

## $L^2$ STABILITY ANALYSIS OF THE CENTRAL DISCONTINUOUS GALERKIN METHOD AND A COMPARISON BETWEEN THE CENTRAL AND REGULAR DISCONTINUOUS GALERKIN METHODS

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**Abstract.** We prove stability and derive error estimates for the recently introduced central discontinuous Galerkin method, in the context of linear hyperbolic equations with possibly discontinuous solutions. A comparison between the central discontinuous Galerkin method and the regular discontinuous Galerkin method in this context is also made. Numerical experiments are provided to validate the quantitative conclusions from the analysis.

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### 1. INTRODUCTION

We continue our study of the recently introduced central discontinuous Galerkin (DG) method [8]. In this paper we prove stability and derive error estimates of central DG approximation for linear hyperbolic equations. We also compare the central DG method with the regular DG method in this context.

The central DG method is based on two existing techniques: the central scheme framework (*e.g.* [7,9]) and the DG framework (*e.g.* [2,4]). It uses overlapping cells and hence duplicative information, but avoids numerical fluxes (Riemann solvers) which is a distinct advantage of central schemes. The central DG scheme also avoids the excessive numerical dissipation for small time steps, common to some of the earlier central schemes, by a suitable choice of the numerical dissipation. Being a variant of the DG method, it shares many of its advantages,

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such as compact stencil, easy parallel implementation, etc. The central DG method performs well in numerical simulations of linear and nonlinear scalar and systems of conservation laws [8].

We proceed with a description of the regular DG method and the central DG method. For simplicity, we consider a scalar one dimensional conservation law

$$u_t + f(u)_x = 0, \quad (x, t) \in [a, b] \times [0, T] \quad (1.1)$$

with periodic or compactly supported boundary conditions. The DG and central DG methods can be defined for nonlinear, multi-dimensional and system cases, and with other boundary conditions as well, see [4,8].

Let  $\{x_j\}$  be a partition of  $[a, b]$  with  $h_{j+\frac{1}{2}} = x_{j+1} - x_j$  and  $h = \max_j h_{j+\frac{1}{2}}$ . The mesh is regular, in the sense that  $\max_j h_{j+\frac{1}{2}} / \min_j h_{j+\frac{1}{2}}$  is upper-bounded by a fixed constant during mesh refinements. Denote  $x_{j+\frac{1}{2}} = \frac{1}{2}(x_{j+1} + x_j)$ ,  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ , and  $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$ .  $V_h$  is the set of piecewise polynomials of degree  $k$  over the subintervals  $\{I_j\}$  with no continuity assumed across the subinterval boundaries. Likewise,  $W_h$  is the set of piecewise polynomials of degree  $k$  over the subintervals  $\{I_{j+\frac{1}{2}}\}$  with no continuity assumed across the subinterval boundaries.

The regular DG method is defined using the space  $V_h$  only. The semi-discrete version is as follows. Find  $u_h(\cdot, t) \in V_h$  such that for any  $\varphi_h \in V_h$  and for all  $j$ ,

$$\begin{aligned} \int_{I_j} \partial_t u_h \varphi_h dx &= \int_{I_j} f(u_h) \partial_x \varphi_h dx - \hat{f}(u_h(x_{j+\frac{1}{2}}^-, t), u_h(x_{j+\frac{1}{2}}^+, t)) \varphi_h(x_{j+\frac{1}{2}}^-) \\ &\quad + \hat{f}(u_h(x_{j-\frac{1}{2}}^-, t), u_h(x_{j-\frac{1}{2}}^+, t)) \varphi_h(x_{j-\frac{1}{2}}^+) \end{aligned} \quad (1.2)$$

where  $\hat{f}(u^-, u^+)$  is a monotone numerical flux, namely it is increasing in the first argument and decreasing in the second argument, or symbolically  $\hat{f}(\uparrow, \downarrow)$ ; it is consistent with the physical flux  $\hat{f}(u, u) = f(u)$ ; and it is at least Lipschitz continuous with both arguments. For monotone fluxes suitable for the DG method, see, e.g. [2]. For systems the monotone numerical flux is replaced by a numerical flux obtained through an exact or approximate Riemann solver, see, e.g. [10].

The central DG method is defined on overlapping cells and uses both spaces  $V_h$  and  $W_h$ . We start with the description of the method for the fully discrete version, with forward Euler time discretization from  $t_n$  to  $t_{n+1} = t_n + \tau$  (here the time step  $\tau$  could change with  $n$ , but for simplicity of notations we will use a constant  $\tau$ ). The scheme is defined by the following procedure: Find  $u_h^{n+1} \in V_h$  and  $v_h^{n+1} \in W_h$ , such that for any  $\varphi_h \in V_h$  and  $\psi_h \in W_h$ ,

$$\begin{aligned} \int_{I_j} u_h^{n+1} \varphi_h dx &= \theta \int_{I_j} v_h^n \varphi_h dx + (1 - \theta) \int_{I_j} u_h^n \varphi_h dx \\ &\quad + \tau \left( \int_{I_j} f(v_h^n) \partial_x \varphi_h dx - f(v_h^n(x_{j+\frac{1}{2}})) \varphi_h(x_{j+\frac{1}{2}}^-) + f(v_h^n(x_{j-\frac{1}{2}})) \varphi_h(x_{j-\frac{1}{2}}^+) \right) \end{aligned} \quad (1.3)$$

$$\begin{aligned} \int_{I_{j+\frac{1}{2}}} v_h^{n+1} \psi_h dx &= \theta \int_{I_{j+\frac{1}{2}}} u_h^n \psi_h dx + (1 - \theta) \int_{I_{j+\frac{1}{2}}} v_h^n \psi_h dx \\ &\quad + \tau \left( \int_{I_{j+\frac{1}{2}}} f(u_h^n) \partial_x \psi_h dx - f(u_h^n(x_{j+1})) \psi_h(x_{j+1}^-) + f(u_h^n(x_j)) \psi_h(x_j^+) \right) \end{aligned} \quad (1.4)$$

where  $\theta = \frac{\tau}{\tau_{\max}}$  and  $\tau_{\max}$  is an upper bound for the time step size due to the CFL restriction, that is,  $\tau_{\max} = ch$  with a given constant CFL number  $c$  dictated by stability. Computationally, the scheme (1.3)–(1.4) is used with

the forward Euler replaced by a Runge-Kutta method of suitable temporal accuracy, *e.g.* the SSP Runge-Kutta methods in [5,11]. The time step  $\tau$  can be chosen arbitrarily subject to the stability restriction  $\tau \leq \tau_{\max}$ , hence  $0 \leq \theta \leq 1$ . If  $\theta = 1$ , the scheme is similar in spirit to the original central scheme [9].

We now take the limit  $\tau \rightarrow 0$  to obtain the semi-discrete version of the central DG scheme: Find  $u_h(\cdot, t) \in V_h$  and  $v_h(\cdot, t) \in W_h$ , such that for any  $\varphi_h \in V_h$  and  $\psi_h \in W_h$ ,

$$\int_{I_j} \partial_t u_h \varphi_h dx = \frac{1}{\tau_{\max}} \int_{I_j} (v_h - u_h) \varphi_h dx + \int_{I_j} f(v_h) \partial_x \varphi_h dx - f\left(v_h\left(x_{j+\frac{1}{2}}, t\right)\right) \varphi_h\left(x_{j+\frac{1}{2}}^-\right) + f\left(v_h\left(x_{j-\frac{1}{2}}, t\right)\right) \varphi_h\left(x_{j-\frac{1}{2}}^+\right) \tag{1.5}$$

$$\int_{I_{j+\frac{1}{2}}} \partial_t v_h \psi_h dx = \frac{1}{\tau_{\max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h) \psi_h dx + \int_{I_{j+\frac{1}{2}}} f(u_h) \partial_x \psi_h dx - f\left(u_h\left(x_{j+1}, t\right)\right) \psi_h\left(x_{j+1}^-\right) + f\left(u_h\left(x_j, t\right)\right) \psi_h\left(x_j^+\right). \tag{1.6}$$

Notice that, unlike the regular DG scheme (1.2), the central DG scheme (1.5)–(1.6) does not need a numerical flux to define the interface values of the solution, since the evaluation of the solution at the interface is in the middle of the staggered mesh, hence in the continuous region of the solution. The first term on the right side of (1.5) or (1.6) is a numerical dissipation term. This will become clear when we discuss the stability of the scheme in Section 2.1. In all the DG schemes (1.2), (1.3)–(1.4) and (1.5)–(1.6), the initial condition is taken as the  $L^2$  projection of the PDE initial condition into the relevant finite element space.

The organization of the paper is as follows. In Section 2, we first analyze the  $L^2$  stability and give an *a priori* error estimate for the semi-discrete central DG scheme (1.5)–(1.6) for the linear hyperbolic equation, using similar techniques of stability and error analysis for standard DG methods [3,6]. We then perform a Fourier analysis for the semi-discrete central DG scheme (1.5)–(1.6) for the linear hyperbolic equation with uniform meshes for piecewise constant and linear elements, using the techniques in [13]. This analysis is more explicit and allows us to compare the errors quantitatively between the central and standard DG schemes, which is performed in Section 3. In Section 3 we also perform numerical experiments to verify such quantitative conclusions. We give a few concluding remarks in Section 4.

## 2. ANALYSIS OF THE CENTRAL DG SCHEME

For the purpose of analysis we will consider the linear hyperbolic equation, namely (1.1) with  $f(u) = au$  for a constant  $a$ . Our analysis can be easily generalized to multi-dimensional linear equations and hyperbolic linear systems. In particular, these results apply to hyperbolic equations with propagating discontinuities.

Without loss of generality we consider the following linear hyperbolic equation

$$u_t + u_x = 0, \quad (x, t) \in [a, b] \times [0, T]. \tag{2.1}$$

We study the  $L^2$  stability of the central DG scheme (1.5)–(1.6) for the equation (2.1) in Section 2.1, and compare the result with that for the regular DG scheme in [6]. In Section 2.2 we provide an  $L^2$  *a priori* error estimate for smooth solutions, and compare the result with that for the regular DG scheme in [3]. In Section 2.3 we give an quantitative error estimate for the central DG scheme for polynomial degree up to 1 using Fourier analysis, similar to the technique used in [12,13].

2.1.  $L^2$  stability

**Theorem 2.1.** *The numerical solution  $u_h$  and  $v_h$  of the central DG scheme (1.5)–(1.6) for the equation (2.1) satisfies the following  $L^2$  stability condition*

$$\frac{1}{2} \frac{d}{dt} \int_a^b ((u_h)^2 + (v_h)^2) dx = -\frac{1}{\tau_{\max}} \int_a^b (u_h - v_h)^2 dx \leq 0. \tag{2.2}$$

*Proof.* Taking the test functions  $\varphi_h = u_h$  and  $\psi_h = v_h$  in (1.5) and (1.6) respectively, summing up over  $j$ , observing  $f(u) = u$  and the periodic (or compactly supported) boundary condition, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_a^b ((u_h)^2 + (v_h)^2) dx &= \frac{1}{\tau_{\max}} \int_a^b (v_h u_h - (u_h)^2 + u_h v_h - (v_h)^2) dx + \sum_j \left[ \int_{I_j} v_h \partial_x u_h dx + \int_{I_{j+\frac{1}{2}}} u_h \partial_x v_h dx \right. \\ &\quad - v_h \left(x_{j+\frac{1}{2}}, t\right) u_h \left(x_{j+\frac{1}{2}}^-, t\right) + v_h \left(x_{j-\frac{1}{2}}, t\right) u_h \left(x_{j-\frac{1}{2}}^+, t\right) \\ &\quad \left. - u_h \left(x_{j+1}, t\right) v_h \left(x_{j+1}^-, t\right) + u_h \left(x_j, t\right) v_h \left(x_j^-, t\right) \right] \\ &= -\frac{1}{\tau_{\max}} \int_a^b (u_h - v_h)^2 dx + \sum_j \left[ \int_{x_{j-\frac{1}{2}}}^{x_j} \partial_x (u_h v_h) dx + \int_{x_j}^{x_{j+\frac{1}{2}}} \partial_x (u_h v_h) dx \right. \\ &\quad - v_h \left(x_{j+\frac{1}{2}}, t\right) u_h \left(x_{j+\frac{1}{2}}^-, t\right) + v_h \left(x_{j-\frac{1}{2}}, t\right) u_h \left(x_{j-\frac{1}{2}}^+, t\right) \\ &\quad \left. - u_h \left(x_{j+1}, t\right) v_h \left(x_{j+1}^-, t\right) + u_h \left(x_j, t\right) v_h \left(x_j^-, t\right) \right] \\ &= -\frac{1}{\tau_{\max}} \int_a^b (u_h - v_h)^2 dx \leq 0. \quad \square \end{aligned}$$

**Remark 2.1.** The proof of Theorem 2.1 is similar to the proof of the cell entropy inequality for the regular DG method in [6]. However, we cannot prove a similar  $L^2$  stability result for the central DG scheme when applied to the nonlinear scalar conservation law (1.1), even though the proof of Theorem 2.1 can be easily generalized to multi-dimensional central DG schemes for linear equations. This is in contrary to the cell entropy inequality for regular DG schemes, which holds for arbitrary nonlinear scalar conservation laws [6].

**Remark 2.2.** Theorem 2.1 indicates that the energy dissipation term is  $\frac{1}{\tau_{\max}} \int_a^b (u_h - v_h)^2 dx$ , that is, it is directly related to the difference of the two duplicative representations  $u_h$  and  $v_h$  of the solution in overlapping cells. In contrast, for the regular DG method, the energy dissipation term is directly related to the jumps of the numerical solution at cell interfaces.

2.2.  $L^2$  a priori error estimate

In this subsection we use the standard DG techniques [3] to obtain an a priori  $L^2$  error estimate for the central DG scheme.

**Theorem 2.2.** *The numerical solution  $u_h$  and  $v_h$  of the central DG scheme (1.5)–(1.6) for the equation (2.1) with a smooth initial condition  $u(\cdot, 0) \in H^{k+1}$  satisfies the following  $L^2$  error estimate*

$$\|u - u_h\|^2 + \|u - v_h\|^2 \leq Ch^{2k} \tag{2.3}$$

where  $u$  is the exact solution of (2.1),  $k$  is the polynomial degree in the finite element spaces  $V_h$  and  $W_h$ , and the constant  $C$  depends on the  $(k + 1)$ -th order Sobolev norm of the initial condition  $\|u(\cdot, 0)\|_{H^{k+1}}$  as well as on the final time  $t$ , but is independent of the mesh size  $h$ .

*Proof.* Let us first introduce the standard notation

$$\begin{aligned}
 B_j(u_h, v_h; \varphi_h, \psi_h) &= \int_{I_j} \partial_t u_h \varphi_h dx - \frac{1}{\tau_{\max}} \int_{I_j} (v_h - u_h) \varphi_h dx - \int_{I_j} v_h \partial_x \varphi_h dx \\
 &\quad + v_h(x_{j+\frac{1}{2}}, t) \varphi_h(x_{j+\frac{1}{2}}^-) - v_h(x_{j-\frac{1}{2}}, t) \varphi_h(x_{j-\frac{1}{2}}^+) \\
 &\quad + \int_{I_{j+\frac{1}{2}}} \partial_t v_h \psi_h dx - \frac{1}{\tau_{\max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h) \psi_h dx - \int_{I_{j+\frac{1}{2}}} u_h \partial_x \psi_h dx \\
 &\quad + u_h(x_{j+1}, t) \psi_h(x_{j+1}^-) - u_h(x_j, t) \psi_h(x_j^+).
 \end{aligned} \tag{2.4}$$

Clearly, we have:

$$B_j(u_h, v_h; \varphi_h, \psi_h) = 0 \tag{2.5}$$

for all  $j$  and all  $\varphi_h \in V_h$  and  $\psi_h \in W_h$ . It is also clear that the exact solution  $u$  of the PDE (2.1) satisfies

$$B_j(u, u; \varphi_h, \psi_h) = 0 \tag{2.6}$$

for all  $j$  and all  $\varphi_h \in V_h$  and  $\psi_h \in W_h$ . Subtracting (2.5) from (2.6), we obtain the error equation

$$B_j(u - u_h, u - v_h; \varphi_h, \psi_h) = 0 \tag{2.7}$$

for all  $j$  and all  $\varphi_h \in V_h$  and  $\psi_h \in W_h$ . □

We now define  $P$  and  $Q$  as the standard  $L^2$  projection into  $V_h$  and  $W_h$  respectively. That is, for each  $j$ ,

$$\int_{I_j} (Pw(x) - w(x)) \varphi_h(x) dx = 0 \quad \forall \varphi_h \in \mathbb{P}^k(I_j) \tag{2.8}$$

and

$$\int_{I_{j+\frac{1}{2}}} (Qw(x) - w(x)) \psi_h(x) dx = 0 \quad \forall \psi_h \in \mathbb{P}^k(I_{j+\frac{1}{2}}) \tag{2.9}$$

where  $\mathbb{P}^k(I_j)$  and  $\mathbb{P}^k(I_{j+\frac{1}{2}})$  denote the spaces of polynomials of the degree up to  $k$  in the cell  $I_j$  and the cell  $I_{j+\frac{1}{2}}$  respectively. Standard approximation theory [1] implies, for a smooth function  $w$ ,

$$\|(Pw(x) - w(x))\| + h^{1/2} \|(Pw(x) - w(x))\|_{\Gamma} \leq Ch^{k+1} \tag{2.10}$$

and

$$\|(Qw(x) - w(x))\| + h^{1/2} \|(Qw(x) - w(x))\|_{\Gamma} \leq Ch^{k+1} \tag{2.11}$$

where  $\Gamma$  denotes the set of boundary points of all elements  $I_j$  or  $I_{j+1/2}$  respectively, the norm  $\|\cdot\|_{\Gamma}$  is the standard  $L^2$  norm, and the positive constant  $C$ , here and below, solely depending on  $w(x)$  and its derivatives, is independent of  $h$ .

We also recall that [1], for any  $w_h \in V_h$  or  $w_h \in W_h$ , there exists a positive constant  $C$  independent of  $w_h$  and  $h$ , such that

$$\|\partial_x w_h\| \leq Ch^{-1} \|w_h\|; \quad \|w_h\|_{\Gamma} \leq Ch^{-1/2} \|w_h\| \tag{2.12}$$

where  $\Gamma$  is the set of boundary points of all elements  $I_j$  or  $I_{j+1/2}$ .

We now take:

$$\varphi_h = Pu - u_h, \quad \psi_h = Qu - v_h \tag{2.13}$$

in the error equation (2.7), and denote

$$\varphi^e = Pu - u, \quad \psi^e = Qu - u \tag{2.14}$$

to obtain

$$B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) = B_j(\varphi^e, \psi^e; \varphi_h, \psi_h). \quad (2.15)$$

For the left-hand side of (2.15), we use Theorem 2.1 to conclude

$$\sum_j B_j(\varphi_h, \psi_h; \varphi_h, \psi_h) = \frac{1}{2} \frac{d}{dt} \int_a^b ((\varphi_h)^2 + (\psi_h)^2) dx + \frac{1}{\tau_{\max}} \int_a^b (\varphi_h - \psi_h)^2 dx. \quad (2.16)$$

We then write the right-hand side of (2.15) as a sum of three terms

$$B_j(\varphi^e, \psi^e; \varphi_h, \psi_h) = B_j^1 + B_j^2 + B_j^3 \quad (2.17)$$

where

$$B_j^1 = \frac{1}{\tau_{\max}} \int_{I_j} (\varphi^e - \psi^e) \varphi_h dx + \frac{1}{\tau_{\max}} \int_{I_{j+\frac{1}{2}}} (\psi^e - \varphi^e) \psi_h dx + \int_{I_j} \partial_t \varphi^e \varphi_h dx + \int_{I_{j+\frac{1}{2}}} \partial_t \psi^e \psi_h dx$$

$$B_j^2 = - \int_{I_j} \psi^e \partial_x \varphi_h dx - \int_{I_{j+\frac{1}{2}}} \varphi^e \partial_x \psi_h dx$$

$$B_j^3 = \psi^e(x_{j+\frac{1}{2}}, t) \varphi_h(x_{j+\frac{1}{2}}^-, t) - \psi^e(x_{j-\frac{1}{2}}, t) \varphi_h(x_{j-\frac{1}{2}}^+, t) + \varphi^e(x_{j+1}, t) \psi_h(x_{j+1}^-, t) - \varphi^e(x_j, t) \psi_h(x_j^+, t)$$

and we will estimate each term separately.

By using the simple inequality

$$\alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2), \quad (2.18)$$

the  $L^2$  projection property (2.10)–(2.11) for  $\varphi^e$ ,  $\psi^e$ ,  $\partial_t \varphi^e$  and  $\partial_t \psi^e$ , and the fact that  $\tau_{\max} = O(h)$ , we have:

$$\sum_j B_j^1 \leq \int_a^b (\varphi_h)^2 dx + \int_a^b (\psi_h)^2 dx + Ch^{2k}. \quad (2.19)$$

Likewise, by using the simple inequality (2.18), the  $L^2$  projection property (2.10) for  $\varphi^e$  and  $\psi^e$ , and the first inequality in (2.12) for  $\varphi_h$  and  $\psi_h$ , we have:

$$\sum_j B_j^2 \leq \int_a^b (\varphi_h)^2 dx + \int_a^b (\psi_h)^2 dx + Ch^{2k}. \quad (2.20)$$

Finally, by using the simple inequality (2.18), the  $L^2$  projection property (2.10) for  $\varphi^e$  and  $\psi^e$ , and the second inequality in (2.12) for  $\varphi_h$  and  $\psi_h$ , we have:

$$\sum_j B_j^3 \leq \int_a^b (\varphi_h)^2 dx + \int_a^b (\psi_h)^2 dx + Ch^{2k}. \quad (2.21)$$

Combining (2.19), (2.20) and (2.21) with with (2.16), we obtain from (2.15)

$$\frac{1}{2} \frac{d}{dt} \int_a^b ((\varphi_h)^2 + (\psi_h)^2) dx \leq C \int_a^b ((\varphi_h)^2 + (\psi_h)^2) dx + Ch^{2k}.$$

This, together with the approximation result (2.10), implies the desired error estimate (2.3).

**Remark 2.3.** The error estimate of Theorem 2.2 is sub-optimal. Numerically we observe the optimal  $(k+1)$ -th order accuracy, see [8] and also the numerical results in next section. In contrast, the error estimate for the regular DG method in such one dimensional and also in multi-dimensional tensor product cases is optimal [3]. The major technical difficulty leading to this loss of optimality in the proof is the  $B_j^2$  term. For the regular DG method this term is zero for both the regular  $L^2$  projection and for a special projection which is orthogonal to polynomials of one degree lower and can render the boundary terms in  $B_j^3$  also to vanish. However, for the central DG method, since it involves  $\psi^e$  and  $\partial_x \varphi_h$  and they are defined by polynomials in different cells, it is impossible to make the  $B_j^2$  term vanish no matter how the projection is chosen, although a special projection like that used for regular DG methods can render  $B_j^3$  to be zero. A better technique is needed to prove the optimal  $(k+1)$ -th order accuracy observed numerically, however we have been unable to find such a technique so far.

**Remark 2.4.** The error estimate of Theorem 2.2 can be easily generalized to one dimensional linear hyperbolic systems, multi-dimensional scalar linear hyperbolic equations, and multi-dimensional symmetric linear systems.

### 2.3. A quantitative error estimate via Fourier analysis

In this subsection we perform a Fourier analysis for the semi-discrete central DG scheme (1.5)–(1.6) for the linear hyperbolic equation with uniform meshes for piecewise constant and linear elements, using the techniques in [13]. This analysis is more explicit and allows us to compare the errors quantitatively between the central and standard DG schemes.

For this purpose we rewrite the scheme (1.5)–(1.6) for the linear equation (2.1) as a finite difference scheme on a uniform mesh. Towards this goal we choose the degrees of freedom for the  $k$ -th degree polynomial inside the cells  $I_j$  and  $I_{j+\frac{1}{2}}$  respectively as the point values of the solution, denoted by

$$u_{j+\frac{2i-k}{2(k+1)}}(t), \quad i = 0, \dots, k$$

and

$$v_{j+\frac{2i+1-k}{2(k+1)}}(t), \quad i = 0, \dots, k$$

at the  $k+1$  equally spaced points

$$\left( j + \frac{2i-k}{2(k+1)} \right) h, \quad i = 0, \dots, k$$

and

$$\left( j + \frac{2i+1-k}{2(k+1)} \right) h, \quad i = 0, \dots, k.$$

The schemes written in terms of these degrees of freedom become finite difference schemes on a globally uniform mesh (with a mesh size  $h/(k+1)$ ), however they are not standard finite difference schemes because each point in the group of  $k+1$  points belonging to the cell  $I_j$  or  $I_{j+\frac{1}{2}}$  obeys a different form of the finite difference scheme.

To be more specific, we first concentrate on the simplest piecewise constant  $k = 0$  case. For this case, we choose the degrees of freedom as the point values at the  $N$  uniformly spaced points

$$u_j(t), \quad j = 0, \dots, N-1$$

or

$$v_{j+\frac{1}{2}}(t), \quad j = 0, \dots, N-1.$$

The solution inside the cell  $I_j$  or  $I_{j+\frac{1}{2}}$  is then represented by

$$u_h(x, t) = u_j(t) \varphi_h^0(x)$$

or

$$v_h(x, t) = v_{j+\frac{1}{2}}(t) \psi_h^0(x)$$

where  $\varphi_h^0(x)$  is the constant function which equals 1 inside  $I_j$ , and similarly  $\psi_h^0(x)$  is the constant function which equals 1 inside  $I_{j+\frac{1}{2}}$ . With this representation, taking the test functions  $\varphi_h$  as  $\varphi_h^0$ , and  $\psi_h$  as  $\psi_h^0$ , respectively, we obtain easily the finite difference schemes corresponding to the central DG scheme:

$$\begin{aligned} u'_j &= -\frac{1}{\tau_{\max}}u_j + \left(\frac{1}{2\tau_{\max}} + \frac{1}{h}\right)v_{j-\frac{1}{2}} + \left(\frac{1}{2\tau_{\max}} - \frac{1}{h}\right)v_{j+\frac{1}{2}} \\ v'_{j+\frac{1}{2}} &= -\frac{1}{\tau_{\max}}v_{j+\frac{1}{2}} + \left(\frac{1}{2\tau_{\max}} + \frac{1}{h}\right)u_j + \left(\frac{1}{2\tau_{\max}} - \frac{1}{h}\right)u_{j+1} \end{aligned} \quad (2.22)$$

for  $j = 0, \dots, N-1$ . Here  $u'$  and  $v'$  denote the time derivatives of  $u$  and  $v$ . The scheme can be rewritten into a more compact form

$$\begin{pmatrix} u'_j \\ v'_{j+\frac{1}{2}} \end{pmatrix} = A \begin{pmatrix} u_{j-1} \\ v_{j-\frac{1}{2}} \end{pmatrix} + B \begin{pmatrix} u_j \\ v_{j+\frac{1}{2}} \end{pmatrix} + C \begin{pmatrix} u_{j+1} \\ v_{j+\frac{3}{2}} \end{pmatrix} \quad (2.23)$$

with

$$A = \begin{pmatrix} 0 & \frac{1}{2\tau_{\max}} + \frac{1}{h} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{\tau_{\max}} & \frac{1}{2\tau_{\max}} - \frac{1}{h} \\ \frac{1}{2\tau_{\max}} + \frac{1}{h} & -\frac{1}{\tau_{\max}} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ \frac{1}{2\tau_{\max}} - \frac{1}{h} & 0 \end{pmatrix}. \quad (2.24)$$

We now perform the following standard Fourier analysis for the finite difference scheme (2.23)–(2.24). This analysis depends heavily on the assumption of uniform mesh sizes and periodic boundary conditions. We make an ansatz of the form

$$\begin{pmatrix} u_j(t) \\ v_{j+\frac{1}{2}}(t) \end{pmatrix} = \begin{pmatrix} \hat{u}_m(t) \\ \hat{v}_m(t) \end{pmatrix} e^{imx_j} \quad (2.25)$$

and substitute this into the scheme (2.23)–(2.24) to find the evolution equation for the coefficient vector as

$$\begin{pmatrix} \hat{u}'_m(t) \\ \hat{v}'_m(t) \end{pmatrix} = G(m, h) \begin{pmatrix} \hat{u}_m(t) \\ \hat{v}_m(t) \end{pmatrix} \quad (2.26)$$

where the amplification matrix  $G(m, h)$  is given by

$$G(m, h) = A e^{-imh} + B + C e^{imh} \quad (2.27)$$

with the matrices  $A$ ,  $B$  and  $C$  defined by (2.24). The two eigenvalues of the amplification matrix  $G(m, h)$  are

$$\lambda_1 = -\frac{1}{\tau_{\max}} - \alpha e^{i\frac{mh}{2}}, \quad \lambda_2 = -\frac{1}{\tau_{\max}} + \alpha e^{i\frac{mh}{2}} \quad (2.28)$$

where  $\alpha = \left(\frac{1}{2\tau_{\max}} + \frac{1}{h}\right)e^{-imh} + \frac{1}{2\tau_{\max}} - \frac{1}{h}$ . The general solution of the ODE (2.26) is given by

$$\begin{pmatrix} \hat{u}_m(t) \\ \hat{v}_m(t) \end{pmatrix} = a_1 e^{\lambda_1 t} V_1 + a_2 e^{\lambda_2 t} V_2, \quad (2.29)$$

where the eigenvalues  $\lambda_1$  and  $\lambda_2$  are given by (2.28), and  $V_1$  and  $V_2$  are the corresponding eigenvectors given by

$$V_1 = \begin{pmatrix} -e^{-i\frac{\xi}{2}} \\ 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} e^{-i\frac{\xi}{2}} \\ 1 \end{pmatrix}, \quad (2.30)$$

with  $\xi = mh$ . For accuracy we look at the low modes, in particular at  $m = 1$ . To fit the given initial condition

$$u_j(0) = e^{ix_j}, \quad v_{j+\frac{1}{2}}(0) = e^{ix_{j+\frac{1}{2}}} \quad (2.31)$$



whose imaginary part is the initial condition

$$u_j(0) = \sin(x_j), \quad v_{j+\frac{1}{2}}(0) = \sin(x_{j+\frac{1}{2}}), \tag{2.32}$$

we require, at  $t = 0$ ,

$$\begin{pmatrix} \hat{u}_1(0) \\ \hat{v}_1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\frac{h}{2}} \end{pmatrix},$$

hence we obtain the coefficients  $a_1$  and  $a_2$  in (2.29) as

$$a_1 = 0, \quad a_2 = e^{i\frac{h}{2}}. \tag{2.33}$$

We remark that the usual way of taking initial conditions in a finite element method is *via* an  $L^2$  projection, not by a point value collocation (2.32), however we have verified that this does not affect the final results in the analysis in this paper. We thus have the explicit solutions of the scheme (2.23)–(2.24) with the initial condition (2.31), for example

$$u_j(t) = a_2 e^{ix_j + \lambda_2 t - i\frac{h}{2}} \tag{2.34}$$

with the eigenvalue  $\lambda_2$  given by (2.28) with  $m = 1$  and the coefficient  $a_2$  given by (2.33). By a simple Taylor expansion, we obtain the imaginary part of  $u_j(t)$  to be

$$Im\{u_j(t)\} = \sin(x_j - t) - \left(\frac{t}{8c} \sin(x_j - t)\right) h + O(h^2) \tag{2.35}$$

where  $c = \frac{\tau_{max}}{h} = O(1)$  is the maximum CFL number. This is clearly consistent with the exact solution to first order accuracy. We can similarly check the first order accuracy of  $Im\{v_{j+\frac{1}{2}}(t)\}$ .

We now repeat this analysis for the piecewise linear  $k = 1$  case. The solution inside the cell  $I_j$  or  $I_{j+\frac{1}{2}}$  is then represented by

$$u_h(x, t) = u_{j-\frac{1}{4}}(t)\varphi_h^1(\xi) + u_{j+\frac{1}{4}}(t)\varphi_h^2(\xi)$$

or

$$v_h(x, t) = v_{j+\frac{1}{4}}(t)\psi_h^1(\xi) + v_{j+\frac{3}{4}}(t)\psi_h^2(\xi)$$

where  $\varphi_h^1(\xi) = -\xi + \frac{1}{2}$ ,  $\varphi_h^2(\xi) = \xi + \frac{1}{2}$ ,  $\psi_h^1(\xi) = -\xi + \frac{3}{2}$  and  $\psi_h^2(\xi) = \xi - \frac{1}{2}$ , with  $\xi = \frac{2(x-x_j)}{h}$ . With this representation, taking the test functions  $\varphi_h$  and  $\psi_h$  as  $\varphi_h^1, \varphi_h^2$  and  $\psi_h^1, \psi_h^2$  respectively, and inverting the small  $4 \times 4$  mass matrix by hand, we obtain easily the finite difference scheme corresponding to the central DG scheme as

$$\begin{pmatrix} u'_{j-\frac{1}{4}} \\ u'_{j+\frac{1}{4}} \\ v'_{j+\frac{1}{4}} \\ v'_{j+\frac{3}{4}} \end{pmatrix} = A \begin{pmatrix} u_{j-\frac{5}{4}} \\ u_{j-\frac{3}{4}} \\ v_{j-\frac{3}{4}} \\ v_{j-\frac{1}{4}} \end{pmatrix} + B \begin{pmatrix} u_{j-\frac{1}{4}} \\ u_{j+\frac{1}{4}} \\ v_{j+\frac{1}{4}} \\ v_{j+\frac{3}{4}} \end{pmatrix} + C \begin{pmatrix} u_{j+\frac{3}{4}} \\ u_{j+\frac{5}{4}} \\ v_{j+\frac{5}{4}} \\ v_{j+\frac{7}{4}} \end{pmatrix} \tag{2.36}$$

with

$$\begin{aligned}
 A &= \begin{pmatrix} 0 & 0 & \frac{1}{16\tau_{\max}} + \frac{5}{4h} & \frac{13}{16\tau_{\max}} - \frac{1}{4h} \\ 0 & 0 & \frac{-1}{16\tau_{\max}} - \frac{1}{4h} & \frac{3}{16\tau_{\max}} + \frac{5}{4h} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 B &= \begin{pmatrix} \frac{-1}{\tau_{\max}} & 0 & \frac{3}{16\tau_{\max}} - \frac{5}{4h} & \frac{-1}{16\tau_{\max}} + \frac{1}{4h} \\ 0 & \frac{-1}{\tau_{\max}} & \frac{13}{16\tau_{\max}} + \frac{1}{4h} & \frac{1}{16\tau_{\max}} - \frac{5}{4h} \\ \frac{1}{16\tau_{\max}} + \frac{5}{4h} & \frac{13}{16\tau_{\max}} - \frac{1}{4h} & \frac{-1}{\tau_{\max}} & 0 \\ \frac{-1}{16\tau_{\max}} - \frac{1}{4h} & \frac{3}{16\tau_{\max}} + \frac{5}{4h} & 0 & \frac{-1}{\tau_{\max}} \end{pmatrix}, \\
 C &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{3}{16\tau_{\max}} - \frac{5}{4h} & \frac{-1}{16\tau_{\max}} + \frac{1}{4h} & 0 & 0 \\ \frac{13}{16\tau_{\max}} + \frac{1}{4h} & \frac{1}{16\tau_{\max}} - \frac{5}{4h} & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{2.37}$$

We make an ansatz of the form

$$\begin{pmatrix} u_{j-\frac{1}{4}}(t) \\ u_{j+\frac{1}{4}}(t) \\ v_{j+\frac{1}{4}}(t) \\ v_{j+\frac{3}{4}}(t) \end{pmatrix} = \begin{pmatrix} \hat{u}_{m,-\frac{1}{4}}(t) \\ \hat{u}_{m,\frac{1}{4}}(t) \\ \hat{v}_{m,\frac{1}{4}}(t) \\ \hat{v}_{m,\frac{3}{4}}(t) \end{pmatrix} e^{imx_j} \tag{2.38}$$

and substitute this into the scheme (2.36)–(2.37) to find the evolution equation for the coefficient vector as

$$\begin{pmatrix} \hat{u}'_{m,-\frac{1}{4}}(t) \\ \hat{u}'_{m,\frac{1}{4}}(t) \\ \hat{v}'_{m,\frac{1}{4}}(t) \\ \hat{v}'_{m,\frac{3}{4}}(t) \end{pmatrix} = G(k, h) \begin{pmatrix} \hat{u}_{m,-\frac{1}{4}}(t) \\ \hat{u}_{m,\frac{1}{4}}(t) \\ \hat{v}_{m,\frac{1}{4}}(t) \\ \hat{v}_{m,\frac{3}{4}}(t) \end{pmatrix} \tag{2.39}$$

where the amplification matrix  $G(m, h)$  is given by

$$G(m, h) = A e^{-imh} + B + C e^{imh} \tag{2.40}$$

with the matrices  $A$ ,  $B$  and  $C$  defined by (2.37). The eigenvalues of the amplification matrix  $G(m, h)$  are

$$\begin{aligned}
 \lambda_1 &= \frac{1}{\tau_{\max}} \left( -1 + \frac{\sqrt{2}}{8} e^{-imh} \sqrt{-\alpha_1 - \alpha_2} \right) \\
 \lambda_2 &= \frac{1}{\tau_{\max}} \left( -1 - \frac{\sqrt{2}}{8} e^{-imh} \sqrt{-\alpha_1 - \alpha_2} \right) \\
 \lambda_3 &= \frac{1}{\tau_{\max}} \left( -1 + \frac{\sqrt{2}}{8} e^{-imh} \sqrt{-\alpha_1 + \alpha_2} \right) \\
 \lambda_4 &= \frac{1}{\tau_{\max}} \left( -1 - \frac{\sqrt{2}}{8} e^{-imh} \sqrt{-\alpha_1 + \alpha_2} \right)
 \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= (1 - 4c - 52c^2)e^{imh} - 2(11 - 52c^2)e^{2imh} + (1 + 4c - 52c^2)e^{3imh}, \\ \alpha_2 &= e^{imh}(1 + e^{imh} + 10c(1 - e^{imh})) [-3 - 12c + 4c^2 \\ &\quad + 2(21 - 4c^2)e^{imh} + 4(-3 + 12c + 4c^2)e^{2imh}]^{1/2} \end{aligned} \tag{2.41}$$

with  $c = \frac{\tau_{\max}}{h}$  being the maximum CFL number.

The general solution of the ODE (2.39) is given by

$$\begin{pmatrix} \hat{u}_{m,-\frac{1}{4}}(t) \\ \hat{u}_{m,+\frac{1}{4}}(t) \\ \hat{v}_{m,\frac{1}{4}}(t) \\ \hat{v}_{m,\frac{3}{4}}(t) \end{pmatrix} = a_1 e^{\lambda_1 t} V_1 + a_2 e^{\lambda_2 t} V_2 + a_3 e^{\lambda_3 t} V_3 + a_4 e^{\lambda_4 t} V_4, \tag{2.42}$$

where the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$  are given by (2.41), and  $V_1, V_2, V_3$  and  $V_4$  are the corresponding eigenvectors given by

$$\begin{aligned} V_1 &= \begin{pmatrix} \frac{\sqrt{2}e^{-imh} \sqrt{-\alpha_1 - \alpha_2} (\alpha_7 + 2\alpha_2)}{\alpha_5 (\alpha_6 - \alpha_2)} \\ \frac{\sqrt{-2(\alpha_1 + \alpha_2)} (1 + e^{imh} - 10c(-1 + e^{imh}))}{\alpha_6 - \alpha_2} \\ \frac{\alpha_4 \alpha_2 + \alpha_3}{\alpha_5 (\alpha_6 - \alpha_2)} \\ 1 \end{pmatrix}, \\ V_2 &= \begin{pmatrix} -\frac{\sqrt{2}e^{-imh} \sqrt{-\alpha_1 - \alpha_2} (\alpha_7 + 2\alpha_2)}{\alpha_5 (\alpha_6 - \alpha_2)} \\ -\frac{\sqrt{-2(\alpha_1 + \alpha_2)} (1 + e^{imh} - 10c(-1 + e^{imh}))}{\alpha_6 - \alpha_2} \\ \frac{\alpha_4 \alpha_2 + \alpha_3}{\alpha_5 (\alpha_6 - \alpha_2)} \\ 1 \end{pmatrix}, \\ V_3 &= \begin{pmatrix} \frac{\sqrt{2}e^{-imh} \sqrt{-\alpha_1 + \alpha_2} (-\alpha_7 + 2\alpha_2)}{\alpha_5 (\alpha_6 + \alpha_2)} \\ \frac{\sqrt{2(-\alpha_1 + \alpha_2)} (1 + e^{imh} - 10c(-1 + e^{imh}))}{\alpha_6 + \alpha_2} \\ \frac{-\alpha_4 \alpha_2 + \alpha_3}{\alpha_5 (\alpha_6 + \alpha_2)} \\ 1 \end{pmatrix}, \\ V_4 &= \begin{pmatrix} -\frac{\sqrt{2}e^{-imh} \sqrt{-\alpha_1 + \alpha_2} (-\alpha_7 + 2\alpha_2)}{\alpha_5 (\alpha_6 + \alpha_2)} \\ -\frac{\sqrt{2(-\alpha_1 + \alpha_2)} (1 + e^{imh} - 10c(-1 + e^{imh}))}{\alpha_6 + \alpha_2} \\ \frac{-\alpha_4 \alpha_2 + \alpha_3}{\alpha_5 (\alpha_6 + \alpha_2)} \\ 1 \end{pmatrix}, \end{aligned} \tag{2.43}$$

where  $c = \frac{\tau_{\max}}{h}$  is still the maximum CFL number,  $\alpha_1$  and  $\alpha_2$  are given by (2.41), and the remaining  $\alpha$ 's are given by

$$\begin{aligned} \alpha_3 &= -(7 + 104c + 332c^2 - 80c^3)e^{imh} + (77 + 896c + 532c^2 - 240c^3)e^{2imh} \\ &\quad + (79 - 856c - 68c^2 + 240c^3)e^{3imh} - (5 - 64c + 132c^2 + 80c^3)e^{4imh}, \\ \alpha_4 &= -1 - 20c + (-3 + 20c)e^{imh}, \\ \alpha_5 &= -1 - 4c + (13 + 4c)e^{imh}, \\ \alpha_6 &= (1 + 10c)^2 e^{imh} + (2 - 200c^2)e^{2imh} + (1 - 10c)^2 e^{3imh}, \\ \alpha_7 &= (1 + 10c)e^{imh} - 20c^2 e^{2imh} - (1 - 10c)^2 e^{3imh}. \end{aligned} \tag{2.44}$$

We again look at the low modes to determine accuracy. In particular we look at  $m = 1$ . To fit the given initial condition

$$u_{j-\frac{1}{4}}(0) = e^{ix_{j-\frac{1}{4}}}, \quad u_{j+\frac{1}{4}}(0) = e^{ix_{j+\frac{1}{4}}}, \quad v_{j+\frac{1}{4}}(0) = e^{ix_{j+\frac{1}{4}}}, \quad v_{j+\frac{3}{4}}(0) = e^{ix_{j+\frac{3}{4}}} \quad (2.45)$$

whose imaginary part is our initial condition for (2.1), we require, at  $t = 0$ ,

$$\begin{pmatrix} \hat{u}_{m,-\frac{1}{4}}(0) \\ \hat{u}_{m,+\frac{1}{4}}(0) \\ \hat{v}_{m,\frac{1}{4}}(0) \\ \hat{v}_{m,\frac{3}{4}}(0) \end{pmatrix} = \begin{pmatrix} e^{i\frac{-h}{4}} \\ e^{i\frac{h}{4}} \\ e^{i\frac{h}{4}} \\ e^{i\frac{3h}{4}} \end{pmatrix}.$$

This gives us the coefficients  $a_1, a_2, a_3$  and  $a_4$  in the solution (2.42). We thus have the explicit solution of the scheme (2.36)–(2.37) with the initial condition (2.45), for example

$$u_{j-\frac{1}{4}} = a_1 e^{ix_j + \lambda_1 t - i\frac{h}{4}} V_1 + a_2 e^{ix_j + \lambda_2 t - i\frac{h}{4}} V_2 + a_3 e^{ix_j + \lambda_3 t - i\frac{h}{4}} V_3 + a_4 e^{ix_j + \lambda_4 t - i\frac{h}{4}} V_4 \quad (2.46)$$

with the eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  given by (2.41) and the eigenvectors  $V_1, V_2, V_3, V_4$  given by (2.43) with  $m = 1$ , and the coefficients  $a_1, a_2, a_3$  and  $a_4$  obtained above by fitting the initial condition. Through a simple Taylor expansion, we obtain the imaginary part of  $u_{j-\frac{1}{4}}(t)$  to be

$$\text{Im}\{u_{j-\frac{1}{4}}(t)\} = \sin(x_{j-\frac{1}{4}} - t) - \frac{\sin(x_{j-\frac{1}{4}} - t)}{30} h^2 + O(h^3) \quad (2.47)$$

for a fixed choice of  $\tau_{\max} = 0.4h$ . Results for other choices of  $\tau_{\max}$  also indicate the same second order accuracy. The results for  $\text{Im}\{u_{j+\frac{1}{4}}(t)\}$ ,  $\text{Im}\{v_{j+\frac{1}{4}}(t)\}$  and  $\text{Im}\{v_{j+\frac{3}{4}}(t)\}$  are similar.

In principle this analysis can be performed for higher order polynomials in the central DG scheme, however the algebra becomes prohibitively complicated.

### 3. A COMPARISON BETWEEN THE CENTRAL DG AND STANDARD DG METHODS

In [13], results similar to those in (2.35) and (2.47) were obtained for the regular DG scheme (1.2) applied to the linear equation (2.1) with an upwind numerical flux. For the piecewise constant  $k = 0$  case, the result for the regular DG scheme is

$$\text{Im}\{u_j(t)\} = \sin(x_j - t) - \left(\frac{t}{2} \sin(x_j - t)\right) h + O(h^2), \quad (3.1)$$

and for the piecewise linear  $k = 1$  case, it is

$$\text{Im}\{u_{j-\frac{1}{4}}(t)\} = \sin(x_{j-\frac{1}{4}} - t) - \frac{\sin(x_{j-\frac{1}{4}} - t)}{24} h^2 + O(h^3). \quad (3.2)$$

We are now in a position to make a quantitative comparison between the regular and central DG schemes. For the piecewise constant  $k = 0$  case we have the following conclusions:

- (1) The semi-discrete versions of the regular and central DG schemes are both stable. When discretized with the first order forward Euler method, the CFL numbers for the DG method and for the central DG method are 1.0 and 0.5, respectively (this can be verified by an easy von Neumann analysis). When discretized with the second order nonlinearly stable Runge-Kutta method [11], the CFL numbers for the DG method and for the central DG method are 1.0 and 0.87, respectively. Thus the central DG method

TABLE 1.  $L^2$  and  $L^\infty$  errors and orders of accuracy of the first order central DG method.

$h$	Numerical solution				Predicted by analysis			
	$L^2$ error	Order	$L^\infty$ error	Order	$L^2$ error	Order	$L^\infty$ error	Order
$2\pi/80$	1.88 E-01	—	2.65 E-01	—	2.17 E-01	—	3.07 E-01	—
$2\pi/160$	1.01 E-01	0.90	1.43 E-01	0.90	1.09 E-01	1.00	1.53 E-01	1.00
$2\pi/320$	5.23 E-02	0.95	7.40 E-02	0.95	5.42 E-02	1.00	7.67 E-02	1.00
$2\pi/640$	2.67 E-02	0.97	3.77 E-02	0.97	2.71 E-02	1.00	3.84 E-02	1.00
$2\pi/1280$	1.35 E-02	0.99	1.90 E-02	0.99	1.36 E-02	1.00	1.92 E-02	1.00

TABLE 2.  $L^2$  and  $L^\infty$  errors and orders of accuracy of the first order regular DG method.

$h$	Numerical solution				Predicted by analysis			
	$L^2$ error	Order	$L^\infty$ error	Order	$L^2$ error	Order	$L^\infty$ error	Order
$2\pi/80$	4.42 E-01	—	6.25 E-01	—	6.94 E-01	—	9.81 E-01	—
$2\pi/160$	2.74 E-01	0.69	3.88 E-01	0.69	3.47 E-01	1.00	4.91 E-01	1.00
$2\pi/320$	1.54 E-01	0.83	2.18 E-01	0.83	1.73 E-01	1.00	2.45 E-01	1.00
$2\pi/640$	8.17 E-02	0.91	1.16 E-01	0.91	8.68 E-02	1.00	1.23 E-01	1.00
$2\pi/1280$	4.21 E-02	0.96	5.95 E-02	0.96	4.34 E-02	1.00	6.14 E-02	1.00

has a smaller CFL number. Considering that the central DG method uses duplicative information and hence is twice as costly as the regular DG method for the same mesh, we could compare the regular DG scheme on mesh size  $\frac{h}{2}$  with the central DG scheme on mesh size  $h$  to equalize the cost. In such a context the CFL number would be the same for the two DG methods with the first order forward Euler time discretization.

- (2) They are both first order accurate. By a comparison of (2.35) and (3.1), the leading errors for the central and regular DG methods for the first mode (*i.e.* for the  $\sin(x)$  initial condition) have a ratio  $1/(4c)$ . That is, the central DG method has a smaller error than the standard DG method on the same mesh when  $c = \frac{\tau_{\max}}{h} > \frac{1}{4}$ . If we compare the regular DG scheme on mesh size  $\frac{h}{2}$  with the central DG scheme on mesh size  $h$  to equalize the cost, then the central DG method has a smaller error than the standard DG method when  $c = \frac{\tau_{\max}}{h} > \frac{1}{2}$ .

We now compute the DG and central DG solutions to (2.1) with  $u(x, 0) = \sin(x)$  as the initial condition and with periodic boundary conditions, up to  $t = 25$  (about four periods later in time), to verify the quantitative comparison above. In our computation we take  $\tau_{\max} = 0.8h$ , namely  $c = 0.8$ . We take a small time step  $\tau = 0.01h$  to reduce the effect from the time discretization. In order to be consistent with the error analysis above, the errors are computed for  $u_h$  at the points  $x_j$ . The  $L^2$  and  $L^\infty$  errors and order of accuracy of the central DG and regular DG methods are listed in Tables 1 and 2 respectively. We also list the predicted errors by the analysis, namely the leading terms in the Taylor expansions in (2.35) and (3.1) in these tables. We can see that the predicted errors and the actual errors are very close, validating our quantitative analysis in Section 2.3.

Likewise, for the piecewise linear case  $k = 1$ , we have the following conclusions:

- (1) The semi-discrete versions of the regular and central DG schemes are both stable. When discretized with the second order nonlinearly stable Runge-Kutta method [11], the CFL numbers for the DG method and for the central DG method are 0.33 and 0.45, respectively (this can be verified by an easy von Neumann analysis). Thus the central DG method has a larger CFL number. If we compare the regular DG scheme on mesh size  $\frac{h}{2}$  with the central DG scheme on mesh size  $h$  to equalize the cost, then the central DG method has an even larger CFL number.

TABLE 3. The leading error term  $bh^2 \sin(x_{j-\frac{1}{4}} - t)$  for the central DG method for different  $c = \frac{\tau_{\max}}{h}$ .

$c$	0.2	0.3	0.4	0.5	0.6	0.8
$b$	$\frac{1}{60}$	$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{24}$	$\frac{1}{20}$	$\frac{2}{30}$

TABLE 4.  $L^2$  and  $L^\infty$  errors and orders of accuracy of the second order central DG method.

$h$	Numerical solution				Predicted by analysis			
	$L^2$ error	Order	$L^\infty$ error	Order	$L^2$ error	Order	$L^\infty$ error	Order
$2\pi/20$	1.20 E-02	—	1.37 E-02	—	4.10 E-03	—	4.11 E-03	—
$2\pi/40$	1.53 E-03	2.98	1.93 E-03	2.82	4.11 E-04	2.00	4.11 E-04	2.00
$2\pi/80$	1.91 E-04	3.00	2.94 E-04	2.72	1.03 E-04	2.00	1.03 E-04	2.00
$2\pi/160$	2.57 E-05	2.89	4.96 E-05	2.57	2.57 E-05	2.00	2.57 E-05	2.00
$2\pi/320$	6.43 E-06	2.00	9.42 E-06	2.40	6.43 E-06	2.00	6.43 E-06	2.00

TABLE 5.  $L^2$  and  $L^\infty$  errors and orders of accuracy of the second order regular DG method.

$h$	Numerical solution				Predicted by analysis			
	$L^2$ error	Order	$L^\infty$ error	Order	$L^2$ error	Order	$L^\infty$ error	Order
$2\pi/20$	1.06 E-02	—	1.46 E-02	—	4.10 E-03	—	4.11 E-03	—
$2\pi/40$	1.34 E-03	2.98	2.36 E-03	2.63	1.03 E-03	2.00	1.03 E-03	2.00
$2\pi/80$	2.57 E-04	2.39	4.24 E-04	2.47	2.57 E-04	2.00	2.57 E-04	2.00
$2\pi/160$	6.42 E-05	2.00	8.53 E-05	2.32	6.43 E-05	2.00	6.43 E-05	2.00
$2\pi/320$	1.61 E-05	2.00	1.82 E-05	2.19	1.61 E-05	2.00	1.61 E-05	2.00

- (2) They are both second order accurate. The central DG method has different leading errors  $bh^2 \sin(x_{j-\frac{1}{4}} - t)$  for different  $c = \frac{\tau_{\max}}{h}$ , with the  $\sin(x)$  initial condition. We list the corresponding  $b$  with different values of  $c$  in Table 3.

By a comparison of Table 3 and (3.2), the leading errors for the central and regular DG methods for the first mode (*i.e.* for the  $\sin(x)$  initial condition) are equal when  $c = \frac{\tau_{\max}}{h} = 0.5$ . The central DG method has a smaller error than the standard DG method on the same mesh when  $c < 0.5$ . If we compare the regular DG scheme on mesh size  $\frac{h}{2}$  with the central DG scheme on mesh size  $h$  to equalize the cost, then the central DG method has a smaller error than the standard DG method when  $c < 0.3$ .

We now compute the DG and central DG solutions to (2.1) with a  $u(x, 0) = \sin(x)$  initial condition and periodic boundary conditions, up to  $t = 25$  (about four periods later in time), to verify the quantitative comparison above. In our computation we take  $\tau_{\max} = 0.2h$ , namely  $c = 0.2$ . We take a small time step  $\tau = 0.01h$  to reduce the effect from the time discretization. In order to be consistent with the error analysis above, the errors are computed for  $u_h$  at the points  $x_{j-\frac{1}{4}}$  and  $x_{j+\frac{1}{4}}$ . The  $L^2$  and  $L^\infty$  errors and order of accuracy of the central DG and regular DG methods are listed in Tables 4 and 5 respectively. We also list the predicted errors by the analysis, namely the leading terms in the Taylor expansions in Table 3 and (3.2) in these tables. We can see again that the predicted errors and the actual errors are very close, validating our quantitative analysis in Section 2.3.

Finally, in Table 6 we carry out a numerical Fourier analysis to obtain the CFL numbers when the central DG scheme is coupled with Runge-Kutta methods of various orders, and compare them with the results of the regular DG method with the Runge-Kutta time discretization (RKDG methods, Tab. 2.2 in [4]). We observe that, except for the case of  $k = 0$  (piecewise constant), for all other  $k$  the central DG method has a larger CFL number than the RKDG method with the same Runge-Kutta time discretization.

TABLE 6. CFL numbers for the  $(k + 1)$ -th order RKDG method and central DG method with Runge-Kutta method of order  $\nu$ .

$k$	Regular DG					Central DG				
	0	1	2	3	4	0	1	2	3	4
$\nu = 1$	1.00	—	—	—	—	0.50	—	—	—	—
$\nu = 2$	1.00	0.33	—	—	—	0.86	0.45	—	—	—
$\nu = 3$	1.25	0.40	0.20	0.13	0.08	1.12	0.58	0.33	0.22	0.17
$\nu = 4$	1.39	0.46	0.23	0.14	0.10	1.19	0.79	0.47	0.31	0.25

#### 4. CONCLUDING REMARKS

We have performed an  $L^2$  stability analysis and an *a priori* error estimate for the recently introduced central discontinuous Galerkin method when applied to linear hyperbolic equations. We have also performed a Fourier type error analysis which is more quantitative and allows us to make a comparison with the regular discontinuous Galerkin method. It is verified that, even though the central discontinuous Galerkin method uses duplicative representation of the solution, hence involves twice the computational cost and storage requirement than the regular discontinuous Galerkin method, it is more accurate for certain choices of a dissipation parameter for the same mesh. The stability analysis and error estimates do not seem to be easily generalizable to nonlinear hyperbolic equations. Further analysis in this direction is needed. A comprehensive comparison of the numerical performance of the central discontinuous Galerkin method and the regular discontinuous Galerkin method for nonlinear multi-dimensional systems of hyperbolic conservation laws would also be very useful.

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