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FOREWORD

On the occasion of the International Conference on Nonlinear Hyperbolic Problems held in St. Etienne, France, 1986 it was decided to start a two years cycle of conferences on this very rapidly expanding branch of mathematics and it's applications in Continuum Mechanics and Aerodynamics. The second conference took place in Aachen, FRG, March 14-18, 1988. The number of more than 200 participants from more than 20 countries all over the world and about 100 invited and contributed papers, well balanced between theory, numerical analysis and applications, do not leave any doubt that it was the right decision to start this cycle of conferences, of which the third will be organized in Sweden in 1990.

This volume contains sixty eight original papers presented at the conference, twenty two of them dealing with the mathematical theory, e.g. existence, uniqueness, stability, behaviour of solutions, physical modelling by evolution equations. Twenty two articles in numerical analysis are concerned with stability and convergence to the physically relevant solutions such as schemes especially deviced for treating shocks, contact discontinuities and artificial boundaries. Twenty four papers contain multidimensional computational applications to nonlinear waves in solids, flow through porous media and compressible fluid flow including shocks, real gas effects, multiphase phenomena, chemical reactions etc.

The editors and organizers of the Second International Conference on Hyperbolic Problems would like to thank the Scientific Committee for the generous support of recommending invited lectures and selecting the contributed papers of the conference.

The meeting was made possible by the efforts of many people to whom the organizers are most grateful. It is a particular pleasure to acknowledge the help of Riikka Tuominen for preparing the abstract book and Bert Pohl for his dedicated help organizing the conference. It is also a pleasure to thank Sylvie Wiertz, Angela Schneider, Gabriele Goblet and Thomas Hoerkens for preparing these proceedings. Finally the organizers are indebted to the host organizations Rheinisch Westfälische Technische Hochschule Aachen and the city of Aachen and to those organizations which provided the needed financial support for the conference: Control Data GmbH, Cray Research GmbH, Deutsche Forschungsgemeinschaft, Diehl GmbH & Co., Digital Equipment GmbH, FAHO Gesellschaft von Freunden der Aachener Hochschule, IBM Deutschland GmbH, Mathematisch-Naturwissenschaftliche Fakultät der RWTH, Ministerium für Wissenschaft und Forschung des Landes Nordrhein-Westfalen, Office of Naval Research Branch of London, Rheinmetall GmbH, US Air Force EOARD, US Army European Research Office of London, Wegmann GmbH & Co.

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SIMPLE STABILITY CRITERIA FOR DIFFERENCE APPROXIMATIONS OF HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS

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SUMMARY

In this note we discuss new, simple stability criteria for a wide class of finite difference approximations for initial-boundary value problems associated with the hyperbolic system $\partial u/\partial t = A\partial u/\partial x + Bu + \mathbf{f}$ in the quarter plane $x \geqslant 0$, $t \geqslant 0$. With these criteria, stability is easily achieved for a multitude of examples that incorporate and generalize most of the cases studied in recent literature.

Consider the first order system of hyperbolic partial differential equations

$$\partial u(x,t)/\partial t = A\partial u(x,t)/\partial x + Bu(x,t) + \mathbf{f}(x,t), \quad x \geqslant 0, \ t \geqslant 0, \tag{1a}$$

where $u(x,t) = \left(u^{(1)}(x,t),\dots,u^{(n)}(x,t)\right)'$ is the unknown vector (prime denoting the transpose), $\mathbf{f}(x,t) = \left(f^{(1)}(x,t),\dots,f^{(n)}(x,t)\right)'$ is a given n-vector, and A and B are fixed $n \times n$ matrices such that A is diagonal of the form

$$A = \operatorname{diag}\left(A^{I},A^{II}\right), \quad A^{I} > 0, \quad A^{II} < 0, \tag{2}$$

with A^{I} and A^{II} of orders $k \times k$ and $(n-k) \times (n-k)$, respectively.

The solution of (1a) is uniquely determined if we prescribe intial values

$$u(x,0), \quad x \geqslant 0 \tag{1b}$$

and boundary conditions

$$u^{II}(0,t) = Su^{I}(0,t) + \mathbf{g}(t), \quad t \geqslant 0, \tag{1c}$$

where S is a fixed $(n-k) \times k$ coupling matrix, $\mathbf{g}(t)$ a given $(n-k)$-vector, and

$$u^{I} = \left(u^{(1)},\dots,u^{(k)}\right)', \quad u^{II} = \left(u^{(k+1)},\dots,u^{(n)}\right)' \tag{3}$$

a partition of u into its outflow and inflow components, respectively, corresponding to the partition of A in (2).

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Introducing a mesh size $\Delta x > 0$, $\Delta t > 0$, such that $\lambda \equiv \Delta t/\Delta x$ is constant, and using the notation $v_\nu(t) = v(\nu\Delta x, t)$, we approximate (1a) by a general, basic difference scheme — explicit or implicit, dissipative or not, two-level or multilevel — of the form

$$Q_{-1}v_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q_\sigma v_\nu(t-\sigma\Delta t) + \Delta t b_\nu(t), \quad \nu = r, r+1, \dots,$$

$$Q_\sigma = \sum_{j=-r}^{p} A_{j\sigma}E^j, \quad Ev_\nu = v_{\nu+1}, \quad \sigma = -1, \dots, s, \tag{4}$$

where the $n \times n$ coefficient matrices $A_{j\sigma}$ are polynomials in λA and ΔtB, and the n-vectors $b_\nu(t)$ depend on $f(x,t)$ and its derivatives.

The difference equations in (4) have a unique solution $v_\nu(t+\Delta t)$ if we provide initial values

$$v_\nu(\mu\Delta t), \quad \mu = 0, \dots, s, \quad \nu = 0, 1, 2, \dots, \tag{5}$$

and specify, at each time level $t = \mu\Delta t$, $\mu = s, s+1, \dots$, boundary values $v_\nu(t+\Delta t)$, $\nu = 0, \dots, r-1$. Such boundary values are determined by conditions of the form

$$T_{-1}^{(\nu)}v_\nu(t+\Delta t) = \sum_{\sigma=0}^{q} T_\sigma^{(\nu)}v_\nu(t-\sigma\Delta t) + \Delta t d_\nu(t), \quad \nu = 0, \dots, r-1,$$

$$T_\sigma^{(\nu)} = \sum_{j=0}^{m} C_{j\sigma}^{(\nu)}E^j, \quad \sigma = -1, \dots, q, \tag{6a}$$

where the $n \times n$ matrices $C_{j\sigma}^{(\nu)}$ depend on A, ΔtB and S, and the n-vectors $d_\nu(t)$ are functions of $f(x,t)$, $g(t)$ and their derivatives.

Our intention is to interpret the difficult and often stubborn Gustafsson-Kreiss-Sundström (GKS) stability criterion in [4] in order to obtain simple and convenient stability criteria for approximation (4)-(6a). While we were unable to meet this goal for general boundary conditions of type (6a), we managed to achieve rather satisfactory results under the further assumption that, in accordance with the partition of A in (2), the $C_{j\sigma}^{(\nu)}$ are of the form

$$C_{j\sigma}^{(\nu)} = \begin{bmatrix} C_{j\sigma}^{II} & C_{j\sigma}^{I\ II(\nu)} \\ C_{j\sigma}^{II\ I(\nu)} & C_{j\sigma}^{II\ II(\nu)} \end{bmatrix}, \tag{6b}$$

where

$$\text{the } C_{j\sigma}^{II} \text{ are independent of } \nu, \tag{6c}$$

$$\text{the } C_{j\sigma}^{II} \text{ are diagonal when } B = 0, \tag{6d}$$

$$\text{the } C_{j\sigma}^{I\ II(\nu)} = 0 \text{ when } B = 0, \tag{6e}$$

$$C_{j\sigma}^{II\ II(\nu)} = 0 \text{ for } j > 0 \text{ and } \sigma > -1 \text{ when } B = 0. \tag{6f}$$

The essence of (6c)-(6e) is that for $B = 0$, the outflow boundary conditions are *translatory* (i.e., determined at all boundary points by the same coefficients), *separable* (i.e., split into independent scalar conditions for the different outflow unknowns), and independent of inflow values. Assumption (6f) implies that for $B = 0$ the inflow values at the boundary depend essentially on the outflow.

It should be pointed out that our outflow boundary conditions are quite general, despite the apparent restrictions in (6c)-(6e). Indeed, (6c) is not much of a restriction, since in practice the outflow boundary conditions are translatory. In particular, if the numerical boundary consists of a single point, then the boundary conditions are translatory by definition, so (6c) holds automatically. The restrictions in (6d),(6e) pose no great difficulties either, since they are satisfied by all reasonable boundary conditions, where for $B = 0$ the $C_{j\sigma}^{II}$ usually reduce to polynomials in the block A^I, and the $C_{j\sigma}^{I\ II(\nu)}$ vanish.

We realize that in view of the restriction in (6f) our inflow boundary conditions are not quite as general as the outflow ones. They can, however, be constructed to any degree of accuracy (see [1]); and if the boundary consists of a single point, then such conditions can be achieved in a trivial manner, simply by duplicating the analytic condition (1c), i.e.,

$$v_0^{II}(t+\Delta t) = Sv_0^{I}(t+\Delta t) + g(t+\Delta t).$$

Throughout our work we assume, of course, that the basic scheme (4) is stable for the pure Cauchy problem, and that the other assumptions which guarantee the validity of the GKS theory in [4] hold.

The first step in our analysis was to reduce the above stability question to that of a scalar, homogeneous problem. This is obtained by considering the outflow scalar equation

$$\partial u/\partial t = a\partial u/\partial x, \quad x \geqslant 0, \quad t \geqslant 0, \quad a = \text{constant} > 0, \tag{7}$$

for which the basic scheme (1.4) reduces to the homogeneous scheme

$$Q_{-1}v_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q_\sigma v_\nu(t-\sigma\Delta t), \quad \nu = r, r+1, \ldots$$
$$Q_\sigma = \sum_{j=-r}^{p} a_{j\sigma}E^j, \quad \sigma = -1, \ldots, s, \tag{8a}$$

and the boundary conditions (1.6) reduce to translatory conditions of the form

$$T_{-1}v_\nu(t+\Delta t) = \sum_{\sigma=0}^{q} T_\sigma v_\nu(t-\sigma\Delta t), \quad \nu = 0, \ldots, r-1,$$
$$T_\sigma = \sum_{j=0}^{m} c_{j\sigma}E^j, \quad \sigma = -1, \ldots, q, \tag{8b}$$

where $a_{j\sigma}$ and $c_{j\sigma}$ are scalar coefficients.

Referring to (8) as the *basic approximation*, we proved:

THEOREM 1 [3, Theorem 1.1]. *Approximation* (4)-(6) *is stable if and only if the reduced outflow scalar approximation* (8) *is stable for every eigenvalue* a > 0 *of* A^I. *That is, approximation* (4)-(6) *is stable if and only if the scalar outflow components of its principal part are all stable.*

This reduction theorem implies that from now on we may restrict our stability study to the basic approximation (8).

In order to introduce our stability criteria for the basic approximation, we use the coefficients of the basic scheme (8a) to define the *basic characteristic function*

$$P(z,\kappa) = \sum_{j=-r}^{p} \left[a_{j,-1} - \sum_{\sigma=0}^{s} a_{j\sigma} z^{-\sigma-1} \right] \kappa^j .$$

Similarly, using the coefficients of the boundary conditions in (8b) we define the *boundary characteristic function*

$$R(z,\kappa) = \sum_{j=0}^{m} \left[c_{j,-1} - \sum_{\sigma=0}^{q} c_{j\sigma} z^{-\sigma-1} \right] \kappa^j .$$

Now putting

$$\Omega(z,\kappa) \equiv |P(z,\kappa)| + |R(z,\kappa)| ,$$

it is not difficult to combine Theorems 3.1' and 3.2' of [3] in order to obtain:

THEOREM 2. *The basic approximation* (8) *is stable if:*

(i) either

$$\left. \frac{\partial P(z,\kappa)}{\partial z} \cdot \frac{\partial P(z,\kappa)}{\partial \kappa} \right|_{z=\kappa=-1} < 0 \tag{10a}$$

or

$$\Omega(z=-1,\kappa=-1) > 0. \tag{10b}$$

(ii) $$\Omega(z,\kappa) > 0 \text{ for all } |z| = |\kappa| = 1,\ \kappa \neq 1,\ (z,\kappa) \neq (-1,-1), \tag{10c}$$

$$\Omega(z,\kappa=1) > 0 \text{ for all } |z| = 1,\ z \neq 1, \tag{10d}$$

$$\Omega(z,\kappa) > 0 \text{ for all } |z| \geq 1,\ 0 < |\kappa| < 1. \tag{10e}$$

The advantage of this setting of Theorem 2 is clarified by the following lemma, in which we provide helpful sufficient conditions for each of the four inequalities in (10b-e) to hold:

LEMMA 1 [3, Theorem 2.2].

(*i*) *Inequalities* (10b,c) *hold if either the basic scheme* (8a) *or the boundary conditions* (8b) *are dissipative.*

(*ii*) *Inequality* (10d) *holds if any of the following is satisfied:*
(*a*) *The basic scheme is two-level.*
(*b*) *The basic scheme is three-level and*

$$\Omega(z=-1,\kappa=1) > 0. \qquad (11)$$

(*c*) *The boundary conditions are two-level and at least zero-order accurate as an approximation of equation* (7).

(*d*) *The boundary conditions are three-level, at least zero-order accurate, and* (11) *is satisfied.*

(*iii*) *Inequality* (10e) *holds if the boundary conditions fulfill the von Neumann condition, and are either explicit or satisfy*

$$T_{-1}(\kappa) \equiv \sum_{j=0}^{m} c_{j,-1}\kappa^{j} \neq 0 \text{ for } 0 < |\kappa| \leqslant 1.$$

As mentioned earlier, we always assume that the basic scheme is *stable* for the pure Cauchy problem, i.e.,

(i) The basic scheme fulfills the von Neumann condition; that is, the roots $z(\kappa)$ of the equation

$$P(z,\kappa) = 0$$

satisfy

$$|z(\kappa)| \leqslant 1 \quad \text{for all } \kappa \text{ with } |\kappa| = 1.$$

(ii) If $|\kappa| = 1$ and $z(\kappa)$ is a root of $P(z,\kappa)$ with $|z(\kappa)| = 1$, then $z(\kappa)$ is a simple root of $P(z,\kappa)$.

As usual, we say that the basic scheme is *dissipative* if the roots of $P(z,\kappa)$ satisfy

$$|z(\kappa)| < 1 \text{ for all } \kappa \text{ with } |\kappa| = 1,\ \kappa \neq 1.$$

Analogous definitions hold for the boundary conditions with $P(z,\kappa)$ replaced by $R(z,\kappa)$. Clearly, both for the basic scheme and the boundary conditions, dissipativity implies the von Neumann condition.

The stability criteria obtained in Theorem 2 depend both on the basic scheme and the boundary conditions, but not on the intricate and often complicated interaction between the two. Consequently, Theorem 2, aided by Lemma 1, provides in many cases a convenient alternative to the celebrated GKS stability criterion in [4].

Having the new criteria, one can now easily establish stability for a host of examples that incorporate and generalize most of the cases studied in recent literature (e.g., [3]). We conclude this note with three of these examples:

EXAMPLE 1. Consider an arbitrary basic scheme, and let the boundary conditions be generated by either the explicit, first-order accurate, right-sided Euler scheme:

$$v_\nu(t+\Delta t) = v_\nu(t) + \lambda a\left[v_{\nu+1}(t) - v_\nu(t)\right], \quad 0 < \lambda a < 1, \quad \nu = 0,\dots,r-1, \tag{12}$$

or by its implicit analogue:

$$v_\nu(t+\Delta t) = v_\nu(t) + \lambda a\left[v_{\nu+1}(t+\Delta t) - v_\nu(t+\Delta t)\right], \quad \lambda a > 0, \quad \nu = 0,\dots,r-1. \tag{13}$$

These two-level boundary conditions are dissipative (see [1], Examples 3.5 and 3.6), hence fulfill the von Neumann condition. Further, for (13) we have

$$\mathrm{Re}\left[T_{-1}(\kappa)\right] = 1 + \lambda a\left[1 - \mathrm{Re}(\kappa)\right] \neq 0, \quad |\kappa| \leqslant 1.$$

By Lemma 1, therefore, inequalities (10b-e) hold, and Theorem 2 implies stability.

EXAMPLE 2. Take an arbitrary two-level basic scheme, and define the boundary conditions by horizontal extrapolation of order ℓ-1:

$$v_\nu(t+\Delta t) = \sum_{j=1}^{\ell} \binom{\ell}{j}(-1)^{j+1} v_{\nu+j}(t+\Delta t), \quad \nu = 0,\dots,r-1.$$

Here,

$$R(z,\kappa) = \sum_{j=1}^{\ell} \binom{\ell}{j}(-1)^{j+1}\kappa^j = (1-\kappa)^{\ell},$$

so $R(z,\kappa) \neq 0$ for $\kappa \neq 1$, which directly gives (10b,c,e). Moreover, since the basic scheme is two-level, Lemma *1(ii)(a)* implies (10d), and Theorem 2 again proves stability.

It is interesting to note (e.g. [2]) that this result may fail, both for dissipative and nondissipative basic schemes, if the basic scheme consists of more than two time levels.

EXAMPLE 3. Consider the Leap-Frog scheme

$$v_\nu(t+\Delta t) = v_\nu(t-\Delta t) + \lambda a\left[v_{\nu+1}(t) - v_{\nu-1}(t)\right], \quad 0 < \lambda a < 1, \quad \nu = 1,2,3,\dots,$$

with oblique extrapolation of order ℓ-1 at the boundary:

$$v_0(t+\Delta t) = \sum_{j=1}^{\ell} \binom{\ell}{j}(-1)^{j+1} v_j\left[t - (j-1)\Delta t\right].$$

We have

$$P(z,\kappa) = 1 - z^{-2} - \lambda a z^{-1}(\kappa - \kappa^{-1}),$$

so

$$\frac{\partial P}{\partial z} \cdot \frac{\partial P}{\partial \kappa}\bigg|_{z=\kappa=-1} = \frac{-1}{\lambda a} < 0.$$

Also,

$$\Omega(z,\kappa) \geqslant |P(z,\kappa)| > 0 \quad \text{for } z = \kappa \neq \pm 1,$$

and

$$\Omega(z,\kappa) \geqslant |R(z,\kappa)| = |1 - z^{-1}\kappa|^{\ell} > 0 \quad \text{for } z \neq \kappa.$$

Hence, (10a,c-e) hold, and by Theorem 2 stability follows.

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