

## A MINIMUM ENTROPY PRINCIPLE IN THE GAS DYNAMICS EQUATIONS \*

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Let  $\mathbf{u}(\bar{x}, t)$  be a weak solution of the Euler equation, governing the inviscid polytropic gas dynamics; in addition,  $\mathbf{u}(\bar{x}, t)$  is assumed to respect the usual entropy conditions connected with the conservative Euler equations. We show that such *entropy solutions* of the gas dynamics equations satisfy a *minimum entropy principle*, namely, that the spatial minimum of their specific entropy,  $\text{Ess inf}_{\bar{x}} S(\mathbf{u}(\bar{x}, t))$ , is an *increasing* function of time. This principle equally applies to discrete approximations of the Euler equations such as the Godunov-type and Lax–Friedrichs schemes. Our derivation of this minimum principle makes use of the fact that there is a *family* of generalized entropy functions connected with the conservative Euler equations.

### 1. Introduction

Many phenomena in continuum mechanics are modeled by hyperbolic systems of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{k=1}^d \frac{\partial f^{(k)}}{\partial x_k} = 0, \quad (\bar{x} = (x_1, \dots, x_d), t) \in \mathbb{R} \times [0, \infty), \quad (1.1)$$

where  $\mathbf{f}^{(k)} \equiv \mathbf{f}^{(k)}(\mathbf{u}) = (f_1^{(k)}, \dots, f_N^{(k)})^T$  are smooth nonlinear flux mappings of the  $N$ -vector of conservative variables  $\mathbf{u} \equiv \mathbf{u}(\bar{x}, t) = (u_1, \dots, u_N)^T$ . Friedrichs and Lax [3] have observed that the hyperbolic nature of such models is revealed by the property of most of those systems being endowed with a *generalized entropy function*: A smooth convex mapping  $U(\mathbf{u})$  augmented with *entropy flux* mappings  $\bar{F} \equiv \bar{F}(\mathbf{u}) = (F^{(1)}(\mathbf{u}), \dots, F^{(d)}(\mathbf{u}))$ , such that the following compatibility relations hold

$$U_{\mathbf{u}}^T \mathbf{f}_u^{(k)} = F_u^{(k)T}, \quad k = 1, 2, \dots, d. \quad (1.2)$$

Multiplying (1.1) by  $U_{\mathbf{u}}^T$  and employing (1.2), one arrives at an equivalent formulation of the compatibility relations (1.2), namely, that *under the smooth regime* we have on top of (1.1) the additional conservation of entropy

$$\frac{\partial U}{\partial t} + \sum_{k=1}^d \frac{\partial F^{(k)}}{\partial x_k} = 0. \quad (1.3)$$

Owing to the nonlinearity of the fluxes  $\mathbf{f}^{(k)}(\mathbf{u})$ , solutions of (1.1) may develop singularities at

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a finite time after which one must admit *weak* solutions, i.e., those derived directly from the underlying *integral* conservative equations. Considering (1.1) as a *strong* limit of the regularized problem,

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{k=1}^d \frac{\partial \mathbf{f}^{(k)}}{\partial x_k} = \mu \sum_{k=1}^d \frac{\partial^2 \mathbf{u}}{\partial x_k^2}, \quad \mu \downarrow 0, \quad (1.4)_\mu$$

then, following Lax [9] and Krushkov [8], we postulate as an admissibility criterion for such limit solutions, an entropy stability condition which manifests itself in terms of an *entropy inequality*: We have, in the sense of distributions,

$$\frac{\partial U}{\partial t} + \sum_{k=1}^d \frac{\partial F^{(k)}}{\partial x_k} \leq 0. \quad (1.5)$$

Weak solutions of (1.1), which in addition satisfy the inequality (1.5) for *all* entropy pairs  $(U, \vec{F})$  connected with that system, are called *entropy solutions*.<sup>1</sup> Having a (weakly) nonpositive quantity on the L.H.S. of (1.5) is thus a consequence of viewing these entropy solutions as limits of vanishing dissipativity mechanisms. In particular, the inequality (1.5) implies that the total entropy in the domain decreases in time (we assume entropy outflux through the boundaries)

$$\frac{d}{dt} \int_{\bar{x}} U(\mathbf{u}(\bar{x}, t)) \, d\bar{x} \leq 0. \quad (1.6)$$

In this paper, we consider entropy solutions,

$$\mathbf{u} = (\rho, \mathbf{m}, E)^T \quad (1.7a)$$

of the Euler equations. These equations govern the inviscid polytropic gas dynamics, asserting the conservation of the density  $\rho$ , the momentum  $\mathbf{m} = (m_1, m_2, m_3)^T$ , and the energy  $E$ . Let  $\mathbf{q} \equiv \mathbf{m}/\rho$  denote the velocity field of such motion. Then, expressed in terms of the pressure,  $p$ ,

$$p = (\gamma - 1) \cdot \left[ E - \frac{1}{2} \cdot \rho \cdot |\mathbf{q}|^2 \right], \quad \gamma = \text{adiabatic exponent}, \quad (1.7b)$$

the corresponding fluxes in this case are given by<sup>2</sup>

$$\mathbf{f}^{(k)} = (m_k, q_k \cdot \mathbf{m} + p \cdot \mathbf{e}^{(k)}, q_k (E + p))^T, \quad k = 1, 2, 3. \quad (1.7c)$$

The main result of this paper asserts that entropy solutions of Euler equations satisfy the following principle.

**Minimum Principle.** *Let  $\mathbf{u} \equiv \mathbf{u}(\bar{x}, t)$  be an entropy solution of the gas dynamics equations (1.7) and let*

$$S(\bar{x}, t) \equiv S(\mathbf{u}(\bar{x}, t)) = \ln(pp^{-\gamma}) \quad (1.8)$$

*denote the specific entropy of such solution. Then the following estimate holds*

$$\text{Ess inf}_{|\bar{x}| \leq R} S(\bar{x}, t) \geq \text{Ess inf}_{|\bar{x}| \leq R + t \cdot q_{\max}} S(\bar{x}, t = 0). \quad (1.9)$$

*Here  $q_{\max}$  stands for the maximal speed  $|\mathbf{q}|$  in the domain.*

The proof of this assertion is provided in Section 3. Prior to that we elaborate in Section 2 on the entropy inequality connected with the gas dynamics equations. In particular, Harten [5] has

<sup>1</sup> Krushkov [8, p.241] has termed such solutions simply as *generalized solutions*.

<sup>2</sup> With  $\mathbf{e}^{(k)}$  denoting the unit Cartesian vector  $\mathbf{e}_j^{(k)} = \delta_{kj}$ .

shown that there exists a whole family of entropy pairs associated with these equations, a fact which is essential in our derivation of the minimum principle.

As an immediate consequence of the minimum principle, we conclude that  $\text{Ess inf}_{\bar{x}} S(\bar{x}, t)$  is an increasing function of  $t$  for every entropy solution of (1.7). The following argument sheds additional light on this conclusion in the case of a piecewise-smooth flow. To this end, an arbitrary particle currently located at  $(\bar{x}, t)$  is traced backwards in time into its initial position at  $t = 0$ . Since the specific entropy of such particle remains constant along the particle path—except for its decrease when crossing backwards shock waves, it follows that its value  $S(\bar{x}, t)$  is greater or equal than that of the initial spatial minimum  $\text{Ess inf}_{\bar{x}} S(\bar{x}, t = 0)$ , as asserted. In contrast to the above ‘Lagrangian’ argument, the derivation of the minimum principle outlined below, is purely an ‘Eulerian’ one. It enables us to relax the regularity assumption on the flow, and—since we do not follow the characteristics—it equally applies to discrete approximations of the Euler equations.

In Section 4 we consider approximate solutions of the Euler equations,  $w(\bar{x}_\nu, t)$ , which respect the entropy decrease estimate (1.6),

$$\sum_{\nu} U(w(\bar{x}_\nu, t + \Delta t)) \Delta \bar{x}_\nu \leq \sum_{\nu} U(w(\bar{x}_\nu, t)) \Delta \bar{x}_\nu. \tag{1.10}$$

We note that such approximate solutions are obtained by entropy stable schemes satisfying the cell entropy inequality

$$U(w(\bar{x}_\nu, t + \Delta t)) \leq U(w(\bar{x}_\nu, t)) + \sum_{k=1}^d \frac{1}{\Delta \bar{x}_\nu} [F_{\nu+1/2}^{(k)} - F_{\nu-1/2}^{(k)}], \tag{1.11}$$

e.g., the Godunov-type and Lax–Friedrichs schemes [6]. We have the following.

**Minimum Principle.** Let  $w(\bar{x}_\nu, t)$  be an approximate solution of the gas dynamics equations (1.7) and let

$$S(\bar{x}_\nu, t) \equiv S(w(\bar{x}_\nu, t)) = \ln(p \rho^{-\gamma}) \tag{1.12}$$

denote the specific entropy of such solution. Assume that its total entropy decreases in time, (1.10). Then the following estimate holds

$$S(\bar{x}, t + \Delta t) \geq \min_{\nu} [S(\bar{x}_\nu, t)]. \tag{1.13}$$

In the case of entropy stable schemes, (1.11), a more precise estimate is obtained, which takes into account the support of the schemes’ stencil.

The inequality (1.13) leads to an a priori pointwise estimate on the approximate solution  $w(\bar{x}, t)$ . Such pointwise estimates play an essential role with regard to question of the convergence of entropy stable schemes. In particular, DiPerna [2, Section 7] has recently shown that in certain cases, such (two-sided) estimates are sufficient in order to guarantee the convergence of such schemes.

## 2. Generalized entropy functions of the Euler equations

We consider the Euler equations for polytropic gas

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \mathbf{m} \\ E \end{bmatrix} + \sum_{k=1}^3 \frac{\partial}{\partial x_k} \begin{bmatrix} m_k \\ q_k \mathbf{m} + p \mathbf{e}^{(k)} \\ q_k (E + p) \end{bmatrix} = 0. \tag{2.1}$$

It is well-known, e.g., [1], that for all smooth solutions of (2.1) the specific entropy <sup>3</sup>

$$S(\bar{x}, t) = \ln(p\rho^{-\gamma})$$

remains constant along streamlines, i.e.,

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \sum_{k=1}^3 q_k \frac{\partial S}{\partial x_k} = 0. \quad (2.2a)$$

Let  $h(S)$  be an arbitrary smooth function of  $S$ . Multiplying (2.2a) by  $\rho h'(S)$ —prime denoting  $S$ -differentiation—we find

$$\rho \frac{\partial h(S)}{\partial t} + \sum_{k=1}^3 m_k \frac{\partial h(S)}{\partial x_k} = 0.$$

Adding this to the continuity equation which is premultiplied by  $h(S)$ ,

$$\frac{\partial \rho}{\partial t} h(S) + \sum_{k=1}^3 \frac{\partial m_k}{\partial x_k} h(S) = 0, \quad (2.2b)$$

we obtain, after changing sign, a conservative entropy equation like (1.3) which reads [5]

$$\frac{\partial}{\partial t} [-\rho h(S)] + \sum_{k=1}^3 \frac{\partial}{\partial x_k} [-m_k h(S)] = 0. \quad (2.3)$$

In order to comply with the further requirement of being a generalized entropy function,  $U(\mathbf{u}) = -\rho h(S)$  has to be a *convex* function of the conservative variables  $\mathbf{u} = (\rho, \mathbf{m}, E)^T$ . A straightforward computation carried out by Harten [5, Section 2] in the two-dimensional case shows that the Hessian  $U_{\mathbf{u}\mathbf{u}}$  is positive definite if and only if

$$\rho [h'(S) - \gamma \cdot h''(S)] > 0, \quad h'(S) > 0.$$

Excluding negative densities we may summarize that *there exists a family of (generalized) entropy pairs  $(U, \vec{F})$  associated with Euler equations (2.1),*

$$U(\mathbf{u}) = -\rho h(S), \quad F^{(k)}(\mathbf{u}) = -m_k h(S), \quad k = 1, 2, 3, \quad (2.4a)$$

*generated by the smooth increasing functions  $h(S)$  which satisfy*

$$h'(S) - \gamma \cdot h''(S) > 0. \quad (2.4b)$$

### 3. A minimum entropy principle

Let  $\mathbf{u} = (\rho, \mathbf{m}, E)^T$  be an entropy solution of the gas dynamics equations (2.1). Such a solution is characterized by the entropy inequality (1.5)

$$\frac{\partial U(\mathbf{u})}{\partial t} + \sum_{k=1}^3 \frac{\partial F^{(k)}(\mathbf{u})}{\partial x_k} \leq 0 \quad (3.1)$$

which holds for *all* entropy pairs  $(U, \vec{F})$  connected with the equations. Thus we have

$$\frac{\partial}{\partial t} [-\rho h(S)] + \sum_{k=1}^3 \frac{\partial}{\partial x_k} [-m_k h(S)] \leq 0 \quad (3.2a)$$

<sup>3</sup> After normalization, taking the specific heat constant to be  $c_v = 1$ .

where  $h(S)$  is any smooth increasing function which satisfies

$$h'(S) - \gamma \cdot h''(S) > 0. \tag{3.2b}$$

To derive a minimum principle, we shall make use of an argument due to Lax [9, Section 3]. We begin with the following lemma.

**Lemma 3.1.** *Let  $u$  be an entropy solution of the gas dynamics equations (2.1). Then for all nonpositive smooth increasing functions  $h(S)$  satisfying (3.2b), we have*

$$\int_{|\bar{x}| \leq R} \rho(\bar{x}, t) \cdot h(S(\bar{x}, t)) \, d\bar{x} \geq \int_{|\bar{x}| \leq R+t \cdot q_{\max}} \rho(\bar{x}, 0) \cdot h(S(\bar{x}, 0)) \, d\bar{x}. \tag{3.3}$$

Here  $q_{\max}$  denotes the maximal speed  $|q|$  in the domain.

**Proof.** As in [10, Theorem 4.1] we integrate the entropy inequality (3.2a) over the truncated cone  $C = \{|\bar{x}| \leq R + (t - \tau) \cdot q_{\max} \mid 0 \leq \tau \leq t\}$ ; if we let  $(n_0, \bar{n})$  denote the unit outward normal, then by Green's theorem

$$\int_{\partial C} \rho h(S) \cdot \left[ n_0 + \sum_{k=1}^3 q_k n_k \right] \partial \bar{x} \geq 0. \tag{3.4}$$

The integrals over the top and bottom surfaces give us the difference between the left- and right-hand sides in (3.3), and by (3.4) this difference is bounded from below by

$$- \int_{\text{mantle}} \rho h(S) \cdot \left[ n_0 + \sum_{k=1}^3 q_k n_k \right] \partial \bar{x}.$$

The result follows upon showing that the last quantity is *nonnegative*. Indeed, since by assumption  $-\rho h(S) < 0$ , this is the same thing as

$$n_0 + \sum_{k=1}^3 q_k n_k \geq 0;$$

on the mantle we have

$$(n_0, \bar{n}) = (1 + q_{\max}^2)^{-1/2} (q_{\max}, \bar{x}/|\bar{x}|),$$

and hence

$$\begin{aligned} n_0 + \sum_{k=1}^3 q_k n_k &= (1 + q_{\max}^2)^{-1/2} \left( q_{\max} + \sum_{k=1}^3 \frac{q_k \bar{x}_k}{|\bar{x}|} \right) \\ &\geq (1 + q_{\max}^2)^{-1/2} \left( q_{\max} - \sum_{k=1}^3 \frac{|q_k|^2}{|q|} \right) \geq 0 \end{aligned}$$

as asserted.  $\square$

The discussion in Lemma 3.1 was restricted to smooth functions  $h(S)$ ; by passing to the limit, its conclusion (3.3) follows for any nonpositive nondecreasing function  $h(S)$  satisfying (3.2b), whether smooth or not.

To derive the minimum entropy principle, we make a special choice of such function,  $h(S)$ , given by

$$h(S) = \min[S - S_0, 0], \quad S_0 = \operatorname{Ess\,inf}_{|\bar{x}| \leq R+t \cdot q_{\max}} S(\bar{x}, 0). \tag{3.5}$$

The nonpositive function  $h(S)$  is a nondecreasing concave one, hence admissible by (3.2b), and consequently (3.3) applies

$$\begin{aligned} & \int_{|\bar{x}| \leq R} \rho(\bar{x}, t) \cdot \min[S(\bar{x}, t) - S_0, 0] \, d\bar{x} \\ & \geq \int_{|\bar{x}| \leq R + t \cdot q_{\max}} \rho(\bar{x}, 0) \cdot \min[S(\bar{x}, 0) - S_0, 0] \, d\bar{x}. \end{aligned} \quad (3.6)$$

Now, by the choice of  $S_0$ , the integral on the right of (3.6) vanishes since  $\min[S(\bar{x}, 0) - S_0, 0]$  does. The inequality (3.6) then tells us that the integral on the left is also nonnegative. But since the integrand on the left is by definition nonpositive, this can be the case provided this integrand vanishes almost everywhere; that is, we have for almost all  $\bar{x}$ ,  $|\bar{x}| \leq R$

$$S(\bar{x}, t) \geq S_0 = \operatorname{Ess\,inf}_{|\bar{x}| \leq R + t \cdot q_{\max}} S(\bar{x}, t = 0)$$

and (1.9) follows.

The minimum entropy principle was deduced from the entropy inequality (3.2), which in turn was postulated based on the formal regularization introduced in (1.4). In general, other regularizations equally apply; in particular, Euler equations are usually sought as the *vanishing viscosity limit* of the Navier–Stokes equations (here we take for simplicity the one-dimensional case)<sup>4</sup>

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ m \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} m \\ qm + p \\ q(E + p) \end{bmatrix} = \mu \frac{\partial}{\partial x} \begin{bmatrix} 0 \\ \partial q / \partial x \\ q \partial q / \partial x \end{bmatrix}, \quad \mu \downarrow 0. \quad (3.7)$$

Do the (generalized) entropy inequalities (3.2) remain valid on the basis of such limit? To answer this question we first note that if  $U(\mathbf{u})$  is *any* entropy function, then thanks to its convexity the mapping  $\mathbf{u} \rightarrow \mathbf{v} \equiv U_{\mathbf{u}}$  is one-to-one, and hence one can make the change of variables  $\mathbf{u} = \mathbf{u}(\mathbf{v})$ . Harten [5] has shown that such change of variable by *each* member of the family of entropy functions (2.4) puts the viscosity terms on the right of (3.7) into a *negative* semidefinite form. This makes apparent the *dissipative* effect of these viscosity terms. Indeed, if  $T = [E/\rho - \frac{1}{2} \cdot |q|^2]/(\gamma - 1)$  denotes the absolute temperature, then direct manipulation of (3.7) yields, e.g., [1, Section 63], [12, Section 6.10],

$$\frac{\partial}{\partial t} [\rho h(S)] + \frac{\partial}{\partial x} [mh(S)] = \mu \cdot h'(S) \frac{q_x^2}{T}, \quad (3.8)$$

from which we recover the entropy inequality (3.2a) for *all smooth increasing functions*  $h(S)$ . We note that the convexity condition was *not* assumed in this case. The merit of using the convexity condition, however, is that it enables us to deal with more general artificial viscosity terms, other than those appearing in the Navier–Stokes equations. Such artificial viscosity terms are frequently encountered in finite-difference approximations to the Euler equations; specific examples of this kind are studied in the next section.

Finally we would like to remark on the previously mentioned Navier–Stokes equations. Our discussion above took into account only the viscosity contribution, neglecting heat conduction.

<sup>4</sup> With  $\mu$  combining the two viscosity coefficients in the general Navier–Stokes equations.

Hughes et al. [7] have shown that when the heat flux is also added, compare (3.7),

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ m \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} m \\ qm + p \\ q(E + p) \end{bmatrix} = \mu \frac{\partial}{\partial x} \begin{bmatrix} 0 \\ \partial q / \partial x \\ q \partial q / \partial x \end{bmatrix} + \kappa \frac{\partial}{\partial x} \begin{bmatrix} 0 \\ 0 \\ \partial T / \partial x \end{bmatrix} \quad (3.9)$$

with  $\kappa$  denoting the heat conductivity constant, then only the ‘physical’ entropy,  $U(\mathbf{u}) = -\rho S$  survives as the one which puts the additional heat flux into a *symmetric* negative-definite form. We would like to note in this connection the different limit behavior of the Navier–Stokes flows depending on the viscosity and heat conductivity; Gilbarg [4] has shown that as  $\kappa \rightarrow 0$  keeping  $\mu$  fixed, we are led to a continuous thermally nonconducting shock layer, whereas for  $\mu \rightarrow 0$  with  $\kappa$  fixed the convergence is to a (generally) discontinuous nonviscous shock layer. Consequently, the viscosity rather than the heat flux should play the major rule in an appropriate regularization model for the Euler equations.

#### 4. Discrete approximations of the Euler equations

In this section we consider approximate solutions of the Euler equations,  $w(x_\nu, t)$ , whose total entropy decreases in time, compare (1.10)

$$\sum_\nu U(w(\bar{x}_\nu, t + \Delta t)) \Delta \bar{x}_\nu \leq \sum_\nu U(w(\bar{x}_\nu, t)) \Delta \bar{x}_\nu. \quad (4.1)$$

Estimate (4.1) holds for *all* entropy functions  $U = -\rho h(S)$  in (2.4). By passing to the limit, this applies to our previous choice of the function  $h(S)$  in (3.5)

$$h(S) = \min[S - S_0, 0], \quad (4.2a)$$

this time with a constant  $S_0$  which is taken to be

$$S_0 = \min_\nu S(w(\bar{x}_\nu, t)). \quad (4.2b)$$

By our choice of  $S_0$ , we have  $U(w(\bar{x}_\nu, t)) = 0$ . The inequality (4.1) tells us that the left-hand side is therefore, nonnegative; consequently

$$S(\bar{x}, t + \Delta t) - S_0 \geq h(S(x, t + \Delta t)) \geq 0$$

and (1.13) follows.

Approximate solutions which fulfill the required estimate (4.1) can be obtained by entropy stable schemes satisfying the cell entropy inequality (1.11)

$$U(w(\bar{x}_\nu, t + \Delta t)) \leq U(w(\bar{x}_\nu, t)) + \sum_{k=1}^d \frac{1}{\Delta x_\nu} [F_{\nu+1/2}^{(k)} - F_{\nu-1/2}^{(k)}]. \quad (4.3)$$

Examples of such entropy stable schemes include the Godunov-type and Lax–Friedrichs schemes, e.g., [6]. A more precise minimum principle follows in these cases, taking into account the support of the schemes’ stencil. In particular, the (one-dimensional) Godunov scheme results from averaging of two neighboring Riemann problems [6], each of which satisfies (1.9). Consequently we have the following.

**Minimum Principle** (of the Godunov scheme). *Let  $w(x_\nu, t)$  the Godunov approximate solution to the Euler equations (2.1). Assume that the appropriate CFL condition is met. Then the following estimate holds*

$$S(w(x_\nu, t + \Delta t)) \geq \min_{\nu-1 \leq j \leq \nu+1} S(w(x_j, t)). \quad (4.4)$$

Since the Lax–Friedrichs scheme coincides with a staggered Godunov’s solver, the same conclusion, (4.4), holds. Another way to see this is outlined below; it makes no reference to Riemann’s solution and can be generalized to the multidimensional problem.

To this end, we approximate the (for simplicity—one-dimensional) Euler equations with the Lax–Friedrichs scheme

$$\begin{aligned} w(x_\nu, t + \Delta t) &= \frac{1}{2} [w(x_{\nu+1}, t) + w(x_{\nu-1}, t)] - \frac{1}{2}\lambda [f(w(x_{\nu+1}, t)) - f(w(x_{\nu-1}, t))], \\ \lambda &\equiv \Delta t / \Delta x. \end{aligned} \quad (4.5)$$

We remark that the Lax–Friedrichs scheme can be derived from center differencing of the regularization model (1.4) $_{\Delta x}$ . Lax has shown [9, Theorem 1.2] that if  $\lambda \equiv \Delta t / \Delta x$  is sufficiently small, then solutions of this difference scheme satisfy the following cell entropy inequality

$$\begin{aligned} U(w(x_\nu, t + \Delta t)) &= \frac{1}{2} [U(w(x_{\nu+1}, t)) + U(w(x_{\nu-1}, t))] \\ &\quad - \frac{1}{2}\lambda [F(w(x_{\nu+1}, t)) - F(w(x_{\nu-1}, t))] \end{aligned} \quad (4.6)$$

for all entropy pairs  $(U, F) = (-\rho h(S), -mh(S))$  in (2.4). By passing to the limit, this applies to our previous choice of the function  $h(S)$  in (3.5)

$$h(S) = \min[S - S_0, 0], \quad (4.7a)$$

this time, with a constant  $S_0$  which is taken to be

$$S_0 = \min[S(x_{\nu+1}, t), S(x_{\nu-1}, t)]. \quad (4.7b)$$

The inequality (4.6) now reads

$$\begin{aligned} \rho(x_\nu, t + \Delta t) \cdot h(S(x_\nu, t + \Delta t)) \\ \geq \left[ \frac{1}{2}(1 + \lambda q(x_{\nu-1}, t)) \rho(x_{\nu-1}, t) \cdot h(S(x_{\nu-1}, t)) \right. \\ \left. + \frac{1}{2}(1 - \lambda q(x_{\nu+1}, t)) \rho(x_{\nu+1}, t) \cdot h(S(x_{\nu+1}, t)) \right]. \end{aligned} \quad (4.8)$$

By our choice of the function  $h(S)$  in (4.7), we have  $h(S(x_{\nu \pm 1}, t)) = 0$ . The inequality (4.8) tells us that the left-hand side is therefore nonnegative; consequently

$$0 \leq h(S(x_\nu, t + \Delta t)) \leq S(x_\nu, t + \Delta t) - S_0$$

and the following minimum principle follows

$$S(w(x_\nu, t + \Delta t)) \geq \min S(w(x_{\nu \pm 1}, t)).$$

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