A Central Differencing Simulation of the Orszag–Tang Vortex System

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***Abstract*—The Orszag–Tang vortex system describes the transition to supersonic turbulence for the equations of magnetohydrodynamics (MHD) in two space dimensions. The complex interaction between various shock waves traveling at different speed regimes that characterizes the solution of this test problem requires the use of numerical schemes capable of detecting and resolving accurately steep gradients while avoiding the onset of spurious oscillations. A simulation of the Orszag–Tang MHD vortex system computed with a third-order semi-discrete central scheme (Kurganov and Tadmor, 2000), (Balbas and Tadmor, submitted to *SIAM Journal of Scientific Computing*) is presented below. The central differencing approach avoids any detailed knowledge of the characteristic structure of the hyperbolic model, resulting in simple to implement, yet robust, *black-box* numerical schemes, (Balbas and Tadmor, 2004).**

***Index Terms*—High-resolution central schemes, ideal magnetohydrodynamics (MHD) equations, Jacobian-free form, multidimensional conservation laws, nonoscillatory.**

I. Orszag–Tang Vortex System

THE RESULTS presented in Fig. 1 display the evolution of the density contours of a plasma whose two-dimensional flow is modeled by the equations of ideal magnetohydrodynamics

$$\rho_t = -\nabla \cdot (\rho \mathbf{v}) \tag{1a}$$

$$(\rho \mathbf{v})_t = -\nabla \cdot \left[\rho \mathbf{v}\mathbf{v}^\top + \left(p + \frac{1}{2}B^2\right) I_{3\times 3} - \mathbf{B}\mathbf{B}^\top\right] \tag{1b}$$

$$e_t = -\nabla \cdot \left[\left(\frac{\gamma}{\gamma-1}p + \frac{1}{2}\rho v^2\right)\mathbf{v} - (\mathbf{v}\times\mathbf{B})\times\mathbf{B}\right] \tag{1c}$$

$$\mathbf{B}_t = \nabla \times (\mathbf{v}\times\mathbf{B}). \tag{1d}$$

Here, ρ and e are scalar quantities representing the mass density and the total energy, and $\mathbf{v}$ and $\mathbf{B}$ represent the velocity and magnetic fields, respectively. The pressure p is coupled to the energy by the equation of state, $e = (1/2)\rho v^2 + (1/2)B^2 + p/(\gamma - 1)$, where γ is the (fixed) ratio of specific heats.

Equations (1) are evolved subject to the initial conditions

$$\begin{aligned} \rho(x,z,0) &= \gamma^2 \\ p(x,z,0) &= \gamma \\ v_x(x,z,0) &= -\sin z \\ v_z(x,z,0) &= \sin x \\ B_x(x,z,0) &= -\sin z \\ B_z(x,z,0) &= \sin 2x \end{aligned} \tag{2}$$

where $\gamma = 5/3$. With this data, the root mean square values of the velocity and magnetic fields are both 1, the initial average Mach number is 1, and the average plasma beta is $10/3$.

This test problem considers the evolution of a compressible vortex system whose evolution involves the interaction between several shock waves traveling at various speed regimes (see [4] and references therein). The problem is solved in $[0, 2\pi] \times [0, 2\pi]$, with periodic boundary conditions in both x and z directions using a uniform grid with 288×288 points.

II. Numerical Scheme

The system (1) and (2) is solved using the semidiscrete central schemes of Kurganov and Tadmor [1], which evolve the cell averages $\bar{\mathbf{u}}(t)$

$$\frac{d}{dt}\bar{\mathbf{u}}_{j,k}(t) = -\frac{H^x_{j+\frac{1}{2},k}(\mathbf{u}(t)) - H^x_{j-\frac{1}{2},k}(\mathbf{u}(t))}{\Delta x} - \frac{H^z_{j,k+\frac{1}{2}}(\mathbf{u}(t)) - H^z_{j,k-\frac{1}{2}}(\mathbf{u}(t))}{\Delta z}. \tag{3}$$

Here, $\mathbf{u}(t) = \mathbf{u} := (\rho, \rho\mathbf{v}, \mathbf{B}, e)$, and $H^x(\mathbf{u}(t))$ and $H^z(\mathbf{u}(t))$ stand for the nonlinear numerical fluxes resulting from the discretization of the system (1). Central schemes avoid dimensional splitting and eliminate the need of intricate Riemann solvers [3]. The above semidiscrete formulation (3) accepts a variety of alternatives to implement the two main steps of the numerical scheme: a piecewise nonoscillatory reconstruction of the point values of $\mathbf{u}(t)$ from the neighboring cell averages, $\{\bar{\mathbf{u}}(t)_{j,k}\}$, and the evolution of the semidiscrete ODE (3). For the results in Fig. 1, we employ a third-order central weighted essentially nonoscillatory (CWENO) reconstruction, [5], [2], and a third-order SSP Runge–Kutta solver [6], [2].

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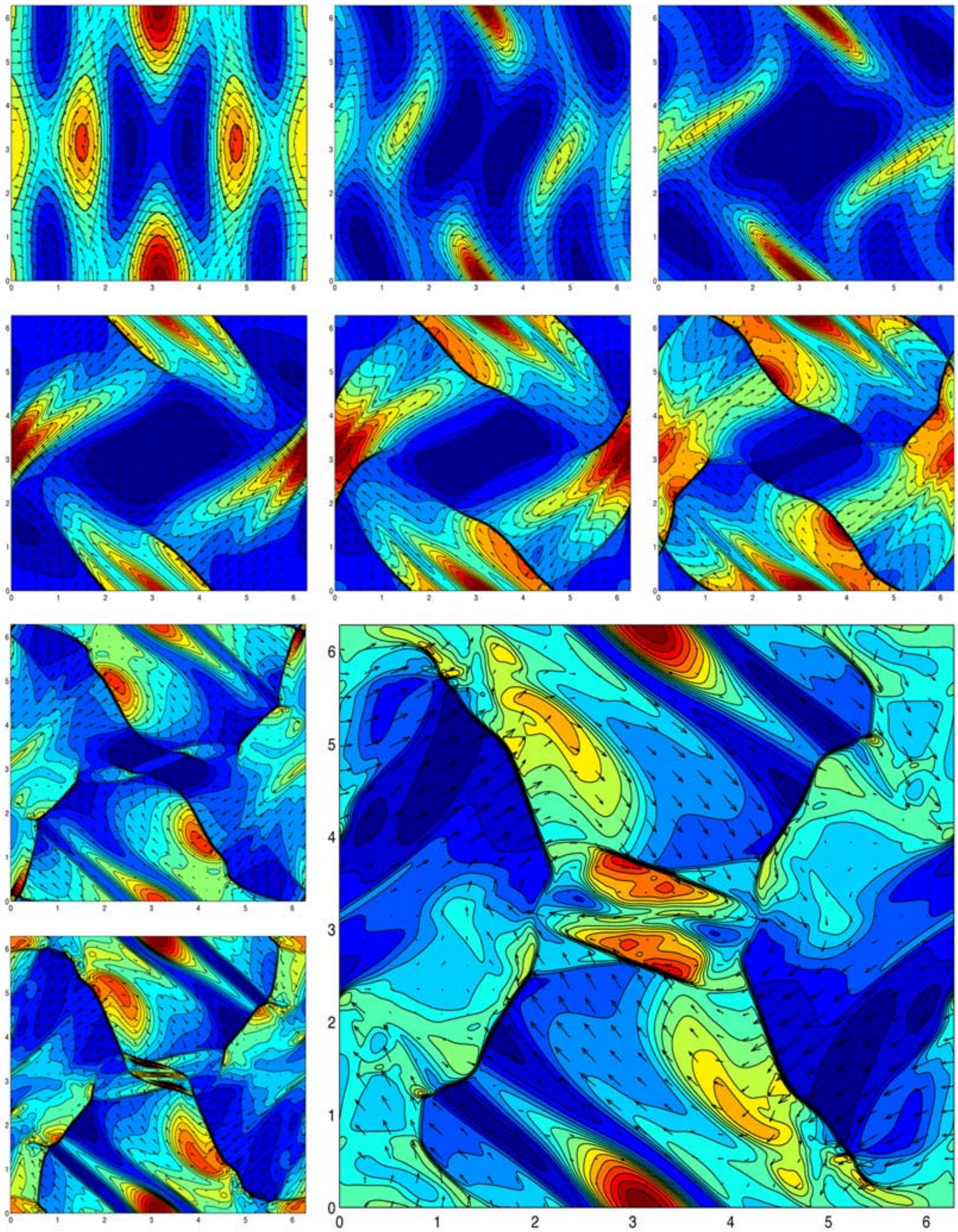

Fig. 1. Evolution of density with superimposed xz-velocity field, from left to right and top to bottom: density contours from $t = 0.003$: top-left corner to $t = 2.7$: lower left corner, displayed at (approximately) 0.375 time intervals. Large image in the lower right corner displays the solution at $t = 3$, density ranges from 1.2 to 6.1, and the maximum value of $\|v\|$ is 1.7. There are 16 contour lines: red—high value; blue—low value.

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