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On the entropy stability of difference schemes: a comparison principle and a homotopy approach

Eitan Tadmor

ABSTRACT. We study the entropy stability of semi- and fully-discrete difference approximations to nonlinear hyperbolic conservation laws, and related time-dependent problems governed by additional dissipative and dispersive forcing terms.

1. Entropy-conservative and entropy-stable schemes

We consider semi-discrete conservative schemes of the form

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\Big], \tag{1}$$

serving as consistent approximations to systems of conservation laws of the form

$$\frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0, \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{2}$$

where $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}),\ldots,f_N(\mathbf{u}))^\top$ are smooth flux functions of the N-vector of conservative variables[1] $\mathbf{u}(x,t) = (u_1(x,t),\ldots,u_N(x,t))^\top$. Here, $\mathbf{u}_\nu(t)$ denotes the discrete solution along the grid line (x_ν,t) with $\Delta x_\nu := \frac{1}{2}(x_{\nu+1} - x_{\nu-1})$ being the variable meshsize, and $\mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\mathbf{u}_{\nu-p+1},\ldots,\mathbf{u}_{\nu+p})$ being the Lipschitz-continuous numerical flux consistent with the differential flux, $\mathbf{f}(\mathbf{u},\mathbf{u},\ldots,\mathbf{u}) \equiv \mathbf{f}(\mathbf{u})$. The numerical flux, $\mathbf{f}(\cdot,\cdot,\ldots,\cdot)$, involves a stencil of $2p$ neighboring grid values, and as such could be clearly distinguished from the (same notation of) the differential flux, $\mathbf{f}(\cdot)$.

We are concerned here with the question of *entropy stability* of such schemes. Here we assume that system (2) is equipped with a convex *entropy function*, $U(\mathbf{u})$, such that

$$U_{\mathbf{uu}}A = [U_{\mathbf{uu}}A]^\top, \quad A(\mathbf{u}) := \mathbf{f}_{\mathbf{u}}(\mathbf{u}). \tag{3}$$

Thus, the Hessian of an entropy function symmetrizes the system (2) upon multiplication 'on the left' [1]. An alternative procedure, which respects

[1]Here and below, scalars are distinguished from vectors, which are denoted by **bold** letters.

both strong and weak solutions of (2), is to symmetrize 'on the right', where (3) is replaced by the equivalent statement $A(U_{\mathbf{uu}})^{-1} = \big[A(U_{\mathbf{uu}})^{-1}\big]^\top$.

To this end, [5] (see also [2]) suggested the following procedure. Define the *entropy variables* $\mathbf{v} \equiv \mathbf{v}(\mathbf{u}) := \nabla_{\mathbf{u}} U(\mathbf{u})$. Thanks to the convexity of $U(\mathbf{u})$, the mapping $\mathbf{u} \to \mathbf{v}$ is one-to-one and hence we can make the change of variables $\mathbf{u} = \mathbf{u}(\mathbf{v})$, which puts the system (2) into its equivalent symmetric form

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{v}) + \frac{\partial}{\partial x}\mathbf{g}(\mathbf{v}) = 0, \quad \mathbf{g}(\mathbf{v}) := \mathbf{f}(\mathbf{u}(\mathbf{v})). \tag{4}$$

Here, $\mathbf{u}(\cdot)$ and $\mathbf{g}(\cdot)$ become the temporal and spatial fluxes in the independent entropy variables, $\mathbf{v}$, and the system (4) is symmetric in the sense that the Jacobians of these fluxes are, namely $H(\mathbf{v}) := \mathbf{u}_{\mathbf{v}}(\mathbf{v}) = H^\top(\mathbf{v}) > 0$ and $B(\mathbf{v}) := \mathbf{g}_{\mathbf{v}}(\mathbf{v}) = B^\top(\mathbf{v})$. Indeed, (3) holds if and only if there exists an entropy flux function, $F = F(\mathbf{u})$, such that the compatibility relation, $U_{\mathbf{u}}^\top \mathbf{f}_{\mathbf{u}} = F_{\mathbf{u}}^\top$ holds. Consequently, we have

$$\mathbf{u}(\mathbf{v}) = \nabla_{\mathbf{v}}\phi(\mathbf{v}), \quad \phi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{u}(\mathbf{v})\rangle - U(\mathbf{u}(\mathbf{v})) \tag{5}$$

$$\mathbf{g}(\mathbf{v}) = \nabla_{\mathbf{v}}\psi(\mathbf{v}), \quad \psi(\mathbf{v}) := \langle \mathbf{v}, \mathbf{g}(\mathbf{v})\rangle - F(\mathbf{u}(\mathbf{v})), \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. Hence the Jacobians $H(\mathbf{v})$ and $B(\mathbf{v})$ are symmetric, being the Hessians of $\phi(\mathbf{v})$ and $\psi(\mathbf{v})$. The latter, so-called potential functions, $\phi(\mathbf{v})$ and $\psi(\mathbf{v})$, are significant tools in our discussion below. Observe that the symmetry of $B = AH$ amounts to the symmetrization 'on the right' indicated above.

Let (U, F) be an entropy pair associated with the system (2). We ask whether the scheme (1) is *entropy-stable* with respect to such a pair, in the sense of satisfying a discrete entropy inequality analogous to the entropy inequality $U(\mathbf{u})_t + F(\mathbf{u})_x \leq 0$,

$$\frac{d}{dt}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}\Big] \leq 0. \tag{7}$$

Here, $F_{\nu+\frac{1}{2}} = F(\mathbf{u}_{\nu-p+1}, \ldots, \mathbf{u}_{\nu+p})$ is a consistent numerical entropy flux so that $F(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) = F(\mathbf{u})$. If, in particular, equality holds in (7), we say that the scheme (1) is *entropy-conservative.* The answer to this question of entropy stability provided in [8] consists of two main ingredients: (i) the use of the entropy variables and (ii) the comparison with appropriate *entropy-conservative* schemes. We conclude this section with a brief overview.

By making the change of variables $\mathbf{u}_\nu(t) = \mathbf{u}(\mathbf{v}_\nu(t))$, the scheme (1) recasts into the equivalent form

$$\frac{d}{dt}\mathbf{u}_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[\mathbf{g}_{\nu+\frac{1}{2}} - \mathbf{g}_{\nu-\frac{1}{2}}\Big], \quad \mathbf{u}_\nu(t) = \mathbf{u}(\mathbf{v}_\nu(t)), \tag{8}$$

with a numerical flux $\mathbf{g}_{\nu+\frac{1}{2}} = \mathbf{g}(\mathbf{v}_{\nu-p+1}, \ldots, \mathbf{v}_{\nu+p}) := \mathbf{f}(\mathbf{u}(\mathbf{v}_{\nu-p+1}), \ldots, \mathbf{u}(\mathbf{v}_{\nu+p}))$, consistent with the differential flux, $\mathbf{g}(\mathbf{v}, \mathbf{v}, \ldots, \mathbf{v}) = \mathbf{g}(\mathbf{v}) \equiv \mathbf{f}(\mathbf{u}(\mathbf{v}))$. Define $F_{\nu+\frac{1}{2}} := \frac{1}{2}\Big\langle [\mathbf{v}_\nu + \mathbf{v}_{\nu+1}], \mathbf{g}_{\nu+\frac{1}{2}}\Big\rangle - \frac{1}{2}\big[\psi(\mathbf{v}_\nu) + \psi(\mathbf{v}_{\nu+1})\big]$, then the following identity holds, [8],

$$\frac{d}{dt}U(\mathbf{u}_\nu(t)) + \frac{1}{\Delta x_\nu}\Big[F_{\nu+\frac12} - F_{\nu-\frac12}\Big] =$$
$$= \frac12\Big[\Big\langle \Delta\mathbf{v}_{\nu+\frac12}, \mathbf{g}_{\nu+\frac12}\Big\rangle - \Delta\psi_{\nu+\frac12}\Big] + \frac12\Big[\Big\langle \Delta\mathbf{v}_{\nu-\frac12}, \mathbf{g}_{\nu-\frac12}\Big\rangle - \Delta\psi_{\nu-\frac12}\Big].$$

Here $\Delta\psi_{\nu+\frac12} := \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)$ denotes the difference of entropy flux potential, (6), of two neighboring grid values $\mathbf{v}_\nu$ and $\mathbf{v}_{\nu+1}$. Thanks to (6), $F_{\nu+\frac12}$ is a consistent entropy flux and this brings us to the next result.

Theorem 1.1. *[Tadmor 1987] The conservative scheme (8) is entropy-stable (respectively, entropy-conservative) if, and for three-point schemes* ($p = 1$) *only if,*

(9)
$$\Big\langle \Delta\mathbf{v}_{\nu+\frac12}, \mathbf{g}_{\nu+\frac12}\Big\rangle \le \Delta\psi_{\nu+\frac12}, \quad \text{and respectively,} \quad \Big\langle \Delta\mathbf{v}_{\nu+\frac12}, \mathbf{g}_{\nu+\frac12}\Big\rangle = \Delta\psi_{\nu+\frac12}.$$

2. Scalar examples

We discuss the entropy stability of *scalar* schemes of the form

$$\frac{d}{dt}u_\nu(t) = -\frac{1}{\Delta x_\nu}\Big[g_{\nu+\frac12} - g_{\nu-\frac12}\Big], \quad u_\nu(t) \equiv u(v_\nu(t)). \tag{10}$$

For a more convenient formulation, let us define for $\Delta v_{\nu+\frac12} \neq 0$

$$Q_{\nu+\frac12} = \frac{f(u_\nu) + f(u_{\nu+1}) - 2g_{\nu+\frac12}}{\Delta v_{\nu+\frac12}}, \quad \Delta v_{\nu+\frac12} := v_{\nu+1} - v_\nu. \tag{11}$$

Our scheme recasts into the equivalent *viscosity form*

(12)
$$2\Delta x_\nu \frac{d}{dt}u_\nu(t) = -\Big[f(u_{\nu+1}) - f(u_{\nu-1})\Big] + \Big[Q_{\nu+\frac12}\Delta v_{\nu+\frac12} - Q_{\nu-\frac12}\Delta v_{\nu-\frac12}\Big],$$

which reveals the role of $Q_{\nu+\frac12}$ as the numerical viscosity coefficient (*e.g.*, [6]).

According to (9), scalar entropy-conservative schemes are uniquely determined by the numerical flux $g_{\nu+\frac12} = g^*_{\nu+\frac12}$, that is,

(13)
$$g^*_{\nu+\frac12} := \frac{\Delta\psi_{\nu+\frac12}}{\Delta v_{\nu+\frac12}} \equiv \int_{\xi=-\frac12}^{\frac12} g\Big(v_{\nu+\frac12}(\xi)\Big)d\xi, \quad v_{\nu+\frac12}(\xi) := \frac12(v_\nu + v_{\nu+1}) + \xi\Delta v_{\nu+\frac12}.$$

Noting that $g^*_{\nu+\frac12} = \int_{\xi=-\frac12}^{\frac12} \frac{d}{d\xi}(\xi)\cdot g\Big(v_{\nu+\frac12}(\xi)\Big)d\xi$, we find upon integration by parts that entropy-conservative schemes admit the viscosity form (12), with a viscosity coefficient $Q_{\nu+\frac{1}{2}} = Q^*_{\nu+\frac{1}{2}}$ given by[2]

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi g'\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi. \tag{14}$$

The entropy-conservative scheme then takes the form

$$\begin{aligned}2\Delta x_\nu \frac{d}{dt}u_\nu(t) &= -2\Big[g^*_{\nu+\frac{1}{2}} - g^*_{\nu-\frac{1}{2}}\Big] = \\ &= -\Big[f(u_{\nu+1}) - f(u_{\nu-1})\Big] + \Big[Q^*_{\nu+\frac{1}{2}}\Delta v_{\nu+\frac{1}{2}} - Q^*_{\nu-\frac{1}{2}}\Delta v_{\nu-\frac{1}{2}}\Big].\end{aligned}$$

The entropy stability portion of Theorem 1.1 can now be restated in the following form, [8].

Corollary 2.1. *The conservative scheme (12) is entropy-stable if – and for three-point schemes ($p = 1$), only if it contains more viscosity than the entropy-conservative one (14), that is,*

$$Q_{\nu+\frac{1}{2}} \geq Q^*_{\nu+\frac{1}{2}}. \tag{15}$$

Corollary 2.1 enables us to verify the entropy stability of first- second-order accurate schemes. A host of example can be found in [9] and we quote here the prototype example of

Example 2.2. *[Lax and Wendroff 1960] We start by noting that integration by parts of (14) yields*

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{d}{d\xi}\Big(\xi^2 - \frac{1}{4}\Big)g'\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{4} - \xi^2\Big)g''\Big(v_{\nu+\frac{1}{2}}(\xi)\Big)d\xi \cdot \Delta v_{\nu+\frac{1}{2}}.$$

Thus, the viscosity coefficients of the entropy-conservative schemes are in fact of order $\mathcal{O}(|\Delta v_{\nu+\frac{1}{2}}|)$, and this implies their second-order accuracy.

We consider the genuinely nonlinear case, where $f(u)$ is, say, convex. A quadratic entropy stability is sufficient in this case, to single out the unique physically relevant solution. In particular, the choice of the quadratic entropy function $U(u) = \frac{1}{2}u^2$ leads to entropy variables that coincide with the conservative ones, $g(v) = f(u)$. The entropy-conservative coefficient Q^* in example 2.2 amounts to,

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big(\frac{1}{4} - \xi^2\Big)f''\Big(u_{\nu+\frac{1}{2}}(\xi)\Big)d\xi \cdot \Delta u_{\nu+\frac{1}{2}}.$$

By convexity, $Q^*_{\nu+\frac{1}{2}}$ is negative whenever $\Delta u_{\nu+\frac{1}{2}}$ is, and hence numerical viscosity is required only in the case of rarefactions where $\Delta u_{\nu+\frac{1}{2}} > 0$. To see how much viscosity is required in this case, we use the fact that the integrand on the right of Q^* is positive, leading to the upper bound

$$Q^*_{\nu+\frac{1}{2}} \leq \frac{1}{4}\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f''\Big(u_{\nu+\frac{1}{2}}(\xi)\Big)d\xi \cdot \Delta u_{\nu+\frac{1}{2}} = \frac{1}{4}\big[a(u_{\nu+1}) - a(u_\nu)\big]^+,\ a(u) = f'(u). \tag{16}$$

[2]We use primes to indicate differentiation with respect to primary dependent variables, *e.g.*, $\mathbf{g}' = \mathbf{g}_\mathbf{v}(\mathbf{v}), \mathbf{f}'' = \mathbf{f}_{\mathbf{uu}}(\mathbf{u})$, *etc.*

The resulting viscosity coefficient on the right is the second-order Lax-Wendroff viscosity proposed in [4], $Q^{LW}_{\nu+\frac{1}{2}} = \frac{1}{4}\big[a(u_{\nu+1}) - a(u_\nu)\big]^+$.

3. Systems of conservation laws

We study the entropy stability of the semi-discrete schemes that are consistent with the *system* of conservation laws (4). The schemes assume the following viscosity form:

(17)

$$2\Delta x_\nu \frac{d}{dt}\mathbf{u}_\nu(t) = -\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] + \Big[Q_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} - Q_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big].$$

To extend our scalar entropy stability analysis to systems of conservation laws we proceed as before, by comparison with certain entropy-conservative schemes. Unlike the scalar problem, however, we now have more than one way to meet the entropy conservation requirement (9). The various ways differ in their choice of the path of integration in phase space. In this section, we restrict our attention to the simplest choice along the *straight path* $\mathbf{v}_{\nu+\frac{1}{2}}(\xi) = \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}$. The corresponding entropy-conservative flux is given by

$$\text{(18)}\quad \mathbf{g}^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi, \mathbf{v}_{\nu+\frac{1}{2}}(\xi) := \frac{1}{2}(\mathbf{v}_\nu + \mathbf{v}_{\nu+1}) + \xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}.$$

Indeed, the entropy conservation requirement (1) is fulfilled in this case, since, in view of (6),

$$\Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}^*_{\nu+\frac{1}{2}}\Big\rangle = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \Big\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)\Big\rangle d\xi = \int_{\mathbf{v}_\nu}^{\mathbf{v}_{\nu+1}} \langle d\mathbf{v}, \mathbf{g}(\mathbf{v})\rangle = \Delta\psi_{\nu+\frac{1}{2}}.$$

The entropy-conservative flux (18) was introduced in Tadmor (1986, 1987). As before (see (14)), we integrate by parts to find

$$\mathbf{g}^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \frac{d}{d\xi}(\xi)\mathbf{g}\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi = \frac{1}{2}\big[\mathbf{f}(\mathbf{u}_\nu) + \mathbf{f}(\mathbf{u}_{\nu+1})\big] - \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \xi B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi\Delta\mathbf{v}_{\nu+\frac{1}{2}}.$$

Thus, the entropy-conservative scheme (18) admits the equivalent viscosity form

(19)

$$2\Delta x_\nu \frac{d}{dt}\mathbf{u}_\nu(t) = -\big[\mathbf{f}(\mathbf{u}_{\nu+1}) - \mathbf{f}(\mathbf{u}_{\nu-1})\big] + \Big[Q^*_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} - Q^*_{\nu-\frac{1}{2}}\Delta\mathbf{v}_{\nu-\frac{1}{2}}\Big],$$

with a numerical viscosity matrix coefficient, $Q^*_{\nu+\frac{1}{2}}$, given by

$$\text{(20)}\qquad Q^*_{\nu+\frac{1}{2}} := \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} 2\xi B\Big(\mathbf{v}_{\nu+\frac{1}{2}}(\xi)\Big)d\xi, \quad B(\mathbf{v}) = \mathbf{g}_{\mathbf{v}}(\mathbf{v}).$$

The entropy stability portion of Theorem 1.1 can now be conveniently interpreted as follows.

Corollary 3.1. *The conservative scheme (17) is entropy-stable if – and for three-point schemes* ($p = 1$) *only if – it contains more viscosity than the entropy-conservative one (19), (20), that is,*

$$\left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q^*_{\nu+\frac{1}{2}} \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \leq \left\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, Q_{\nu+\frac{1}{2}} \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle. \tag{21}$$

The entropy-conservative flux (18), and likewise its corresponding viscosity coefficient in (20), cannot be evaluated in a closed form. However, Corollary 3.1 enables us to verify entropy stability by *comparison*, $Q^*_{\nu+\frac{1}{2}} \leq \Re Q_{\nu+\frac{1}{2}}$, with the usual ordering between symmetric matrices. We note in passing that $Q^*_{\nu+\frac{1}{2}}$ is symmetric (since $B(\cdot)$ is) and that, in the generic case, the viscosity coefficient $Q_{\nu+\frac{1}{2}}$ is also symmetric. Examples can be found in [9].

In [9] we introduced a new general family of entropy-conservative schemes which are based on different paths in phase space. This enables us to enforce entropy stability by fine-tuning the amount of numerical viscosity along each subpath carrying different intermediate waves. Moreover, the straight path integration of the entropy-conservative flux (18) does not admit a closed form, whereas the new family of entropy-conservative schemes enjoys an *explicit, closed-form formulation.* To this end, at each cell consisting of two neighboring values $\mathbf{v}_\nu$ and $\mathbf{v}_{\nu+1}$, we let $\{\mathbf{r}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$ be an *arbitrary* set of N linearly independent N-vectors, and let $\{\boldsymbol{\ell}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$ denote the corresponding orthogonal set, $\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \mathbf{r}^k_{\nu+\frac{1}{2}} \rangle = \delta_{jk}$. Next, we introduce the intermediate states, $\{\mathbf{v}^j_{\nu+\frac{1}{2}}\}_{j=1}^N$, starting with $\mathbf{v}^1_{\nu+\frac{1}{2}} = \mathbf{v}_\nu$, and followed by $\mathbf{v}^{j+1}_{\nu+\frac{1}{2}} = \mathbf{v}^j_{\nu+\frac{1}{2}} + \left\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \mathbf{r}^j_{\nu+\frac{1}{2}}$, $j = 1, 2, \ldots, N$, thus defining a path in phase space, connecting $\mathbf{v}_\nu$ to $\mathbf{v}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{v}_\nu + \Delta \mathbf{v}_{\nu+\frac{1}{2}} \equiv \mathbf{v}_{\nu+1}$.

Since $\mathbf{u} \mapsto \mathbf{v}$ is one-to-one, the path is mirrored in the usual phase space of conservative variables, $\{\mathbf{u}^j_{\nu+\frac{1}{2}} := \mathbf{u}(\mathbf{v}^j_{\nu+\frac{1}{2}})\}_{j=1}^{N+1}$, starting with $\mathbf{u}^1_{\nu+\frac{1}{2}} = \mathbf{u}_\nu$ and ending with $\mathbf{u}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{u}_{\nu+1}$. Equipped with this notation we turn to our next result.

Theorem 3.2. *The conservative scheme* $d\mathbf{u}_\nu(t)/dt = -\left[\mathbf{g}^*_{\nu+\frac{1}{2}} - \mathbf{g}^*_{\nu-\frac{1}{2}}\right]/\Delta x_\nu$, *with a numerical flux* $\mathbf{g}^*_{\nu+\frac{1}{2}}$ *given by*

$$\mathbf{g}^*_{\nu+\frac{1}{2}} = \sum_{j=1}^{N} \frac{\psi\left(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\right) - \psi\left(\mathbf{v}^j_{\nu+\frac{1}{2}}\right)}{\langle \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \Delta \mathbf{v}_{\nu+\frac{1}{2}} \rangle} \boldsymbol{\ell}^j_{\nu+\frac{1}{2}}, \tag{22}$$

is an entropy-conservative approximation consistent with (4). Here, ψ *is the entropy flux potential associated with the conserved entropy pair* (U, F).

We refer the reader to [9] for details; a prototype example is the following entropy-stability result of Lax-Wendroff-type scheme for *systems* of conservation laws, compare example 2.2.

Corollary 3.3. *The following Lax–Wendroff-type difference scheme is a second-order accurate entropy-stable approximation of (2):*

$$\frac{d}{dt}\mathbf{u}_\nu(t) =$$

$$-\frac{1}{2\Delta x_\nu}\left[\sum_{j=1}^{N}\left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu+\frac12}\right)+\mathbf{f}\left(\mathbf{u}^{j+1}_{\nu+\frac12}\right),\mathbf{r}^j_{\nu+\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu+\frac12} - \left\langle \mathbf{f}\left(\mathbf{u}^j_{\nu-\frac12}\right)+\mathbf{f}\left(\mathbf{u}^{j+1}_{\nu-\frac12}\right),\mathbf{r}^j_{\nu-\frac12}\right\rangle \boldsymbol{\ell}^j_{\nu-\frac12}\right]+$$

$$+\frac{1}{8\Delta x_\nu}\left[\sum_{j=1}^{N}\left[a_j\left(\mathbf{u}^{j+1}_{\nu+\frac12}\right)-a_j\left(\mathbf{u}^{j}_{\nu+\frac12}\right)\right]^+\boldsymbol{\ell}^j_{\nu+\frac12} - \sum_{j=1}^{N}\left[a_j\left(\mathbf{u}^{j+1}_{\nu-\frac12}\right)-a_j\left(\mathbf{u}^{j}_{\nu-\frac12}\right)\right]^+\boldsymbol{\ell}^j_{\nu-\frac12}\right].$$

Note that no artificial dissipation is added in shocks and in particular, the scheme has the desirable property of keeping the sharpness of shock profiles.

4. A homotopy approach for fully discrete schemes

We turn to study the cell entropy inequality associated with *fully-discrete* schemes (here and below, κ denotes the fixed mesh ratio $\kappa := \frac{\Delta t}{\Delta x}$):

$$\begin{aligned}\mathbf{u}^{n+1}_\nu =& \mathbf{u}^n_\nu - \frac{\kappa}{2}\left[\mathbf{f}\left(\mathbf{u}^n_{\nu+1}\right)-\mathbf{f}\left(\mathbf{u}^n_{\nu-1}\right)\right]+\\ &+\frac{\kappa}{2}\left[P_{\nu+\frac12}\left(\mathbf{u}^n_{\nu+1}-\mathbf{u}^n_\nu\right)-P_{\nu-\frac12}\left(\mathbf{u}^n_\nu-\mathbf{u}^n_{\nu-1}\right)\right].\end{aligned} \tag{23}$$

We note that the viscosity part of (23) is expressed in terms of the conservative variables rather than the entropy variables in the corresponding semi-discrete scheme (17).

We set $\mathbf{u}^{n+1}_{\nu\pm\frac12} := \mathbf{u}^n_\nu \mp \kappa\left[\mathbf{f}\left(\mathbf{u}^n_{\nu\pm1}\right)-\mathbf{f}\left(\mathbf{u}^n_\nu\right)\right]+\kappa P_{\nu\pm\frac12}\left(\mathbf{u}^n_{\nu\pm1}-\mathbf{u}^n_\nu\right)$ and decompose $\mathbf{u}^{n+1}_\nu$ as the average of two terms $\mathbf{u}^{n+1}_\nu = \left(\mathbf{u}^{n+1}_{\nu+\frac12}+\mathbf{u}^{n+1}_{\nu-\frac12}\right)/2$. We study the entropy inequality for each term. This decomposition into left- and right-handed stencils in the context of cell entropy inequality was first introduced in [6]. We begin by considering $\mathbf{u}^{n+1}_{\nu+\frac12}$.

To this end, we set $\mathbf{u}^n_{\nu+\frac12}(s) := \mathbf{u}^n_\nu + s(\mathbf{u}^n_{\nu+1}-\mathbf{u}^n_\nu)$ and the following inequality is sought (here and below, $\Delta\mathbf{u} := \mathbf{u}^n_{\nu+1}-\mathbf{u}^n_\nu$, $\Delta F(\mathbf{u}^n_\nu) := F\left(\mathbf{u}^n_{\nu+1}\right)-F\left(\mathbf{u}^n_\nu\right),\ldots$):

$$\begin{aligned}\mathcal{I}_+ :=& U\left(\mathbf{u}^{n+1}_{\nu+\frac12}\right)-U\left(\mathbf{u}^n_\nu\right)+\\ &+\kappa\Delta F\left(\mathbf{u}^n_\nu\right)-\kappa\int_{s=0}^{1}\left\langle U'\left(\mathbf{u}^n_{\nu+\frac12}(s)\right),P_{\nu+\frac12}\Delta\mathbf{u}\right\rangle ds \le 0.\end{aligned} \tag{24}$$

We refer to the last statement as a *quasi-cell entropy inequality* since the last expression on the right is not conservative. To verify (24) we proceed as follows. We set $\mathbf{u}^{n+1}_{\nu+\frac12}(s) := \mathbf{u}^n_\nu - \kappa\left[\mathbf{f}\left(\mathbf{u}^n_{\nu+\frac12}(s)\right)-\mathbf{f}(\mathbf{u}^n_\nu)\right]+\kappa P_{\nu+\frac12}\left(\mathbf{u}^n_{\nu+\frac12}(s)-\right.$

$\mathbf{u}^n_\nu\Big)$. Noting that $\mathbf{u}^{n+1}_{\nu+\frac12}(0)=\mathbf{u}^n_\nu$ and $\mathbf{u}^{n+1}_{\nu+\frac12}(1)=\mathbf{u}^{n+1}_{\nu+\frac12}$, we compute

$$U\Big(\mathbf{u}^{n+1}_{\nu+\frac12}\Big)-U(\mathbf{u}^n_\nu)=\int_{s=0}^1\frac{d}{ds}U\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(s)\Big)ds=$$
$$=\int_{s=0}^1\Big\langle U'\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(s)\Big),\Big(-\kappa A\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(s)\Big)+\kappa P_{\nu+\frac12}\Big)\Delta\mathbf{u}\Big\rangle ds,$$

$$\kappa\Delta F(\mathbf{u}^n_\nu)=\kappa\int_{s=0}^1\Big\langle F'\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big),\Delta\mathbf{u}^n_\nu\Big\rangle ds=\kappa\int_{s=0}^1\Big\langle U'\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big)A\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big),\Delta\mathbf{u}\Big\rangle ds.$$

Adding the last two equalities yields

$$\mathcal{I}_+=\int_{s=0}^1\Big\langle U'\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(s)\Big)-U'\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big),-\kappa\Big(P_{\nu+\frac12}-A\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big)\Big)\Delta\mathbf{u}\Big\rangle ds.$$

Next, we introduce $\mathbf{u}^n_{\nu+\frac12}(r,s):=\mathbf{u}^n_{\nu+\frac12}(s)+r\Big(\mathbf{u}^n_\nu-\mathbf{u}^n_{\nu+\frac12}(s)\Big)\equiv\mathbf{u}^n_\nu+s(1-r)\Delta\mathbf{u}^n_\nu$, and we set

$$\mathbf{u}^{n+1}_{\nu+\frac12}(r,s)=\mathbf{u}^n_{\nu+\frac12}(r,s)-\kappa\Big(\mathbf{f}\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big)-\mathbf{f}\Big(\mathbf{u}^n_{\nu+\frac12}(r,s)\Big)\Big)+$$
$$+\kappa P_{\nu+\frac12}\Big(\mathbf{u}^n_{\nu+\frac12}(s)-\mathbf{u}^n_{\nu+\frac12}(r,s)\Big)$$

so that $\mathbf{u}^{n+1}_{\nu+\frac12}(0,s)=\mathbf{u}^n_{\nu+\frac12}(s)$ and $\mathbf{u}^{n+1}_{\nu+\frac12}(1,s)=\mathbf{u}^{n+1}_{\nu+\frac12}(s)$. This then yields

$$U'\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(s)\Big)-U'\Big(\mathbf{u}^n_{\nu+1}(s)\Big)=\int_{r=0}^1\frac{d}{dr}U'\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(r,s)\Big)dr=$$
$$=-s\int_{r=0}^1U''\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(r,s)\Big)dr\Big(I+\kappa A\Big(\mathbf{u}^n_{\nu+\frac12}(r,s)\Big)-\kappa P_{\nu+\frac12}\Big)\Delta\mathbf{u}.$$

Inserting the last expression into the right-hand side of the $\mathcal{I}_+$-equation above, we end up with

$$\mathcal{I}_+=-\int_{r,s=0}^1 s\Big\langle\Big(I+\kappa A\Big(\mathbf{u}^n_{\nu+\frac12}(r,s)\Big)-\kappa P_{\nu+\frac12}\Big)\Delta\mathbf{u},$$
(25)
$$U''\Big(\mathbf{u}^{n+1}_{\nu+\frac12}(r,s)\Big)\Big(-\kappa A\Big(\mathbf{u}^n_{\nu+\frac12}(s)\Big)+\kappa P_{\nu+\frac12}\Big)\Delta\mathbf{u}\Big\rangle drds.$$

A detailed scalar entropy stability analysis can be found in [9]; we extend our discussion to symmetric systems of conservation laws, associated with the quadratic entropy, $U(\mathbf{u})=|\mathbf{u}|^2/2$. We start by setting $C(s):=P_{\nu+\frac12}-A(\mathbf{u}^n_{\nu+\frac12}(s))$, and noting the definition of the (r,s)-variables, we find that $A\Big(\mathbf{u}^n_{\nu+\frac12}(r,s)\Big)-P_{\nu+\frac12}=-C((1-r)s))$. Change of variables, $t:=(1-r)s$, in (25) then yields

$$\mathcal{I}_+=\int_{s=0}^1\int_{t=0}^s\Big\langle\Big(I-\kappa C(t)\Big)\Delta\mathbf{u},\kappa C(s)\Delta\mathbf{u}\Big\rangle dtds. \tag{26}$$

We now make the first requirement of positivity, assuming $C(\cdot)\geq 0$; then the positivity of $\mathcal{I}_+$ follows if and only if the corresponding eigenvalues satisfy $\lambda\big[C(s)\big(I-\kappa C(t)\big)\big] \geq 0$. But $C(s)\big(I-\kappa C(t)\big)$ is similar to $C^{\frac{1}{2}}(s)\big(I-\kappa C(t)\big)C^{\frac{1}{2}}$, which is congruent to, and hence by Sylvester's theorem has the same number of nonnegative eigenvalues as, $I-\kappa C(t)$. This leads to the second requirement, $\kappa\lambda(C(\cdot)) \leq 1$. Recall that $C(s) = P_{\nu+\frac{1}{2}} - A\big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\big)$ is symmetric, and hence the last two requirements amount to the same CFL condition:

$$\kappa A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big) \leq \kappa P_{\nu+\frac{1}{2}} \leq I + \kappa A\Big(\mathbf{u}_{\nu+\frac{1}{2}}(r,s)\Big).$$

In a similar manner, the CFL condition $-\kappa A\Big(\mathbf{u}^n_{\nu-\frac{1}{2}}(s)\Big) \leq \kappa P_{\nu-\frac{1}{2}} \leq I - \kappa A\Big(\mathbf{u}_{\nu-\frac{1}{2}}(r,s)\Big)$ yields the quasi-cell entropy inequality for $\mathbf{u}^{n+1}_{\nu-\frac{1}{2}}$, and the following conclusion.

Corollary 4.1. *[Tadmor 2003]. Consider the fully discrete scheme (23) consistent with the symmetric system (2) and assume the CFL condition*

$$\kappa\Big|A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big)\Big| \leq \kappa P_{\nu+\frac{1}{2}} \leq I - \kappa\Big|A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(r,s)\Big)\Big| \tag{27}$$

is fulfilled. Then the following cell entropy inequality holds for the quadratic entropy pair $U(\mathbf{u}) = |\mathbf{u}|^2/2, F(\mathbf{u}) = \int^{\mathbf{u}} \mathbf{f}(\mathbf{w})d\mathbf{w} - \langle \mathbf{u}, \mathbf{f}(\mathbf{u})\rangle$*:*

$$U\big(\mathbf{u}^{n+1}_\nu\big) \leq \frac{1}{2}\Big(U\Big(\mathbf{u}^{n+1}_{\nu+\frac{1}{2}}\Big) + U\Big(\mathbf{u}^{n+1}_{\nu-\frac{1}{2}}\Big)\Big) \leq U\big(\mathbf{u}^n_\nu\big) - \frac{\kappa}{2}\big(F\big(\mathbf{u}^n_{\nu+1}\big) - F\big(\mathbf{u}^n_{\nu-1}\big)\big) +$$
$$+\frac{\kappa}{2}\bigg(\int_{s=0}^1 \Big\langle U'\Big(\mathbf{v}^n_{\nu+\frac{1}{2}}(s)\Big), P_{\nu+\frac{1}{2}}\Delta\mathbf{u}^n_\nu\Big\rangle ds - \int_{s=0}^1 \Big\langle U'\Big(\mathbf{u}^n_{\nu-\frac{1}{2}}(s)\Big), P_{\nu-\frac{1}{2}}\Delta\mathbf{u}^n_{\nu-1}\Big\rangle ds\bigg).$$

We demonstrate the application of Corollary 4.1 with the prototype example of upwind schemes (examples of centered schemes can be found in [3] and [9, §8]).

Example 4.2. *[Upwind schemes]. We set* $P_{\nu+\frac{1}{2}} = p\big(A\big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\big)\big)$ *with* $p(\cdot)$ *being any viscosity function satisfying* $p(\cdot) \geq |\cdot|$*. The typical example is the upwind scheme*

$$\mathbf{u}^{n+1}_\nu = \mathbf{u}^n_\nu - \frac{\kappa}{2}\big[\mathbf{f}\big(\mathbf{u}^n_{\nu+1}\big) - \mathbf{f}\big(\mathbf{u}^n_{\nu-1}\big)\big] +$$
$$+ \frac{\kappa}{2}\Big[\Big(\sup_s \Big|A\Big(\mathbf{u}^n_{\nu+\frac{1}{2}}(s)\Big)\Big|\Big)\Delta\mathbf{u}^n_{\nu+\frac{1}{2}} - \Big(\sup_s \Big|A\Big(\mathbf{u}^n_{\nu-\frac{1}{2}}(s)\Big)\Big)\Big|\Delta\mathbf{u}^n_{\nu-\frac{1}{2}}\Big].$$

We find that the upwind scheme is entropy-stable for the quadratic entropy function (for symmetric systems) and for all convex entropies (for scalar equations), provided the CFL condition (27) holds, which amounts to $\kappa\sup_{s,\lambda}\Big|\lambda\Big(A\Big(\mathbf{u}_{\nu+\frac{1}{2}}(s)\Big)\Big)\Big| \leq \frac{1}{2}$.

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Center for Scientific Computation and Mathematical Modeling (CSCAMM)
Department of Mathematics and Institute for Physical Sciences & Technology
University of Maryland
College Park, MD 20742 USA
E-mail address: tadmor@cscamm.umd.edu