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THE CONVERGENCE RATE OF APPROXIMATE SOLUTIONS FOR NONLINEAR SCALAR CONSERVATION LAWS*

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Abstract. Let $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of approximate solutions for the nonlinear scalar conservation law $u_t + f(u)_x = 0$ with C_0^1-initial data. Assume that $\{v^\varepsilon(x,t)\}$ are *Lip*$^+$-*stable* in the sense that they satisfy Oleinik's E-entropy condition. It is shown that if these approximate solutions are *Lip*$'$-*consistent*, i.e., if $\|v^\varepsilon(\cdot,0) - u(\cdot,0)\|_{Lip'(x)} + \|v_t^\varepsilon + f(v^\varepsilon)_x\|_{Lip'(x,t)} = \mathcal{O}(\varepsilon)$, then they converge to the entropy solution, and the convergence rate estimate $\|v^\varepsilon(\cdot,t) - u(\cdot,t)\|_{Lip'(x)} = \mathcal{O}(\varepsilon)$ holds. Consequently, the familiar L^p-type and new pointwise error estimates are derived.

These convergence rate results are demonstrated in the context of entropy satisfying finite-difference and Glimm's schemes.

Key words. conservation laws, entropy stability, weak consistency, error estimates, post-processing, finite-difference approximations, Glimm scheme

AMS(MOS) subject classifications. 35L65, 65M10, 65M15

1. Introduction.

We are concerned here with the convergence *rate* of approximate solutions for the nonlinear scalar conservation law, $u_t + f(u)_x = 0$ with C_0^1-initial data. In this context we first recall Strang's theorem which shows that the classical Lax–Richtmyer linear convergence theory applies for such nonlinear problem, as long as the underlying solution is sufficiently smooth, e.g., [RM, §5]. Since the solutions of the nonlinear conservation law develop spontaneous shock-discontinuities at a finite time, Strang's result does not apply beyond this critical time. Indeed, the Fourier method as well as other L^2-*conservative* schemes provide simple counterexamples of consistent approximations which fail to converge (to the discontinuous entropy solution), despite their linearized L^2-stability, e.g., [Ta4] and [Ta5].

In this paper we extend the linear convergence theory into the weak regime. The extension is based on the usual two ingredients of stability and consistency. On the one hand, the counterexamples mentioned above show that one must *strengthen* the linearized L^2-stability requirement. We assume that the approximate solutions are *Lip*$^+$-stable in the sense that they satisfy a one-sided Lipschitz condition, in agreement with Oleinik's E-condition for the entropy solution. On the other hand, the lack of smoothness requires to *weaken* the consistency requirement, which is measured here in the *Lip*$'$-(semi)norm. In §2 we prove for *Lip*$^+$-stable approximate solutions, that their *Lip*$'$-convergence rate to the entropy solution is of the same order as their *Lip*$'$-consistency. The *Lip*$'$-convergence rate is then converted into stronger L^p-convergence rate estimates. In particular, we recover the usual L^1-convergence rate of order one half, and we obtain new pointwise error estimates that depend on the *local* smoothness of the entropy solution.

In §3 we implement these error estimates for finite-difference approximations, using a finite-element representation, which is interesting for its own sake. In §4 we apply these error estimates for the Glimm scheme. Other applications of the current

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framework to spectral viscosity approximations and various viscosity regularizations can be found in [Ta6], [ST], and [Ta7].

2. Approximate solutions. We study approximate solutions of the scalar, genuinely nonlinear conservation law

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}f(u(x,t))=0, \qquad f''\geq \alpha>0, \tag{2.1}$$

with compactly supported initial conditions prescribed at $t=0$,

$$u(x,t=0)=u_0(x). \tag{2.2}$$

Let $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of approximate solutions of the conservation law (2.1), (2.2) in the following sense.

DEFINITION. A. We say that $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ are *conservative* solutions if

$$\int_x v^\varepsilon(x,t)dx=\int_x u_0(x)dx, \qquad t\geq 0. \tag{2.3}$$

B. We say that $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ are *Lip′-consistent* with the conservation law (2.1), (2.2) if the following estimates are fulfilled*:

(i) consistency with the initial conditions (2.2),

$$\|v^\varepsilon(x,0)-u_0(x)\|_{Lip'}\leq K_0\cdot\varepsilon, \tag{2.4a}$$

(ii) consistency with the conservation law (2.1),

$$\|v_t^\varepsilon(x,t)+f(v^\varepsilon(x,t))_x\|_{Lip'(x,[0,T])}\leq K_T\cdot\varepsilon. \tag{2.4b}$$

We are interested in the *convergence rate* of the approximate solutions, $v^\varepsilon(x,t)$, as their small parameter $\varepsilon\downarrow 0$. This requires an appropriate stability definition for such approximate solutions. Recall that the entropy solution of the nonlinear conservation law (2.1), (2.2) satisfies the a priori estimate [BO], [Ta6]

$$\|u(\cdot,t)\|_{Lip^+}\leq\frac{1}{\|u_0\|_{Lip^+}^{-1}+\alpha t}, \qquad t\geq 0. \tag{2.5}$$

The case $\|u_0\|_{Lip^+}=\infty$ is included in (2.5), and it corresponds to the exact $\sim t^{-1}$ decay rate of an initial rarefaction.

DEFINITION. We say that $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ are *Lip^+-stable* if there exists a constant $\beta\geq 0$ (independent of t and ε) such that the following estimate, analogous to (2.5), is fulfilled:

$$\|v^\varepsilon(\cdot,t)\|_{Lip^+}\leq\frac{1}{\|v^\varepsilon(\cdot,0)\|_{Lip^+}^{-1}+\beta t}, \qquad t\geq 0. \tag{2.6}$$

Remarks. 1. The case of an initial rarefaction subject to the quadratic flux $f(u)=\frac{\alpha}{2}u^2$ demonstrates that the a priori decay estimate of the exact entropy solution

*We let $\|\phi\|_{Lip}, \|\phi\|_{Lip^+}$ and $\|\phi\|_{Lip'}$ denote, respectively, $\operatorname{ess\,sup}_{x\neq y}\left|\frac{\phi(x)-\phi(y)}{x-y}\right|$, $\operatorname{ess\,sup}_{x\neq y}\left[\frac{\phi(x)-\phi(y)}{x-y}\right]_+$, and $\sup_\psi(\phi-\hat{\phi}_0,\psi)/\|\psi\|_{Lip}$, where $\hat{\phi}_0=\int_{\mathrm{supp}\phi}\phi$.

in (2.5) is sharp. A comparison of (2.6) with (2.5) shows that a *necessary* condition for the convergence of $\{v^\varepsilon\}_{\varepsilon>0}$ is

$$0 \le \beta \le \alpha, \tag{2.7}$$

for otherwise, the decay rate of $\{v^\varepsilon(\cdot,t)\}$ (and hence of its $\varepsilon \to 0$ limit) would be *faster* than that of the exact entropy solution.

2. The case $\beta > 0$ in (2.6) corresponds to a *strict* Lip^+-stability in the sense that $\|v^\varepsilon(\cdot,t)\|_{Lip^+}$ decays in time, in agreement with the decay of rarefactions indicated in (2.5).

3. In general, any a priori bound

$$\|v^\varepsilon(\cdot,t)\|_{Lip^+} \le \text{Const}_T < \infty, \qquad 0 \le t \le T, \tag{2.8}$$

is a sufficient stability condition for the convergence results discussed below. In particular, we allow for $\beta = 0$ in (2.6), as long as the approximate initial conditions are Lip^+-bounded. We remark that the restriction of Lip^+-bounded initial data is indeed necessary for convergence, in view of the counterexample of Roe's scheme discussed in §3. Unless stated otherwise, we therefore restrict our attention to the class of Lip^+-bounded (i.e., rarefaction-free) initial conditions, where

$$L_0^+ := \max(\|u_0\|_{Lip^+}, \|v^\varepsilon(\cdot,0)\|_{Lip^+}) < \infty. \tag{2.9}$$

Finally, we remark that in case of strict Lip^+-stability, i.e., in case (2.6) holds with $\beta > 0$, then we can remove this restriction of Lip^+-bounded initial data and our convergence results can be extended to include general L^∞_{loc}-initial conditions. The discussion of this case will be dealt elsewhere.

We begin with the following theorem, which is at the heart of matter.

THEOREM 2.1. A. *Let $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative, Lip^+-stable approximate solutions of the conservation law* (2.1), (2.2), *subject to the Lip^+-bounded initial conditions* (2.9). *Then the following error estimate holds:*

$$\begin{aligned}\|v^\varepsilon(\cdot,T) - u(\cdot,T)\|_{Lip'} \\ \le C_T\left[\|v^\varepsilon(\cdot,0) - u_0(\cdot)\|_{Lip'} + \|v^\varepsilon_t + f(v^\varepsilon)_x\|_{Lip'(x,[0,T])}\right],\end{aligned} \tag{2.10a}$$

where

$$C_T \sim (1+\beta L_0^+ T)^\eta, \qquad \eta := \frac{\max f''}{\beta} \ge 1.$$

B. *In particular, if the family $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ is also Lip'-consistent of order $\mathcal{O}(\varepsilon)$, i.e.,* (2.4a), (2.4b) *hold, then $v^\varepsilon(x,t)$ converges to the entropy solution $u(x,t)$ and the following convergence rate estimate holds:*

$$\|v^\varepsilon(\cdot,T) - u(\cdot,T)\|_{Lip'} \le M_T \cdot \varepsilon, \qquad M_T := (K_0 + K_T)(1+\beta L_0^+ T)^\eta. \tag{2.10b}$$

Proof. We proceed along the lines of [Ta6]. The difference, $e^\varepsilon(x,t) := v^\varepsilon(x,t) - u(x,t)$, satisfies the error equation

$$\frac{\partial}{\partial t}e^\varepsilon(x,t) + \frac{\partial}{\partial x}[\bar{a}_\varepsilon(x,t)e^\varepsilon(x,t)] = F^\varepsilon(x,t), \tag{2.11}$$

where $\overline{a}_\varepsilon(x,t)$ stands for the mean value

$$\overline{a}_\varepsilon(x,t) = \int_{\xi=0}^{1} a[\xi v^\varepsilon(x,t) + (1-\xi)u(x,t)]d\xi, \qquad a(\cdot) \equiv f'(\cdot),$$

and $F^\varepsilon(x,t)$ is the truncation error,

$$F^\varepsilon(x,t) := v_t^\varepsilon(x,t) + f(v^\varepsilon(x,t))_x.$$

Given an arbitrary $\phi(x)\epsilon W_0^{1,\infty}$, we let $\{\phi^\varepsilon(x,t)\}_{0\leq t\leq T}$ denote the solution of the backward transport equation

$$\phi_t^\varepsilon(x,t) + \overline{a}_\varepsilon(x,t)\phi_x^\varepsilon(x,t) = 0, \qquad t \leq T, \tag{2.12a}$$

corresponding to the endvalues, $\phi(x)$, prescribed at $t = T$,

$$\phi^\varepsilon(x,T) = \phi(x). \tag{2.12b}$$

Here, the following a priori estimate holds [Ta6, Thm. 2.2]:

$$\|\phi^\varepsilon\cdot,t)\|_{Lip} \leq \exp\left(\int_t^T \|\overline{a}_\varepsilon(\cdot,\tau)\|_{Lip^+}d\tau\right)\cdot\|\phi(x)\|_{Lip}, \qquad 0\leq t\leq T. \tag{2.13}$$

The Lip^+-stability of the entropy solution (2.5) and its approximate solutions in (2.6), provide us with the one-sided Lipschitz upper-bound required on the right-hand side of (2.13):

$$\|\overline{a}_\varepsilon(\cdot,\tau)\|_{Lip^+} \leq \frac{\max f''}{2}[\|v^\varepsilon(\cdot,\tau)\|_{Lip^+} + \|u(\cdot,\tau)\|_{Lip^+}] \leq \frac{\max f''}{[L_0^+]^{-1}+\beta\tau}. \tag{2.14}$$

Equipped with (2.13), (2.14) we conclude

$$\begin{aligned}\|\phi^\varepsilon(\cdot,t)\|_{Lip} &\leq \frac{(1+\beta L_0^+T)^\eta}{(1+\beta L_0^+t)^\eta}\|\phi(x)\|_{Lip}\\ &\leq C_T\|\phi(x)\|_{Lip}, \qquad 0\leq t\leq T, \quad C_T := (1+\beta L_0^+T)^\eta,\end{aligned} \tag{2.15a}$$

and employing (2.12a) we also have

$$\begin{aligned}\|\phi^\varepsilon(x,\cdot)\|_{Lip[0,T]} &\leq |a|_\infty \max_{0\leq t\leq T}\|\phi^\varepsilon(\cdot,t)\|_{Lip(x)}\\ &\leq |a|_\infty C_T\|\phi(x)\|_{Lip}, \qquad |a|_\infty := \max|f'|.\end{aligned} \tag{2.15b}$$

Of course, (2.12) is just the adjoint problem of the error equation (2.11) which gives us

$$(e^\varepsilon(\cdot,T),\phi(\cdot)) = (e^\varepsilon(\cdot,0),\phi^\varepsilon(\cdot,0)) + (F^\varepsilon(x,t),\phi^\varepsilon(x,t))_{L^2(x,[0,T])}. \tag{2.16}$$

Conservation implies that $\hat{e}_0^\varepsilon \equiv \int e^\varepsilon(x,0)dx = 0$ and by (2.15a) we find

$$\begin{aligned}|(e^\varepsilon(\cdot,0),\phi^\varepsilon(\cdot,0))| &\leq \|e^\varepsilon(\cdot,0)\|_{Lip'}\|\phi^\varepsilon(\cdot,0)\|_{Lip}\\ &\leq (1+\beta L_0^+T)^\eta\|e^\varepsilon(\cdot,0)\|_{Lip'}\cdot\|\phi(x)\|_{Lip};\end{aligned} \tag{2.17a}$$

similarly, conservation implies that $\hat{F}_0^\varepsilon \equiv \int_{x,[0,T]} F^\varepsilon(x,t)dx\,dt = 0$ and by (2.15a), (2.15b) we find
(2.17b)

$$|(F^\varepsilon(x,t),\phi^\varepsilon(x,t))_{L^2(x,[0,T])}| \leq \|F^\varepsilon(x,t)\|_{Lip'(x,[0,T])}\|\phi^\varepsilon(x,t)\|_{Lip(x,[0,T])}$$

$$\leq (1+|a|_\infty)C_T\|F^\varepsilon(x,t)\|_{Lip'(x,[0,T])}\|\phi(x)\|_{Lip}.$$

The error estimate (2.10a) follows from the last two estimates together with (2.16). □

The Lip'-convergence rate estimate (2.10b) can be extended to more familiar $W^{s,p}_{\text{loc}}$-convergence rate estimates. The rest of this section is devoted to three corollaries which summarize these extensions.

We begin by noting that the conservation and Lip^+-stability of $v^\varepsilon(\cdot,t)$ imply that $v^\varepsilon(\cdot,T)$—and consequently that the error, $v^\varepsilon(\cdot,T)-v(\cdot,T)$, have bounded variation,

$$\|v^\varepsilon(\cdot,T)-v(\cdot,T)\|_{BV} \leq \text{Const}\frac{1}{[L_0^+]^{-1}+\beta T}. \tag{2.18}$$

We can now interpolate between the BV-regularity (2.18) and the Lip'-error estimate (2.10b), with the help of Sobolev inequality stating that for all $s \in [-1,\frac{1}{p}]$ we have (e.g., [Fr, Thm. 9.3])

$$\|D_x w\|_{W^{s,p}} \leq \text{Const}\cdot\|D_x w\|^{1-\eta}_{W^{1,1}}\cdot\|w\|^\eta_{L^1}, \qquad \eta = \frac{1-sp}{2p}, \quad 1\leq p\leq\infty.$$

Applying the latter to the *primitive* of $v^\varepsilon(\cdot,T)-v(\cdot,T)$ we conclude the following.

COROLLARY 2.2. *Let $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative, Lip′-consistent, and Lip^+-stable approximate solutions of the conservation law* (2.1), (2.2), *with Lip^+-bounded initial conditions* (2.9). *Then the following convergence rate estimates hold:*

$$\|v^\varepsilon(\cdot,T)-u(\cdot,T)\|_{W^{s,p}} \leq \text{Const}_T\cdot\varepsilon^{\frac{1-sp}{2p}}, \qquad -1\leq s\leq\frac{1}{p}, \quad 1\leq p\leq\infty. \tag{2.19}$$

The error estimate (2.19) with $(s,p)=(0,1)$ yields L^1-convergence rate of order $\mathcal{O}(\sqrt{\varepsilon})$, which is familiar from the setup of monotone difference approximations [Ku], [Sa]. Of course, uniform convergence (which corresponds to $(s,p)=(0,\infty)$) fails in this case, due to the possible presence of shock discontinuities in the entropy solution $u(\cdot,t)$. Instead, we seek pointwise convergence away from the singular support of $u(\cdot,t)$. To this end, we employ a $C_0^1(-1,1)$-unit mass mollifier of the form $\zeta_\delta(x) = \frac{1}{\delta}\zeta(\frac{x}{\delta})$. The error estimate (2.10) asserts that

$$|(v^\varepsilon(\cdot,T)*\zeta_\delta)(x)-(u(\cdot,T)*\zeta_\delta)(x)| \leq M_T\frac{\varepsilon}{\delta^2}\left\|\frac{d\zeta}{dx}\right\|_{L^\infty}. \tag{2.20}$$

Moreover, if $\zeta(x)$ is chosen so that

$$\int x^k\zeta(x)dx = 0 \quad \text{for } k=1,2,\ldots,p-1, \tag{2.21a}$$

then a straightforward error estimate based on Taylor's expansion yields

$$|(u(\cdot,T)*\zeta_\delta)(x)-u(x,T)| \leq \frac{\delta^p}{p!}\|\zeta\|_{L^1}\cdot|u^{(p)}|_{\text{loc}}, \tag{2.21b}$$

where $|u^{(p)}|_{\text{loc}}$ measures the degree of *local* smoothness of $u(\cdot,t)$,

$$|u^{(p)}|_{\text{loc}} := \left\| \frac{\partial^p}{\partial x^p} u(\cdot,T) \right\|_{L^\infty_{\text{loc}}(x+\delta\cdot\text{supp}\zeta)}.$$

The last two inequalities imply the following corollary.

COROLLARY 2.3. *Let $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative, Lip′-consistent, and Lip$^+$-stable approximate solutions of the conservation law* (2.1), (2.2), *with Lip$^+$-bounded initial conditions* (2.9). *Then, for any p-order mollifier $\zeta_\delta(x) \equiv \frac{1}{\delta}\zeta(\frac{x}{\delta})$ satisfying* (2.21a), *the following convergence rate estimate holds*:

$$(2.22)\quad |(v^\varepsilon(\cdot,T) * \zeta_\delta)(x) - u(x,T)| \le \text{Const}_T \left(1 + \frac{|u^{(p)}|_{\text{loc}}}{p!}\right) \cdot \varepsilon^{\frac{p}{p+2}}, \qquad \delta \sim \varepsilon^{\frac{1}{p+2}}.$$

Corollary 2.3 shows that by *post-processing* the approximate solutions $v^\varepsilon(\cdot,t)$, we are able to recover the pointwise values of $u(x,t)$ with an error as close to ε as the local smoothness of $u(\cdot,t)$ permits. A similar treatment enables the recovery of the derivatives of $u(x,t)$ as well, consult [Ta6, §4].

The particular case $p=1$ in (2.22), deserves special attention. In this case, post-processing of the approximate solution with *arbitrary* C_0^1-unit mass mollifier $\zeta(x)$, gives us

$$(2.23)\quad |(v^\varepsilon(\cdot,T) * \zeta_\delta)(x) - u(x,T)| \le \text{Const} \cdot (1 + |u_x(\cdot,T)|_{\text{loc}}) \cdot \sqrt[3]{\varepsilon}, \qquad \delta \sim \sqrt[3]{\varepsilon}.$$

We claim that the pointwise convergence rate of order $\mathcal{O}(\sqrt[3]{\varepsilon})$ indicated in (2.23) holds even *without* post-processing of the approximate solution. Indeed, let us consider the difference

$$v^\varepsilon(x,T) - (v^\varepsilon(\cdot,T) * \zeta_\delta)(x) = \int_y [v^\varepsilon(x,T) - v^\varepsilon(x-y,T)]\zeta_\delta(y)dy$$

$$= \int_y \left[\frac{v^\varepsilon(x,T) - v^\varepsilon(x-y,T)}{-y}\right] \cdot -\frac{y}{\delta}\zeta\left(\frac{y}{\delta}\right) dy.$$

By choosing a positive C_0^1-unit mass mollifier $\zeta(x)$ supported on $(-1,0)$ then, thanks to the *Lip*$^+$-stability condition (2.6), the integrand on the right does not exceed Const $\cdot\, \delta$, and hence

$$(2.24a)\qquad v^\varepsilon(x,T) - (v^\varepsilon(\cdot,T) * \zeta_\delta)(x) \le \text{Const} \cdot \delta\ .$$

Similarly, a different choice of a positive C_0^1-unit mass mollifier $\zeta(x)$ supported on $(0,1)$ leads to

$$(2.24b)\qquad v^\varepsilon(x,T) - (v^\varepsilon(\cdot,T) * \zeta_\delta)(x) \ge \text{Const} \cdot \delta.$$

Each of the last two inequalities (with $\delta \sim \sqrt[3]{\varepsilon}$) together with (2.23) show that the approximate solution itself converges with an $\mathcal{O}(\sqrt[3]{\varepsilon})$-rate, as asserted.

We summarize what we have shown by stating the following.

COROLLARY 2.4. *Let $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ be a family of conservative, Lip′-consistent, and Lip$^+$-stable approximate solutions of the conservation law* (2.1), (2.2), *with Lip$^+$-bounded initial conditions* (2.9). *Then the following convergence rate estimate holds*:

$$(2.25)\qquad \begin{gathered} |v^\varepsilon(x,T) - u(x,T)| \le \text{Const}_{x,T} \cdot \sqrt[3]{\varepsilon}, \\ \text{Const}_{x,T} \sim 1 + |u_x(\cdot,T)|_{L^\infty(x-\sqrt[3]{\varepsilon},\,x+\sqrt[3]{\varepsilon})}. \end{gathered}$$

Remark. The above derivation of pointwise error estimates applies in more general situations. Consider, for example, a family of approximate solutions, $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$, which satisfies a standard L^1 (rather than Lip') error estimate

$$|(v^\varepsilon(\cdot,T) - u(\cdot,T), \phi(\cdot))| \leq \text{Const}_T \cdot \sqrt{\varepsilon}\|\phi\|_{L^\infty}. \tag{2.26}$$

Then our previous arguments show how to post-process $v^\varepsilon(\cdot,T)$ in order to recover the pointwise values of the entropy solution, $u(x,T)$ with an error as close to $\sqrt{\varepsilon}$ as the local smoothness of $u(\cdot,T)$ permits. In particular, using (2.26) with a positive C_0^1-unit mass mollifier, $\zeta_\delta(x) = \frac{1}{\delta}\zeta(\frac{x}{\delta})$ we obtain

$$|(v^\varepsilon(\cdot,T) * \zeta_\delta)(x) - (u(\cdot,T) * \zeta_\delta)(x)| \leq \text{Const}_T \cdot \frac{\sqrt{\varepsilon}}{\delta}\|\zeta\|_{L^\infty}. \tag{2.27}$$

Using this together with

$$|(u(\cdot,T) * \zeta_\delta)(x) - u(x,T)| \leq \delta\|\zeta\|_{L^1} \cdot \|u_x(\cdot,T)\|_{L^\infty_{\text{loc}}(x+\delta\cdot\text{supp}\,\zeta)},$$

we find

$$|(v^\varepsilon(\cdot,T) * \zeta_\delta)(x) - u(x,T)| \leq \text{Const}_T(1 + |u_x(\cdot,T)|_{\text{loc}})\sqrt[4]{\varepsilon}, \qquad \delta \sim \sqrt[4]{\varepsilon}. \tag{2.28}$$

If the approximate solutions $\{v^\varepsilon(x,t)\}_{\varepsilon>0}$ are also Lip^+-stable, then we may augment (2.28) with (2.24) to conclude the pointwise error estimate

$$\begin{gathered} |v^\varepsilon(x,T) - u(x,T)| \leq \text{Const}_{x,T} \cdot \sqrt[4]{\varepsilon}, \\ \text{Const}_{x,T} \sim 1 + |u_x(\cdot,T)|_{L^\infty(x-\sqrt[4]{\varepsilon},x+\sqrt[4]{\varepsilon})}. \end{gathered} \tag{2.29}$$

3. Finite-difference approximations. We want to solve the conservation law (2.1)–(2.2) by difference approximations. To this end we use a grid ($x_\nu := \nu\Delta x, t^n := n\Delta t$) with a fixed mesh ratio $\lambda \equiv \frac{\Delta t}{\Delta x} = \text{Const}$. The approximate solution at these grid points, $v_\nu^n \equiv v(x_\nu, t^n)$, is determined by a conservative difference approximation which takes the following viscosity form, e.g., [Ta2],

$$\begin{aligned} v_\nu^{n+1} &= v_\nu^n - \frac{\lambda}{2}[f(v_{\nu+1}^n) - f(v_{\nu-1}^n)] \\ &\quad + \frac{1}{2}[Q_{\nu+\frac{1}{2}}^n(v_{v+1}^n - v_\nu^n) - Q_{\nu-\frac{1}{2}}^n(v_\nu^n - v_{\nu-1}^n)], \qquad n \geq 0, \end{aligned} \tag{3.1}$$

and is subject to Lip^+-bounded initial conditions,

$$v_\nu^0 = \frac{1}{\Delta x}\int_{x_{\nu-\frac{1}{2}}}^{x_{\nu+\frac{1}{2}}} u_0(\xi)d\xi, \qquad L_0^+ = \|u_0\|_{Lip^+} < \infty. \tag{3.2}$$

Let $v^\varepsilon(x,t)$ be the piecewise linear interpolant of our grid solution, $v^\varepsilon(x_\nu,t^n) = v_\nu^n$, depending on the small discretization parameter $\varepsilon \equiv \Delta x \downarrow 0$. It is given by

$$v^{\Delta x}(x,t) = \sum_{j,m} v_j^m \Lambda_j^m(x,t), \qquad \Lambda_j^m(x,t) := \Lambda_j(x)\Lambda^m(t), \tag{3.3}$$

where $\Lambda_j(x)$ and $\Lambda^m(t)$ denote the usual "hat" functions,

$$\begin{aligned} \Lambda_j(x) &= \frac{1}{\Delta x}\min(x - x_{j-1}, x_{j+1} - x)_+, \\ \Lambda^m(t) &= \frac{1}{\Delta t}\min(t - t^{m-1}, t^{m+1} - t)_+. \end{aligned}$$

To study the convergence rate of $v^{\Delta x}(x,t)$ as $\Delta x \downarrow 0$, we first have to verify the conservation and the *Lip'*-consistency of the difference approximation. To this end we proceed as follows.

We first note that $v^{\Delta x}(x,t)$ are clearly conservative, for by the choice of the initial conditions in (3.2),

$$\int v^{\Delta x}(x,t)dx = \frac{\Delta x}{2}\sum v_\nu^n + v_{v+1}^n = \frac{\Delta x}{2}\sum v_\nu^0 + v_{\nu+1}^0 = \int u_0(x)dx.$$

Moreover, these initial conditions are *Lip'*-consistent—in fact, the following estimate, which is left to the reader, holds: $(v^{\Delta x}(x,0)-u_0(x),\phi(x)) \le \text{Const}\cdot(\Delta x)^2\|u_0(x)\|_{BV}\cdot \|\phi(x)\|_{Lip}$. Finally, we turn to consider the *Lip'*-consistency with the conservation law (2.1). To this end we compare $v^{\Delta}(x,t)$ with certain entropy conservative schemes constructed in [Ta3].

A straightforward computation (carried out in the Appendix) shows that there exists a bounded piecewise-constant function, $D^n(x) = \sum_j D^n_{j+\frac{1}{2}}\chi_{j+\frac{1}{2}}(x)$, ($\chi_{j+\frac{1}{2}}(x) :=$ characteristic function of (x_j, x_{j+1})), such that the difference approximation (3.1) recast into the equivalent form†

(3.4)
$$\left(\frac{\partial}{\partial t}v^{\Delta x}(x,t),\Lambda_\nu^n(x,t)\right)_{\Delta x,t} + \left(\frac{\partial}{\partial x}f(v^{\Delta x}(x,t)),\Lambda_\nu^n(x,t)\right)_{x,\Delta t}$$
$$= \frac{\Delta t}{2}\left(\frac{\partial}{\partial t}v^{\Delta x}(x,t),\frac{\partial}{\partial t}\Lambda_\nu^n(x,t)\right)_{\Delta x,t} - \frac{\Delta x}{2}\left(\frac{\partial}{\partial x}v^{\Delta x}(x,t),\frac{\partial}{\partial x}\Lambda_\nu^n(x,t)\right)_{D(x),\Delta t}.$$

Hence, for arbitrary $\phi\epsilon C_0^\infty$, we may rewrite (3.4) as

$$(v_t^{\Delta x} + f(v^{\Delta x})_x,\phi)_{x,t} = \sum_{k=1}^{4} T_k^{\Delta x}. \tag{3.5}$$

The sum on the right-hand side of (3.5) represents the *truncation error* of the difference approximation (3.1), and according to (3.4), it consists of the following four contributions (here, $\hat\phi(x,t) = \sum_{\nu,n}\phi(x_\nu,t^n)\Lambda_\nu^n(x,t)$ denotes the piecewise-linear interpolant of $\phi(x,t)$):

$$T_1 = -\frac{\Delta x}{2}(v_x^{\Delta x},\hat\phi_x)_{D(x),\Delta t},$$

$$T_2 = \frac{\Delta t}{2}(v_t^{\Delta x},\hat\phi_t)_{\Delta x,t},$$

$$T_3 = (v_t^{\Delta x},\phi)_{x,t} - (v_t^{\Delta x},\hat\phi)_{\Delta x,t},$$

$$T_4 = (f(v^{\Delta x})_x,\phi)_{x,t} - (f(v^{\Delta x})_x,\hat\phi)_{x,\Delta t}.$$

†The Euclidean and weighted L^2-inner products are denoted by $(\rho,\psi)_x = \int\rho(x)\psi(x)dx$ and $(\rho,\psi)_{D(x)} = \int\rho(x)\psi(x)D(x)dx$. The corresponding discrete ℓ^2-inner product reads $(\rho,\psi)_{\Delta x} = \sum_\nu \rho(x_\nu)\psi(x_\nu)\Delta x$. Similar notations are used for (x,t)-functions, e.g., $(\rho,\psi)_{D(x),\Delta t} = \sum_n\int_x\rho(x,t^n)\psi(x,t^n)D(x)dx\Delta t$, $\|\rho\|^p_{L^p(\Delta x,t)} = \int_t\sum_\nu|\rho(x_\nu,t)|^p\Delta x dt$, etc.

We want to show that the difference approximation (3.1) is consistent with the conservation law (2.1), in the sense that the *Lip'*-size of its truncation error is of order $O(\Delta x)$. The required estimates in this direction are collected below. We begin with a straightforward estimate of the first term,

$$\begin{aligned}|T_1| &\leq \frac{\Delta x}{2}\|v_x^{\Delta x}\|_{L^1(|D(x)|,\Delta t)} \cdot \|\hat{\phi}_x\|_{L^\infty(x,\Delta t)} \\ &\leq C_1 \cdot \Delta x\|v^{\Delta x}(x,t)\|_{L^1([0,T],BV(x))} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}.\end{aligned} \tag{3.6a}$$

The difference approximation (3.1) enables us to upper bound time-differences in terms of spatial differences to yield the following upper-bound on the second term:

$$\begin{aligned}|T_2| &\leq \frac{\Delta t}{2}\|v_t^{\Delta x}\|_{L^1(\Delta x,t)} \cdot \|\hat{\phi}_t\|_{L^\infty(\Delta x,t)} \\ &\leq C_2 \cdot \Delta x\|v^{\Delta x}(x,t)\|_{L^1([0,T],BV(x))} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}.\end{aligned} \tag{3.6b}$$

The third contribution to the truncation error we rewrite as

$$T_3 = [(v_t^{\Delta x},\hat{\phi})_{x,t} - (v_t^{\Delta x},\hat{\phi})_{\Delta x,t}] + (v_t^{\Delta x},\phi-\hat{\phi})_{x,t} = T_{31} + T_{32}.$$

We have (abbreviating $\phi_j^m \equiv \phi(x_j,t^m)$):

$$\begin{aligned}T_{31} &= \sum_{\nu,j,n}\frac{1}{2}(v_\nu^{n+1}-v_\nu^{n-1})\phi_j^n(\Lambda_\nu(x),\Lambda_j(x))_x - \sum_{\nu,j,n}\frac{1}{2}(v_\nu^{n+1}-v_\nu^{n-1})\phi_\nu^n\Delta x \\ &= \sum_{\nu,n}\frac{1}{2}(v_\nu^{n+1}-v_\nu^{n-1})\frac{1}{6}(\phi_{\nu+1}^n - 2\phi_\nu^n + \phi_{\nu-1}^n)\Delta x,\end{aligned}$$

and hence T_{31} is upper bounded by

$$|T_{31}| \leq \text{Const.}\Delta x\|v_t^{\Delta x}\|_{L^1(\Delta x,\Delta t)} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}.$$

This, together with the standard interpolation error estimate,

$$|T_{32}| \leq \|v_t^{\Delta x}\|_{L^1(x,t)} \cdot \|\phi-\hat{\phi}\|_{L^\infty(x,t)} \leq \text{Const}\cdot\Delta x\|v_t^{\Delta x}\|_{L^1(x,t)} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])},$$

gives us that the third term does not exceed

$$\begin{aligned}|T_3| &\leq \text{Const}\cdot\Delta x\|v_t^{\Delta x}\|_{L^1(x,t)}\|\phi(x,t)\|_{Lip(x,[0,T])} \\ &\leq C_3\cdot\Delta x\|v^{\Delta x}(x,t)\|_{L^1([0,T],BV(x))} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}.\end{aligned} \tag{3.6c}$$

A similar treatment of the fourth term implies

$$|T_4| \leq C_4\cdot\Delta x\|v^{\Delta x}(x,t)\|_{L^1([0,T],BV(x))} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}. \tag{3.6d}$$

Equipped with the last four estimates (3.6a)–(3.6d), we return to (3.5), obtaining

$$\begin{aligned}&|(v_t^{\Delta x} + f(v^{\Delta x})_x,\phi)_x| \\ &\leq \text{Const}\cdot\Delta x\|v^{\Delta x}(x,t)\|_{L^1([0,T],BV(x))} \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}.\end{aligned} \tag{3.7}$$

This shows that the *Lip'*-consistency estimate (2.4b) holds with $\varepsilon = \Delta x$ and $K_T \sim \|v^{\Delta x}(x,t)\|_{L^1([0,T],BV(x))}$. Thus, Corollaries 2.2–2.4 apply and their various error estimates are put together in the following.

THEOREM 3.1. *Assume that the difference approximation* (3.1)–(3.2) *is* Lip^+*-stable in the sense that the following one-sided Lipschitz condition is fulfilled:*

$$L_n^+ \le \frac{1}{[L_0^+]^{-1} + \beta t^n}, \qquad 0 \le t^n \le T, \tag{3.8}$$
$$L_n^+ \equiv \|v^{\Delta x}(\cdot, t^n)\|_{Lip^+} = \max_\nu \frac{(v_{\nu+1}^n - v_\nu^n)_+}{\Delta x}.$$

Then the following error estimates hold:

(3.9a)
$$\|v^{\Delta x}(\cdot,T) - u(\cdot,T)\|_{W^{s,p}} \le \mathrm{Const}_T \cdot (\Delta x)^{\frac{1-sp}{2p}}, \qquad -1 \le s \le \frac{1}{p}, \qquad 1 \le p \le \infty,$$

$$|v^{\Delta x}(x,T) - u(x,T)| \le \mathrm{Const}_{x,T} \cdot \left[1 + \max_{|\xi - x| \le \sqrt[3]{\Delta x}} |u_x(\xi,T)|\right] \cdot \sqrt[3]{\Delta x}. \tag{3.9b}$$

Examples. The following first-order accurate schemes (identified in a decreasing order according to their numerical viscosity coefficient, $Q_{\nu+\frac12} \equiv Q^n_{\nu+\frac12}$), are frequently referred to in the literature.

$$\text{(3.10a)} \qquad \text{Lax–Friedrichs scheme}: \qquad Q^{LxF}_{\nu+\frac12} \equiv 1,$$

$$\text{(3.10b)} \qquad \text{Engquist–Osher scheme}: \qquad Q^{EO}_{\nu+\frac12} = \frac{\lambda}{v^n_{\nu+1} - v^n_\nu} \int_{v_\nu}^{v_{\nu+1}} |f'(v)|dv,$$

$$\text{(3.10c)} \qquad \text{Godunov scheme}: \qquad Q^{G}_{\nu+\frac12} = \lambda \max_v \left[\frac{f(v^n_{\nu+1}) + f(v^n_\nu) - 2f(v)}{v^n_{\nu+1} - v^n_\nu}\right],$$

$$\text{(3.10d)} \qquad \text{Roe scheme}: \qquad Q^{R}_{\nu+\frac12} = \lambda |a_{\nu+\frac12}|, \quad a_{\nu+\frac12} \equiv a^n_{\nu+\frac12} = \frac{f(v^n_{\nu+1}) - f(v^n_\nu)}{v^n_{\nu+1} - v^n_\nu}.$$

We comment briefly on the Lip^+-stability condition of these schemes. The solution of the Lax–Friedrichs scheme satisfies [Ta1, Eqn. 3.8]

$$L^+_{n+1} \le L^+_n \left(1 - \frac{\alpha}{2}\Delta t L^+_n\right),$$

which in turn implies (by induction) the desired Lip^+-stability (3.8) with $\beta = \frac{\alpha}{2}$, for

$$L_n^+ \le L_{n-1}^+(1 - \beta L_{n-1}^+) < \frac{1}{[L_{n-1}^+]^{-1} + \beta \Delta t} \le \cdots \le \frac{1}{[L_0^+]^{-1} + \beta t^n}. \tag{3.11}$$

A similar framework was used in [GL, §3] to show that Godunov scheme satisfies the Lip^+-stability estimate (3.8) with $\beta = \frac{\alpha}{4}$.

The Lip^+-stability of the Engquist–Osher (E–O) scheme is closely related to Godunov's scheme. The two schemes coincide except for sonic shock cells (where $a(v_{\nu+1}) < 0 < a(v_v))$, which leads to

$$\lambda |a_{\nu+\frac12}| \le Q^G_{\nu+\frac12} \le Q^{EO}_{\nu+\frac12} \le Q^G_{\nu+\frac12} - \mathrm{Const} \cdot (\Delta v_{\nu+\frac12})_- . \tag{3.12}$$

Hence, the forward differences of E–O scheme are upper-bounded by

$$\begin{aligned}
\Delta v^{n+1}_{\nu+\frac{1}{2}} &= (1 - Q^{EO}_{\nu+\frac{1}{2}})\Delta v^n_{\nu+\frac{1}{2}} + \tfrac{1}{2}(Q^G_{\nu+\frac{3}{2}} - \lambda a_{\nu+\frac{3}{2}})\Delta v^n_{\nu+\frac{3}{2}} \\
&\quad + \tfrac{1}{2}(Q^G_{\nu-\frac{1}{2}} + \lambda a_{\nu-\frac{1}{2}})\Delta v^n_{\nu-\frac{1}{2}} \\
&\quad + \tfrac{1}{2}(Q^{EO}_{\nu+\frac{3}{2}} - Q^G_{\nu+\frac{3}{2}})\Delta v^n_{\nu+\frac{3}{2}} + \tfrac{1}{2}(Q^{EO}_{\nu-\frac{1}{2}} - Q^G_{\nu-\frac{1}{2}})\Delta v^n_{\nu-\frac{1}{2}} \qquad (3.13) \\
&\le (1 - Q^{EO}_{\nu+\frac{1}{2}})\Delta v^n_{\nu+\frac{1}{2}} + \tfrac{1}{2}(Q^G_{\nu+\frac{3}{2}} - \lambda a_{\nu+\frac{3}{2}})\Delta v^n_{\nu+\frac{3}{2}} \\
&\quad + \tfrac{1}{2}(Q^G_{\nu-\frac{1}{2}} + \lambda a_{\nu-\frac{1}{2}})\Delta v^n_{\nu-\frac{1}{2}}.
\end{aligned}$$

We distinguish between two cases. If $\Delta v^n_{\nu+\frac{1}{2}} \ge 0$, then the first term on the right of (3.13) does not exceed $(1-Q^G_{\nu+\frac{1}{2}})\Delta v^n_{\nu+\frac{1}{2}}$, and hence the E–O solution satisfies the one-sided Lipschitz condition in this case, because Godunov's solution does. Otherwise, $\Delta v^n_{\nu+\frac{1}{2}}$ and therefore $(1 - Q^{EO}_{\nu+\frac{1}{2}})\Delta v^n_{\nu+\frac{1}{2}}$ is negative, hence

$$\Delta v^{n+1}_{\nu+\frac{1}{2}} \le \tfrac{1}{2}(Q^G_{\nu+\frac{3}{2}} - \lambda a_{\nu+\frac{3}{2}})\Delta v^n_{\nu+\frac{3}{2}} + \tfrac{1}{2}(Q^G_{\nu-\frac{1}{2}} + \lambda a_{\nu-\frac{1}{2}})\Delta v^n_{\nu-\frac{1}{2}}$$

and the *Lip*$^+$-bound follows in view of (3.12) and the CFL condition $\lambda \max|f'| < \frac{1}{2}$.

Finally, for the Roe (or Courant–Isaacson–Rees) scheme, *Lip*$^+$-stability (3.8) with $\beta = 0$ (no decay), was proved in [Br]. Note that the assumption of *Lip*$^+$-bounded initial conditions is essential for convergence to the entropy solution in this case, in view of the discrete steady-state solution, $v^0_\nu = \mathrm{sgn}(\nu + \frac{1}{2})$, which shows that convergence of Roe scheme to the correct entropy rarefaction fails due to the fact that the initial data are *not* *Lip*$^+$-bounded.

Using Theorem 3.1, we conclude the following.

COROLLARY 3.2. *Consider the conservation law* (2.1), (2.2) *with Lip*$^+$*-bounded initial data* (2.9). *Then the Roe, Godunov, Engquist–Osher, and Lax-Friedrichs difference approximations* (3.1), (3.10) *with discrete initial data* (3.2) *converge, and their piecewise-linear interpolants* $v^{\Delta x}(x,t)$, *satisfy the convergence rate estimates* (3.9a), (3.9b).

4. Glimm scheme. We recall the construction of Glimm approximate solution for the conservation law (2.1); see [Gl] and [Sm]. We let $v(x,t)$ be the entropy solution of (2.1) in the slab $t^n \le t < t^{n+1}, n \ge 0$, subject to piecewise constant data $v(x,t^n) = \sum_\nu v^n_\nu \chi_\nu(x)$. To proceed in time, the solution is extended (in a staggered fashion) with a jump discontinuity across the lines $t^{n+1}, n \ge 0$, where $v(x,t^{n+1})$ takes the piecewise constant values

$$v(x,t^{n+1}) = \sum_\nu v^{n+1}_{\nu+\frac{1}{2}}\chi_{\nu+\frac{1}{2}}(x), \qquad v^{n+1}_{\nu+\frac{1}{2}} = v(x_{\nu+\frac{1}{2}} + r^n\Delta x, t^{n+1} - 0). \qquad (4.1)$$

Notice that in each slab, $v(x,t)$ consists of successive noninteracting Riemann solutions provided the CFL condition, $\lambda \cdot \max|a(u)| \le \frac{1}{2}$ is met. This defines the Glimm approximate solution, $v(x,t) \equiv v^\varepsilon(x,t)$, depending on the mesh parameters $\varepsilon = \Delta x \equiv \lambda\Delta t$, and the set of random variables $\{r^n\}$, uniformly distributed in $[-\frac{1}{2}, \frac{1}{2}]$. In the deterministic version of the Glimm scheme, Liu [Li] employs equidistributed rather than a random sequence of numbers $\{r^n\}$. We note that in both versions, we make use of exactly *one* random or equidistributed choice per timestep (independently of the spatial cells), as was first advocated by Chorin [Cho].

It follows that both versions of Glimm scheme share the *Lip*$^+$-stability estimate (2.6). Indeed, since the solution of a scalar Riemann problem remains in the convex hull of its initial data, we may express $v^{n+1}_{\nu+\frac{1}{2}}$ as $(1-\theta^n_{\nu+\frac{1}{2}})v^n_\nu + \theta^n_{\nu+\frac{1}{2}}v^n_{\nu+1}$ for some $\theta^n_{\nu+\frac{1}{2}} \in [0,1]$, and hence

$$v^{n+1}_{\nu+\frac{1}{2}} - v^{n+1}_{\nu-\frac{1}{2}} = \theta^n_{\nu+\frac{1}{2}}\Delta v^n_{\nu+\frac{1}{2}} + (1-\theta^n_{\nu-\frac{1}{2}})\Delta v^n_{\nu-\frac{1}{2}}.$$

We now distinguish between two cases. If either $\Delta v^n_{\nu-\frac{1}{2}}$ or $\Delta v^n_{\nu+\frac{1}{2}}$ is negative, then

$$v^{n+1}_{\nu+\frac{1}{2}} - v^{n+1}_{\nu-\frac{1}{2}} \le \max(\Delta v^n_{\nu+\frac{1}{2}}, \Delta v^n_{\nu-\frac{1}{2}}). \tag{4.2}$$

Otherwise—when both $\Delta v^n_{\nu+\frac{1}{2}}$ and $\Delta v^n_{\nu-\frac{1}{2}}$ are positive, the two values of $v^{n+1}_{\nu+\frac{1}{2}}$ and $v^{n+1}_{\nu-\frac{1}{2}}$ are obtained as sampled values of two consecutive rarefaction waves, and a straightforward computation shows that their difference satisfies (4.2). Thus in either case, the *Lip*$^+$-stability (2.6) holds with $\beta = 0$.

Although Glimm approximate solutions are conservative "on the average," they do not satisfy the conservation requirement (2.3). We therefore need to slightly modify our previous convergence arguments in this case.

We first recall the truncation error estimate for the deterministic version of Glimm scheme [HS, Thm. 3.2],

$$\begin{aligned} &\left(v^{\Delta x}_t + f(v^{\Delta x})_x, \phi(x,t)\right)_{L^2(x,[0,T])} \\ &\qquad \le \mathrm{Const}_T \left[\sqrt{\Delta x}|\ln \Delta x| \cdot \|\phi\|_{L^\infty} + \Delta x \cdot \|\phi(x,t)\|_{Lip(x,[0,T])}\right]. \end{aligned} \tag{4.3}$$

Let $\phi(x,t) = \phi^{\Delta x}(x,t)$ denote the solution of the adjoint error equation (2.12). Applying (4.3) instead of (2.17b) and arguing along the lines of Theorem 2.1, we conclude that Glimm scheme is *Lip*$'$-consistent (and hence has a *Lip*$'$-convergence rate) of order $\sqrt{\Delta x}|\ln \Delta x|$,

$$|(e^{\Delta x}(\cdot,T), \phi(\cdot))| \le \mathrm{Const}_T \left[\sqrt{\Delta x}|\ln \Delta x| \cdot \|\phi\|_{L^\infty} + \Delta x \cdot \|\phi(x)\|_{Lip}\right]. \tag{4.4}$$

To obtain an L^1-convergence rate estimate we employ (4.4) with $\phi_\delta \equiv \phi * \frac{1}{\delta}\zeta\left(\frac{\cdot}{\delta}\right)$ yielding

$$|(e^{\Delta x}(\cdot,T), \phi_\delta)| \le \mathrm{Const}_T \left[\sqrt{\Delta x}|\ln \Delta x| + \frac{\Delta x}{\delta}\right] \|\phi(x)\|_{L^\infty}. \tag{4.5}$$

Using this estimate together with

$$(e^\varepsilon(\cdot,T), [\phi(\cdot) - \phi_\delta(\cdot)]) \equiv (e^\varepsilon(\cdot,T) - e^\varepsilon_\delta(\cdot,T), \phi) \le \mathrm{Const} \cdot \|e^\varepsilon(\cdot,T)\|_{BV} \cdot \delta\|\phi\|_{L^\infty},$$

imply (for $\delta \sim \sqrt{\Delta x}$), the usual L^1-convergence rate of order $O(\sqrt{\Delta x}|\ln \Delta x|)$. As noted in the closing remark of §2, the *Lip*$^+$-stability of Glimm's approximate solutions enables us to convert the L^1-type into pointwise convergence rate estimate.

We close this section by stating the following.

THEOREM 4.1. *Consider the conservation law* (2.1), (2.2) *with sufficiently small* Lip^+*-bounded initial data* (2.9). *Then the deterministic version of Glimm approximate solution* $v^{\Delta x}(x,t)$ (4.1) *converges to the entropy solution* $u(x,t)$, *and the following convergence rate estimates hold:*

$$\|v^{\Delta x}(\cdot,T) - u(\cdot,T)\|_{L^1} \leq \text{Const}_T \cdot \sqrt{\Delta x |\ln \Delta x|}, \tag{4.6}$$

$$|v^{\Delta x}(x,T) - u(x,T)| \leq \text{Const}_{x,T} \cdot \left[1 + \max_{|\xi - x| \leq \sqrt[4]{\Delta x}} |u_x(\xi,T)|\right] \cdot \sqrt[4]{\Delta x |\ln \Delta x|}. \tag{4.7}$$

Remarks. 1. A sharp L^1-error estimate of order $O(\sqrt{\Delta x})$ can be found in [Lu], improving the previous error estimates of [HS].

2. Theorem 4.1 hinges on the truncation error estimate (4.3) which assumes initial data which sufficiently small variation [HS]. Extensions to strong initial discontinuities for Glimm scheme and the front tracking method can be found in [Che, Thms. 4.6 and 5.2].

APPENDIX. We want to show that the piecewise-linear interpolant $v^{\Delta x}(x,t)$ in (3.3) serves as an approximate weak solution of the conservation law (2.1).

Let

$$v^n(x) = \sum_\nu v_\nu^n \Lambda_\nu(x) \quad \text{and} \quad v_\nu(t) = \sum_n v_\nu^n \Lambda^n(t)$$

denote the spatial and temporal interpolants of the discrete grid solution $\{v_\nu^n\}_{\nu,n}$.

Straightforward integration by parts yields [Ta3]

$$\begin{aligned}(f(v^{\Delta x}(x,t))_x, \Lambda_\nu(x))_x &= \tfrac{1}{2}[f(v_{\nu+1}(t)) - f(v_{\nu-1}(t))] \\ &\quad -\tfrac{1}{2}[Q^*_{\nu+\frac{1}{2}}(t)\Delta v_{\nu+\frac{1}{2}}(t) - Q^*_{\nu-\frac{1}{2}}(t)\Delta v_{\nu-\frac{1}{2}}(t)],\end{aligned} \tag{1a}$$

where (we abbreviate $v_{\nu+\frac{1}{2}}(\xi,t) \equiv \frac{1}{2}[v_\nu(t) + v_{\nu+1}(t)] + \xi\Delta v_{\nu+\frac{1}{2}}(t)$)

$$Q^*_{\nu+\frac{1}{2}}(t) = \Delta v_{\nu+\frac{1}{2}}(t) \cdot \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{4} - \xi^2\right) f''(v_{\nu+\frac{1}{2}}(\xi,t))d\xi. \tag{1b}$$

In particular, for $f(v) \equiv v$ we have $Q^* \equiv 0$ and (1a) yields

$$(v_x^{\Delta x}(x,t), \Lambda_\nu(x))_x = \tfrac{1}{2}[v_{\nu+1}(t) - v_{\nu-1}(t)].$$

Exchanging the role of the x and t variables in the last equality we get

$$(v_t^{\Delta x}(x,t), \Lambda^n(t))_t = \tfrac{1}{2}[v^{n+1}(x) - v^{n-1}(x)]. \tag{2}$$

Moreover, with $D(x) = \sum_\nu D_{\nu+\frac{1}{2}}\chi_{\nu+\frac{1}{2}}(x)$ we have

$$\frac{\Delta x}{2}(v_x^{\Delta x}(x,t), (\Lambda_\nu(x))_x)_{D(x)} = -\frac{1}{2}[D_{\nu+\frac{1}{2}}\Delta v_{\nu+\frac{1}{2}}(t) - D_{\nu-\frac{1}{2}}\Delta v_{\nu-\frac{1}{2}}(t)] \tag{3}$$

and by exchanging the role of the x and t variables in (3) we get

$$\frac{\Delta t}{2}(v_t^{\Delta x}(x,t), (\Lambda^n(t))_t)_t = -\frac{1}{2}[v^{n+1}(x) - 2v^n(x) + v^{n-1}(x)]. \tag{4}$$

The equalities (1)–(4) imply

$$(1')\quad \begin{aligned}&(f(v^{\Delta x}(x,t))_x, \Lambda_\nu^n(x,t))_{x,\Delta t}\\ &= \frac{\Delta t}{2}[f(v_{\nu+1}^n) - f(v_{\nu-1}^n)] - \frac{\Delta t}{2}[Q^*_{\nu+\frac12}(t^n)\Delta\nu^n_{\nu+\frac12} - Q^*_{\nu-\frac12}(t^n)\Delta v^n_{\nu-\frac12}],\end{aligned}$$

$$(2')\quad (v_t^{\Delta x}(x,t), \Lambda_\nu^n(x,t))_{\Delta x,t} = \frac{\Delta x}{2}[v_\nu^{n+1} - v_\nu^{n-1}],$$

$$(3')\quad \frac{\Delta x}{2}(v_x^{\Delta x}(x,t), (\Lambda_\nu^n(x,t))_x)_{D(x),\Delta t} = -\frac{\Delta t}{2}[D_{\nu+\frac12}\Delta v^n_{\nu+\frac12} - D_{\nu-\frac12}\Delta v^n_{n-\frac12}],$$

$$(4')\quad \frac{\Delta t}{2}(v_t^{\Delta x}(x,t), (\Lambda_\nu^n(x,t))_t)_{\Delta x,t} = -\frac{\Delta x}{2}[v_\nu^{n+1} - 2v_\nu^n - v_\nu^{n-1}].$$

The difference approximation (3.1) reads

$$(5)\quad \Delta x[v_\nu^{n+1} - v_\nu^n] = -\frac{\Delta t}{2}[f(v_{\nu+1}^n) - f(v_{\nu-1}^n)] + \frac{\Delta x}{2}[Q^n_{\nu+\frac12}\Delta v^n_{\nu+\frac12} - Q^n_{\nu-\frac12}\Delta v^n_{\nu-\frac12}].$$

By (2′) and (4′), the left-hand side (LHS) of (5) equals

$$\begin{aligned}\text{LHS} &= \frac{\Delta x}{2}[v_\nu^{n+1} - v_\nu^{n-1}] + \frac{\Delta x}{2}[v_\nu^{n+1} - 2v_\nu^n + v_\nu^{n-1}]\\ &= (v_t^{\Delta x}(x,t), \Lambda_\nu^n(x,t))_{\Delta x,t} - \frac{\Delta t}{2}(v_t^{\Delta x}(x,t), (\Lambda_\nu^n(x,t))_t)_{\Delta x,t}.\end{aligned}$$

Next, we set $D^n_{\nu+\frac12} \equiv \frac{1}{\lambda}Q^n_{\nu+\frac12} - Q^*_{\nu+\frac12}(t^n)$; then by (1′) and (3′) the right-hand side (RHS) of (5) equals

$$\begin{aligned}\text{RHS} &= -\frac{\Delta t}{2}[f(v_{\nu+1}^n) - f(v_{\nu-1}^n)] + \frac{\Delta t}{2}[Q^*_{\nu+\frac12}\Delta v^n_{\nu+\frac12} - Q^*_{\nu-\frac12}\Delta v^n_{\nu+\frac12}]\\ &\quad + \frac{\Delta t}{2}[D^n_{\nu+\frac12}\Delta v^n_{\nu+\frac12} - D^n_{\nu-\frac12}\Delta v^n_{\nu-\frac12}]\\ &= -(f(v^{\Delta x}(x,t))_x, \Lambda_\nu^n(x,t))_{x,\Delta t} - \frac{\Delta x}{2}(v_x^{\Delta x}(x,t), (\Lambda_\nu^n(x,t))_x)_{D(x),\Delta t}\end{aligned}$$

and (3.4) now follows.

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