = \frac{\partial \phi}{\partial v}, \qquad g(v) = \frac{\partial \psi}{\partial v}, \tag{11}$$

and hence the Jacobians $H(v)$ and $B(v)$ in (9), are the symmetric Hessians of $\phi(v)$ and $\psi(v)$, respectively.

Example 1.1. Consider the Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ m \\ E \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(E+p) \end{pmatrix} = 0, \qquad p = (\gamma - 1)\cdot[E - \frac{m^2}{2\rho}], \tag{12}$$

asserting the conservation of the density ρ, momentum m and energy E. Harten [6] has noted that this system is equipped with a family of entropy pairs; Godunov [7] and Hughes et al. [8] have studied the canonical choice

$$(U = -\rho S,\ F = -mS), \qquad S = \ln(p\rho^{-\gamma}), \tag{13}$$

which leads to the entropy variables

$$v \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{1-\gamma}{p}\begin{pmatrix} E + \frac{p}{\gamma-1}(S - \gamma - 1) \\ -m \\ \rho \end{pmatrix} \tag{14}$$

The inverse mapping $v \to u$ can be found in [8]. We call attention to the fact that the corresponding potential pair in this case is given by $(\phi = (\gamma - 1)\cdot\rho, \psi = (\gamma - 1)\cdot m)$, and hence, in view of (11), Euler equations can be rewritten in the intriguing form [10]

$$\frac{\partial}{\partial t} [\text{grad}_v \rho] + \frac{\partial}{\partial x} [\text{grad}_v m] = 0. \tag{15}$$

Returning to our question of entropy stability, the answer provided in [9], [10], consists of two main ingredients: the use of the entropy variables described above, and the comparison with appropriate entropy conservative schemes. To this end we proceed as follows.

We use the entropy variables--rather than the conservative ones--as our primary dependent quantities, by making the change of variables $u_\nu = u(v_\nu)$, e.g., [11], [12]. The scheme (1) is now equivalently expressed as

$$\frac{d}{dt} u_\nu(t) = -\frac{1}{\Delta x_\nu} [g_{\nu+1/2} - g_{\nu-1/2}], \qquad u_\nu = u(v_\nu(t)), \tag{16}$$

with a numerical flux

$$g_{\nu+1/2} = g(v_{\nu-p+1},\ldots,v_{\nu+p}), \qquad g(\ldots,v,\ldots) = f(\ldots,u(v),\ldots), \tag{17}$$

consistent with the differential one

$$g(v,v,\ldots,v) = g(v), \qquad g(v) = f(u(v)). \tag{18}$$

Defining

$$F_{\nu+1/2} = v^T_{\nu+1} g_{\nu+1/2} - \psi(v_{\nu+1}), \tag{19}$$

then the following identity, originally due to Osher [13], see also [9], holds

$$\frac{d}{dt} U(u_\nu(t)) + \frac{1}{\Delta x_\nu} [F_{\nu+1/2} - F_{\nu-1/2}] = \Delta v^T_{\nu+1/2} g_{\nu+1/2} - \Delta\psi_{\nu+1/2},$$

$$\Delta\psi_{\nu+1/2} \equiv \psi(v_{\nu+1}) - \psi(v_\nu). \tag{20}$$

In view of (10), $F_{\nu+1/2}$ is a consistent numerical entropy flux and this brings us to (see [13], [9, Theorem 5.2])

Theorem 1.2: <u>The conservative scheme (16) is entropy conservative, if and--for three-point schemes</u> (p = 1)--<u>only if, the following equality holds</u>

$$\Delta v^T_{\nu+1/2} g_{\nu+1/2} = \Delta\psi_{\nu+1/2}. \tag{21}$$

2. THE SCALAR PROBLEM

We discuss the entropy stability of scalar conservative schemes, N = 1. Defining

$$Q_{\nu+1/2} = \frac{f(u_\nu) + f(u_{\nu+1}) - 2g_{\nu+1/2}}{\Delta v_{\nu+1/2}}, \qquad \Delta v_{\nu+1/2} = v_{\nu+1} - v_\nu \tag{22}$$

then our scheme recast into the more convenient viscosity form

$$\frac{d}{dt} v_\nu(t) = -\frac{1}{2\Delta x_\nu} [f(u_{\nu+1}) - f(u_{\nu+1})] + \frac{1}{2\Delta x_\nu} [Q_{\nu+\frac{1}{2}} \Delta v_{\nu+\frac{1}{2}} - Q_{\nu-\frac{1}{2}} \Delta v_{\nu-\frac{1}{2}}], \quad (23)$$

thus revealing the role of $Q_{\nu+\frac{1}{2}}$ as the numerical viscosity coefficient [14].

According to (21), the scalar entropy conservative schemes are uniquely determined by $g_{\nu+\frac{1}{2}} = g^*_{\nu+\frac{1}{2}}$ where

$$g^*_{\nu+\frac{1}{2}} = \frac{\Delta\psi_{\nu+\frac{1}{2}}}{\Delta v_{\nu+\frac{1}{2}}} = \int_{\xi=0}^{1} g(v_{\nu+\frac{1}{2}}(\xi))d\xi, \qquad v_{\nu+\frac{1}{2}}(\xi) = v_\nu + \xi \Delta v_{\nu+\frac{1}{2}}. \quad (24)$$

Writing

$$g^*_{\nu+\frac{1}{2}} = \int_{\xi=0}^{1} \frac{d}{d\xi} (\xi - \tfrac{1}{2}) \, g(v_{\nu+\frac{1}{2}}(\xi))d\xi, \quad (25)$$

and integrating by parts, these entropy conservative schemes assume the viscosity form (23), with viscosity coefficient $Q_{\nu+\frac{1}{2}} = Q^*_{\nu+\frac{1}{2}}$, where

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=0}^{1} (2\xi - 1) \, g'(v_{\nu+\frac{1}{2}}(\xi))\xi. \quad (26)$$

We are now ready to characterize entropy stability, by comparison with the above entropy conservative schemes.

Theorem 2.1 [9, Theorem 5.1]: The conservative scheme (23) is entropy stable, if and--for three-point schemes $(p = 1)$--only if, it contains more viscosity than the entropy conservative one (26), i.e.,

$$Q^*_{\nu+\frac{1}{2}} \leq Q_{\nu+\frac{1}{2}}. \quad (27)$$

The last theorem enable us to verify the entropy stability of first—as well as second-order accurate schemes. Indeed, a lengthy calculation which we omit yields [10]

$$Q^*_{\nu+1/2} = \int_{\xi=0}^{1} \int_{\eta=0}^{1} [(\xi-\tfrac{1}{2})^2 + (\eta-\tfrac{1}{2})^2] g''(v_{\nu+1/2}[\xi\eta + (1-\eta)(1-\xi)]) d\eta d\xi \cdot \Delta v_{\nu+1/2} \quad (28)$$

showing that the entropy conservative viscosity is of order $O(|\Delta v_{\nu+1/2}|)$. This implies second order accuracy in view of

Lemma 2.2: <u>Consider the conservative schemes (25) with viscosity coefficient, $Q_{\nu+1/2}$, such that $\frac{Q_{\nu+1/2}}{\Delta v_{\nu+1/2}}$ is Lipschitz continuous. Then these schemes are second-order accurate, in the sense that their truncation is of the order</u>

$$O[|x_{\nu+1} - x_\nu|^2 + |x_\nu - x_{\nu-1}|^2 + |x_{\nu+1} - 2x_\nu + x_{\nu-1}|]. \quad (29)$$

We give two of the examples considered in [10, Section 4].

Example 2.3: Using the simple upper bound, see (28),

$$Q^*_{\nu+1/2} \le \frac{1}{6} \operatorname*{Max}_{v} |g''(v)| \cdot |\Delta v_{\nu+1/2}|, \quad (30)$$

we obtain, on the right of (30), a viscosity coefficient which according to Theorem 2.1 and Lemma 2.2 maintains both, entropy stability and second-order accuracy. Similar viscosity terms were previously derived in a number of special cases, e.g., [15], [16], [17]. We remark that the careful lengthy calculations required in those derivations is due to the delicate balance between the cubic order of entropy loss and the third-order dissipation in this case.

Example 2.4: Consider the genuinely nonlinear case where $f(u)$ is, say, convex. The quadratic entropy function, $U(u) = \frac{1}{2}u^2$, leads to entropy variables which coincide with the conservative ones, $g(v) = f(u)$. Thus, according to (28), viscosity is required only at rarefactions where $\Delta u_{\nu+1/2} > 0$, since $\text{sign}(Q^*_{\nu+1/2}) = \text{sign}(\Delta u_{\nu+1/2}) < 0$ otherwise. A simple second-order accurate entropy stable flux of this kind is given in [10]

$$
f_{\nu+1/2} = \begin{cases} f\left(\frac{1}{2}(u_\nu + u_{\nu+1})\right) & \Delta u_{\nu+1/2} > 0 \\ \frac{1}{2}\left(f(u_\nu) + f(u_{\nu+1})\right) & \Delta u_{\nu+1/2} < 0 \end{cases}
$$

3. SYSTEMS OF CONSERVATION LAWS

We generalize the construction of the scalar entropy conservative schemes (26), to systems of conservation laws, using piecewise linear finite-elements.

To this end, consider the weak formulation of (8)

$$
\int_\Omega w^T \frac{\partial}{\partial t}[u(v)]dxdt = \int_\Omega \frac{\partial w^T}{\partial x} g(v)dxdt. \tag{32}
$$

Let the trial solutions $w \to \hat{w}(x,t) = \sum_k w_k(t)\hat{H}_k(x)$ be chosen out of the typical finite-element set spanned by the C^0 "hat functions"

$$\hat{H}_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}} & x_{k-1} \leq x \leq x_k \\[2ex] \dfrac{x_{k+1} - x}{x_{k+1} - x_k} & x_k \leq x \leq x_{k+1} \end{cases} \quad . \tag{33}$$

The spatial part on the right of (32) yields—after change of variables,

$$\int_{x_{\nu-1}}^{x_{\nu+1}} \frac{\partial \hat{H}_\nu(x)}{\partial x} g[\hat{v}(x,t) = \sum_{\nu-1}^{\nu+1} v_k(t)\hat{H}_k(x)]dx =$$

$$= -[\int_{\xi=0}^{1} g(v_{\nu+1/2}(\xi))d\xi - \int_{\xi=0}^{1} g(v_{\nu-1/2}(\xi))d\xi], \quad v_{\nu+1/2}(\xi) = v_\nu + \xi\Delta v_{\nu+1/2}. \tag{34}$$

A second-order mass lumping on the left of (32) leads to

$$\int_{x_{\nu-1}}^{x_{\nu+1}} \hat{H}_\nu(x) \frac{\partial}{\partial t} u[\hat{v}(x,t) = \sum_{\nu-1}^{\nu+1} v_k(t)H_k(x)] = \Delta x_\nu \frac{\partial}{\partial t} [u(v_\nu(t))] + O(|\Delta v_{\nu+1/2}|^2) \tag{35}$$

Equating (34) and (35) while neglecting the quadratic error terms, we end up with, compare (24),

$$\frac{\partial}{\partial t} u_\nu(t) = -\frac{1}{\Delta x_\nu} [g^*_{\nu+1/2} - g^*_{\nu-1/2}], \qquad g^*_{\nu+1/2} = \int_{\xi=0}^{1} g(v_{\nu+1/2}(\xi))d\xi. \tag{36}$$

Without mass lumping, the global conservation of the entropy is immediate as shown by choosing $\hat{w}(x,t) = \hat{v}(x,t)$, so that in view (4), (32) yields

$$0 = \int_\Omega [\hat{v}^T \frac{\partial}{\partial t} [u(\hat{v})] - \frac{\partial \hat{v}^T}{\partial x} g(\hat{v})]dxdt \equiv \int_\Omega [\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x}]dxdt. \tag{37}$$

With the mass lumping, we obtain the locally entropy conservative scheme (36), satisfying, consult (11),

$$\Delta v^T_{\nu+1/2} g^*_{\nu+1/2} = \int_{\xi=0}^{1} \Delta v_{\nu+1/2}\, g(v_\nu + \xi \Delta v_{\nu+1/2}) d\xi = \int_{v_\nu}^{v_{\nu+1}} dv^T g(v) = \Delta\psi_{\nu+1/2}, \quad (38)$$

in agreement with Theorem 2.1. This entropy conservative scheme can be also rewritten in the viscosity form [9]

$$\frac{d}{dt} u_\nu(t) = \frac{-1}{2\Delta x_\nu} [f(u_{\nu+1}) - f(u_{\nu-1})] + \frac{1}{2\Delta x_\nu} [Q_{\nu+1/2} \Delta v_{\nu+1/2} - Q_{\nu-1/2} \Delta v_{\nu-1/2}] \quad (39)$$

with a numerical viscosity coefficent matrix, $Q_{\nu+1/2} = Q^*_{\nu+1/2}$, where

$$Q^*_{\nu+1/2} = \int_{\xi=0}^{1} (2\xi - 1)\cdot B(v_{\nu+1/2}(\xi)) d\xi, \qquad B(v) \equiv \frac{\partial}{\partial v} [g(v)]. \quad (40)$$

In analogy with Theorem 2.1, we have

Theorem 3.1 [9, Theorem 5.1]: The conservative scheme (39) is entropy stable, if and—for three-point schemes (p = 1)—only if, it contains more viscosity than the entropy conservative one (40), i.e.,

$$\Delta v_{\nu+1/2} Q^*_{\nu+1/2} \Delta v_{\nu+1/2} \leq \Delta v_{\nu+1/2} Q_{\nu+1/2} \Delta v_{\nu+1/2}. \quad (41)$$

If, in particular, $Q_{\nu+1/2}$ is symmetric, then a sufficient entropy stability criterion is

$$Q^*_{\nu+1/2} \leq Q_{\nu+1/2}, \quad (42)$$

where the inequality is understood in the usual sense of order among symmetric matrices.

Example 3.2: Consider the conservative scheme (39) with a numerical viscosity coefficient given by

$$Q_{\nu+1/2} = \int_{\xi=0}^{1} |B(v_{\nu+1/2}(\xi)|d\xi. \tag{43}$$

Here the absolute value of a symmetric matrix is evaluated in the usual fashion from its spectral representation, $B = U^*\Lambda U$,

$$|B(v_{\nu+1/2}(\xi))| = U^*(v_{\nu+1/2}(\xi))|\Lambda(v_{\nu+1/2}(\xi))|U(v_{\nu+1/2}(\xi)). \tag{44}$$

Since

$$(2\xi-1)\Lambda(v_{\nu+1/2}(\xi)) \leq |\Lambda(v_{\nu+1/2}(\xi))|,$$

(42) holds and entropy stability follows.

Away from sonic points, (43) amounts to the usual upwind differencing, e.g., [18]. In the neighborhood of such sonic points, however, an exact evaluation of $Q_{\nu+1/2}$ in (43) may turn out to be a difficult task. Yet, in view of Theorem 3.1, one can use instead simpler upper bounds. In [10] this is achieved using the construction of a whole family of entropy conservative schemes which take into account the characteristic directions associated with the system (2).

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