

## HYDRODYNAMIC ALIGNMENT WITH PRESSURE II. MULTI-SPECIES

BY

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*Dedicated to Constantine Dafermos with friendship and admiration*

**Abstract.** We study the long-time hydrodynamic behavior of systems of multi-species which arise from agent-based description of alignment dynamics. The interaction between species is governed by an array of symmetric communication kernels. We prove that the crowd of different species flocks towards the mean velocity if (i) cross interactions form a heavy-tailed connected array of kernels, while (ii) self-interactions are governed by kernels with singular heads. The main new aspect here is that flocking behavior holds without closure assumption on the specific form of pressure tensors. Specifically, we prove the long-time flocking behavior for connected arrays of multi-species, with self-interactions governed by entropic pressure laws (see E. Tadmor [Bull. Amer. Math. Soc. (2023), to appear]) and driven by fractional  $p$ -alignment. In particular, it follows that such multi-species hydrodynamics approaches a mono-kinetic description. This generalizes the mono-kinetic, “pressure-less” study by He and Tadmor [Ann. Inst. H. Poincaré C Anal. Non Linéaire 38 (2021), pp. 1031–1053].

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**1. Introduction — alignment dynamics of multi-species.**

1.1. *Hydrodynamic description of multi-species.* We study the long-time behavior of the multi-species hydrodynamics

$$\begin{cases} \partial_t \rho_\alpha + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha) = 0, \\ \partial_t (\rho_\alpha \mathbf{u}_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + \mathbb{P}_\alpha) = \mathbf{A}_\alpha(\rho, \mathbf{u}), \end{cases} \quad (t, \mathbf{x}) \in (\mathbb{R}_t, \mathbb{R}^d), \quad (1.1a)$$

subject to initial data  $(\rho_\alpha, \mathbf{u}_\alpha, \mathbb{P}_\alpha)|_{t=0} = (\rho_{\alpha 0}, \mathbf{u}_{\alpha 0}, \mathbb{P}_{\alpha 0})$ . The different species, tagged by the (possibly infinite) index-set  $\alpha \in \mathcal{I}$ , are quantified by their density,  $\rho_\alpha : \mathbb{R}_t \times \mathbb{R}^d \mapsto \mathbb{R}_+$ , momentum,  $\rho_\alpha \mathbf{u}_\alpha : \mathbb{R}_t \times \mathbb{R}^d \mapsto \mathbb{R}^d$ , and pressure tensor,  $\mathbb{P}_\alpha : \mathbb{R}_t \times \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^d$ . Each species occupies a distinct ‘patch’ of mass  $M_\alpha(t)$  supported on  $\mathcal{S}_\alpha(t)$ ,

$$M_\alpha(t) = \int_{\mathcal{S}_\alpha(t)} \rho_\alpha(t, \mathbf{x}) \, d\mathbf{x}, \quad \mathcal{S}_\alpha(t) := \text{supp} \rho_\alpha(t, \cdot).$$

The dynamics is driven by inter-species interactions due to *alignment*, dictated by a symmetric array of symmetric communication kernels,  $\Phi = \{\phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}')\}$ ,

$$\mathbf{A}_\alpha(\rho, \mathbf{u}) := \sum_{\beta \in \mathcal{I}} \int_{\mathcal{S}_\alpha(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (\mathbf{u}_\beta(t, \mathbf{x}') - \mathbf{u}_\alpha(t, \mathbf{x})) \rho_\alpha(t, \mathbf{x}) \rho_\beta(t, \mathbf{x}') \, d\mathbf{x}'. \quad (1.1b)$$

Thus, what distinguishes species  $\alpha$  is the way it communicates with the other species, through symmetric kernels  $\phi_{\alpha\beta}$ ,  $\beta \in \mathcal{I}$ ,

$$\phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') = \phi_{\alpha\beta}(\mathbf{x}', \mathbf{x}), \quad \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') = \phi_{\beta\alpha}(\mathbf{x}, \mathbf{x}'), \quad (1.1c)$$

while self-interactions within the same species are governed by  $\phi_{\alpha\alpha}$ ,  $\alpha \in \mathcal{I}$ . There is a special role for metric kernels where communication is dictated by the distance  $|\mathbf{x} - \mathbf{x}'|$ . In this context we assume the existence of a symmetric array of radially decreasing kernels,  $K := \{k_{\alpha\beta}(r)\}$ , such that

$$\phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \geq k_{\alpha\beta}(|\mathbf{x} - \mathbf{x}'|), \quad k_{\beta\alpha} = k_{\alpha\beta} \geq 0, \quad \alpha, \beta \in \mathcal{I}. \quad (1.1d)$$

We use the standard notation  $\Phi \succeq K$  to abbreviate (1.1d). This covers the prototypical case of *metric kernels*,  $\phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') = k_{\alpha\beta}(|\mathbf{x} - \mathbf{x}'|)$ , with decreasing intensity of communication as a function of the distance, e.g.,  $\phi_{\alpha\beta}(r) = (1+r)^{-\eta}$  in [CS2007a]. In particular, we address general non-decreasing metric kernels,  $\phi_{\alpha\beta}(|\cdot|)$ , in terms of their *decreasing envelope*  $k_{\alpha\beta}(r) := \min\{\phi_{\alpha\beta}(|\mathbf{x}|) \mid |\mathbf{x}| \leq r\}$ . The variety of different classes of communication kernels reflects large literature on collective dynamics which arises in different disciplines [Aok1982, VCBCS1995, CF2003, CS2007a, CDMBC2007, Bal2008, CFL2009, Ka2011, MT2011, GWBL2012, MCEB2015, JJ2015, LZTM2019, MLK2019, ST2020b, ST2021]. The different species are viewed as moving ‘patches’ of different crowds with mass and momentum which interact according to the alignment protocol (1.1). We make the following

three assumptions about these ‘patches’. We assume that the density of species inside their ‘patch’ remains strictly bounded away from vacuum,

$$\min_{\mathbf{x} \in \mathcal{S}_\alpha(t)} \rho_\alpha(t, \mathbf{x}) \geq \rho_- > 0, \quad \forall \alpha \in \mathcal{I}. \tag{H1}$$

Further, we assume that

$$\mathcal{S}_\alpha(t) \text{ have smooth boundary satisfying a Lipschitz or a cone condition, } \forall \alpha \in \mathcal{I}. \tag{H2}$$

Finally, we assume that the boundary of each patch forms a contact discontinuity, governed by Neumann boundary conditions

$$\mathbf{u}_\alpha \cdot \mathbf{n}_\alpha|_{\partial \mathcal{S}_\alpha} = 0, \quad \mathbb{P}_\alpha \mathbf{n}_\alpha|_{\partial \mathcal{S}_\alpha} = 0 \text{ and } \mathbf{q}_\alpha \cdot \mathbf{n}_\alpha|_{\partial \mathcal{S}_\alpha} = 0, \quad \forall \alpha \in \mathcal{I}. \tag{H3}$$

In particular, it follows that there is no flux of mass for each species: integration of (1.1a)<sub>1</sub> implies the mass of each species is conserved

$$M_\alpha(t) = M_{\alpha 0}, \quad M_\alpha(t) = \int_{\mathcal{S}_\alpha} \rho_\alpha(t, \mathbf{x}) \, d\mathbf{x}. \tag{1.2}$$

In particular, the total mass is also conserved  $M := \sum_\alpha M_\alpha(t) = \sum_\alpha M_{\alpha 0}$ . In contrast, the momentum of each species need not necessarily be conserved due to the cross alignment terms between different species on the right of (1.1a)<sub>2</sub>. Instead, the symmetry of  $\phi_{\alpha\beta}(\cdot, \cdot)$  implies that the *total momentum* is conserved<sup>1</sup>

$$\begin{aligned} & \frac{d}{dt} \sum_\alpha \int_{\mathcal{S}_\alpha} \rho_\alpha \mathbf{u}_\alpha \, d\mathbf{x} \\ &= - \int_{\partial \mathcal{S}_\alpha} (\mathbf{u}_\alpha \cdot \mathbf{n}_\alpha \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{n}_\alpha) \, dS \\ &+ \sum_{\alpha, \beta} \iint_{(\mathbf{x}, \mathbf{x}') \in \mathcal{S}_\alpha \times \mathcal{S}_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (\mathbf{u}'_\beta - \mathbf{u}_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' = 0, \end{aligned}$$

and hence

$$\mathbf{m} := \sum_\alpha \mathbf{m}_\alpha(t) = \sum_\alpha \mathbf{m}_{\alpha 0}, \quad \mathbf{m}_\alpha(t) := \int_{\mathcal{S}_\alpha} \rho_\alpha(t, \mathbf{x}) \mathbf{u}_\alpha(t, \mathbf{x}) \, d\mathbf{x}. \tag{1.3}$$

1.2. *The class of entropic pressure laws.* The multi-species system (1.1a) requires a closure for the pressure tensors  $\mathbb{P}_\alpha(t, \mathbf{x}), \alpha \in \mathcal{I}$ . In this context, we recall the notion of *entropic pressure* [Tad2022]. We refer to  $\mathbb{P}_\alpha$  as an *entropic pressure* tensor associated with species  $\alpha$  in (1.1a) if its non-negative trace  $\rho_\alpha e_\alpha(t, \mathbf{x}) := \frac{1}{2} \text{trace}(\mathbb{P}_\alpha(t, \mathbf{x})) \geq 0$  satisfies

$$\partial_t(\rho_\alpha e_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha e_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha) + \text{trace}(\mathbb{P}_\alpha \nabla \mathbf{u}_\alpha) \leq -2 \sum_\beta \int_{\mathcal{S}_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') e_\alpha \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \tag{1.4}$$

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<sup>1</sup>Here and below we abbreviate  $\square' := \square(t, \mathbf{x}')$ .

Here  $\mathbf{q}_\alpha(t, \mathbf{x})$  is a  $C^1$ -flux. The motivation for (1.4) stems from the large-crowd dynamics of the agent-based model proposed in [HT2021], in which different species, each of which consists of  $N_\alpha$  agents with position/velocity  $(\mathbf{x}_i^\alpha(t), \mathbf{v}_i^\alpha(t)) : \mathbb{R}_+ \mapsto \mathbb{R}^d \times \mathbb{R}^d$ , are driven by the Cucker-Smale alignment [CS2007a]

$$\begin{cases} \frac{d}{dt} \mathbf{x}_i^\alpha(t) = \mathbf{v}_i^\alpha(t), \\ \frac{d}{dt} \mathbf{v}_i^\alpha(t) = \sum_{\beta \in \mathcal{I}} \frac{1}{N_\beta} \sum_{j=1}^{N_\beta} \phi_{\alpha\beta}(\mathbf{x}_j^\beta(t), \mathbf{x}_i^\alpha(t)) (\mathbf{v}_j^\beta(t) - \mathbf{v}_i^\alpha(t)), \end{cases} \quad i = 1, 2, \dots, N_\alpha. \quad (1.5)$$

The passage from the agent-based to the hydrodynamic description goes through an intermediate kinetic description which is realized by the empirical distribution,  $f_\alpha(t, \mathbf{x}, \mathbf{v}) := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} \delta_{\mathbf{x}_i^\alpha(t)} \otimes \delta_{\mathbf{v}_i^\alpha(t)}$ . Indeed, the hydrodynamic description (1.1a) is recovered in terms of the first-two limiting moments of  $\{f_\alpha\}$  which are assumed to exist,

$$\rho_\alpha(t, \mathbf{x}) = \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}, \quad \rho_\alpha \mathbf{u}_\alpha(t, \mathbf{x}) = \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} \mathbf{v} f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}.$$

This process of large-crowd limit as  $N_\alpha \rightarrow \infty$  recovers (1.1a) with pressure,  $\mathbb{P}_\alpha$ , given by the second-order moments

$$\mathbb{P}_\alpha(t, \mathbf{x}) = \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} (\mathbf{v} - \mathbf{u}_\alpha) \otimes (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}. \quad (1.6)$$

The formal derivation is outlined in appendix A and it follows the different derivations with different level of rigor in case of single species [HT2008, CFTV2010, CCR2011, FK2019, NP2021, Shv2021, NS2022]. The kinetic description of the pressure in terms of second-order, rank-one moments in (1.6) leads to the notion of *internal energy* which quantifies microscopic fluctuations around the bulk velocity  $\mathbf{u}_\alpha$ ,

$$\rho_\alpha e_\alpha = \frac{1}{2} \text{trace}(\mathbb{P}_\alpha) = \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} \frac{1}{2} |\mathbf{v} - \mathbf{u}_\alpha|^2 f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}.$$

This kinetic description of internal energy formally yields the equality

$$\partial_t(\rho_\alpha e_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha e_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha) + \text{trace}(\mathbb{P}_\alpha \nabla \mathbf{u}_\alpha) = -2 \sum_{\beta \in \mathcal{S}_\beta} \int \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') e_\alpha \rho_\alpha \rho'_\beta \, d\mathbf{x}',$$

with heat flux  $\mathbf{q}_\alpha := \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} \frac{1}{2} |\mathbf{v} - \mathbf{u}_\alpha|^2 (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}$ . Thus, we arrive at the

special case of equality in (1.4). In particular, it covers the “pressure-less” case — the special case of *mono-kinetic closure*

$$f_\alpha(t, \mathbf{x}, \mathbf{v}) \xrightarrow{N_\alpha \rightarrow \infty} \rho_\alpha(t, \mathbf{x}) \delta(\mathbf{v} - \mathbf{u}_\alpha(t, \mathbf{x})),$$

which is realized in terms of zero pressure,  $\mathbb{P}_\alpha = 0$ ,

$$\begin{cases} \partial_t \rho_\alpha + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha) = 0, \\ \partial_t(\rho_\alpha \mathbf{u}_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) = \mathbf{A}_\alpha(\rho, \mathbf{u}), \end{cases} \quad (t, \mathbf{x}) \in (\mathbb{R}_t, \mathbb{R}^d). \quad (1.7)$$

Most of the literature on swarming hydrodynamics of single species *assumes* mono-kinetic closure. The corresponding “pressure-less” multi-species hydrodynamics was studied in [HT2021]. The definition of pressure in terms of the entropy inequality (1.4) is not concerned, however, with the precise details of internal energy, as it lacks a reference to the specific closure with respect to a preferred state of thermal equilibrium. In fact, (1.4) applies to a large class of tensors beyond those which are realizable as second-order moments.

1.3. *Energy dissipation in entropic alignment.* A main consequence of the notion of entropic pressure is to secure the dissipativity of the total energy  $E_\alpha := \frac{|\mathbf{u}_\alpha|^2}{2} + e_\alpha$ . Indeed, manipulating the mass and momentum equations we find

$$\begin{aligned} \partial_t \left( \frac{\rho_\alpha}{2} |\mathbf{u}_\alpha|^2 \right) + \nabla_{\mathbf{x}} \cdot \left( \frac{\rho_\alpha}{2} |\mathbf{u}_\alpha|^2 \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha \right) - \text{trace}(\mathbb{P}_\alpha \nabla \mathbf{u}_\alpha) \\ = - \sum_{\beta} \int_{\mathcal{S}_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \end{aligned} \tag{1.8}$$

Adding the entropic description of the pressure postulated in (1.4) we end up with

$$\begin{aligned} \partial_t (\rho_\alpha E_\alpha) + \nabla_{\mathbf{x}} \cdot \left( \rho_\alpha E_\alpha \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha \right) \\ \leq - \sum_{\beta} \int_{\mathcal{S}_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta + 2e_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \end{aligned} \tag{1.9}$$

Thus, the role of entropic pressure is to complement the energy balance (1.8) in forming an entropy inequality (1.9), which augments the system of hyperbolic balance laws (1.1); we refer to the authoritative book of [Daf2016]. This implies dissipativity of the total energy. Indeed, by the zero Neumann boundary conditions assumed in (H3), it follows that

$$\begin{aligned} \frac{d}{dt} \sum_{\alpha} \int_{\mathcal{S}_\alpha(t)} \rho_\alpha E_\alpha \, d\mathbf{x} \\ \leq - \sum_{\alpha, \beta} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta + 2e_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\ = - \frac{1}{2} \sum_{\alpha, \beta} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}'_\beta - \mathbf{u}_\alpha|^2 + 2e_\alpha + 2e'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}'. \end{aligned} \tag{1.10}$$

For further discussion we refer to [Tad2022, §1].

**2. Swarming and long-time flocking behavior.** We discuss the behavior of a large crowd, possibly infinite number of species,  $\{\rho_\alpha, \mathbf{u}_\alpha, \mathbb{P}_\alpha\}$ , each of which consists of a large crowd of agents,  $\{\mathbf{x}_i^\alpha, \mathbf{v}_i^\alpha\}$  in (1.5). A crowd of species (or agents) is viewed as a *swarm* when it is driven by collective dynamics which coordinates its species (or agents) to aggregate together with emergence of large-scale formations. In the present context of dynamics governed by alignment (1.1), we are concerned with the long-time *flocking*

behavior of the multi-species system (1.1). Flocking refers to the emergence of coherent structure with limiting velocities  $\mathbf{u}_\alpha^\infty$  such that

$$\mathbf{u}_\alpha(t, \mathbf{x}) - \mathbf{u}_\alpha^\infty(t, \mathbf{x}) \xrightarrow{t \rightarrow \infty} 0,$$

with the corresponding limiting densities,  $\rho_\alpha^\infty = \rho_\infty(\mathbf{x} - \mathbf{u}_\alpha^\infty t)$ . Since we ignore attraction, repulsion or external forcing, the limiting behavior of pure alignment should be particularly simple — the different species governed by (1.1a) can only approach the same time-invariant mean velocity

$$\mathbf{u}_\alpha(t, \cdot) \xrightarrow{t \rightarrow \infty} \bar{\mathbf{u}}, \quad \bar{\mathbf{u}} := \frac{\sum_\alpha \mathbf{m}_\alpha}{\sum_\alpha M_\alpha},$$

with a limiting density carried out as a traveling wave  $\rho_\alpha^\infty(\mathbf{x} - \bar{\mathbf{u}}t)$ . Ideally, we seek uniform convergence. In the present context multi-species with pressure, we have no access to uniform bounds on the velocities. Instead a more relaxed notion of  $L^2_\rho$ -convergence becomes accessible by studying *energy fluctuations*,

$$\delta \mathcal{E}(t) := \sum_\alpha \int_{\mathcal{S}_\alpha} \left\{ \frac{1}{2} |\mathbf{u}_\alpha(t, \mathbf{x}) - \bar{\mathbf{u}}|^2 + e_\alpha(t, \mathbf{x}) \right\} \rho_\alpha(t, \mathbf{x}) \, d\mathbf{x}. \tag{2.1}$$

Observing that

$$\delta \mathcal{E}(t) = \sum_\alpha \int_{\mathcal{S}_\alpha} \rho_\alpha E_\alpha \, d\mathbf{x} - \langle \bar{\mathbf{u}}, \sum_\alpha \mathbf{m}_\alpha \rangle + \frac{1}{2} |\bar{\mathbf{u}}|^2 M = \sum_\alpha \int_{\mathcal{S}_\alpha} \rho_\alpha E_\alpha \, d\mathbf{x} - \frac{1}{2} |\bar{\mathbf{u}}|^2 M, \tag{2.2}$$

we conclude that energy fluctuations decay at the same rate as the total energy in (1.10)

$$\frac{d}{dt} \delta \mathcal{E}(t) \leq -\frac{1}{2} \sum_{\alpha, \beta} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}'_\beta - \mathbf{u}_\alpha|^2 + 2e_\alpha + 2e'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}'. \tag{2.3}$$

Our flocking results will be quantified in terms of the decay of energy/energy fluctuations, which in turn implies the decay of both — the macroscopic velocity fluctuations around the *mean velocity*  $\bar{\mathbf{u}}$  and the microscopic (kinetic) fluctuations of the different species around their bulk velocities,  $\rho_\alpha e_\alpha = \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} \frac{1}{2} |\mathbf{v} - \mathbf{u}_\alpha|^2 f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}$ . A second component of flocking behavior requires that alignment is strong enough to keep the dynamics contained in a finite ball, forming the ‘flock’

$$D(t) := \sum_\alpha D_\alpha(t) \leq D_+ < \infty, \quad D_\alpha(t) := \max_{\mathbf{x}, \mathbf{x}' \in \mathcal{S}_\alpha(t)} |\mathbf{x} - \mathbf{x}'|.$$

In practice we may need to address a relaxed notion of flocking which allows a slow time growth,  $D(t) \leq C_D(1 + t)^\gamma$  with some fixed  $\gamma > 0$ .

2.1. *Statement of main results.* The multi-species alignment dynamics (1.1a) is dictated by the array of communication kernels  $\Phi = \{\phi_{\alpha\beta}\}$ . Our flocking results require  $\Phi$  to form a connected array. To this end, it will suffice to consider the radial lower bounds  $\phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \geq k_{\alpha\beta}(|\mathbf{x} - \mathbf{x}'|)$  assumed in (1.1d). The array  $K = \{k_{\alpha\beta}\}$  is viewed

as the adjacency matrix of a *weighted graph*, with a weighted graph Laplacian,  $\Delta_{\#}K(r)$  [HT2021],

$$(\Delta_{\#}K(r))_{\alpha\beta} := \begin{cases} -k_{\alpha\beta}(r)\sqrt{M_{\alpha}M_{\beta}}, & \alpha \neq \beta, \\ \sum_{\gamma \neq \alpha} k_{\alpha\gamma}(r)M_{\gamma}, & \alpha = \beta. \end{cases}$$

Algebraic connectivity is quantified in terms of the *spectral gap*,  $\lambda_2(\Delta_{\#}K)$  [Fie1973, Fie1989],

$$\lambda_2(\Delta_{\#}K) := M \min_{\mathbf{y}} \left\{ \sum_{\alpha} \sum_{\beta \neq \alpha} k_{\alpha\beta} |y_{\alpha} - y_{\beta}|^2 M_{\alpha} M_{\beta} \mid \sum_{\alpha} \sum_{\beta \neq \alpha} |y_{\alpha} - y_{\beta}|^2 M_{\alpha} M_{\beta} = 1 \right\}. \tag{2.4}$$

The graph associated with  $K$  is connected if and only if  $\lambda_2(\Delta_{\#}K) > 0$ . Since the spectral gap is a non-decreasing function of the non-negative entries [HT2021, §3],  $\lambda_2(\Delta_{\#}K) > 0$  also controls the connectivity of the communication kernels,  $\lambda_2(\Delta_{\#}\Phi) \geq \lambda_2(\Delta_{\#}K) > 0$ . Our flocking results require *heavy-tailed connectivity* in the sense that  $\lambda_2(\Delta_{\#}K(r))$  has slow enough decay in a manner made precise in theorem 2.1.

NOTATIONS. Below, we use  $C_K$  and  $C_D$  to denote constants which characterize the heavy-tailed behavior of  $K$  and the dispersion of diameter  $D(t)$ . We let  $C_R$  denote a constant, with different values in different contexts, depending on a spatial scale  $R$ , as well as on the other fixed parameters on the problem  $\eta, \gamma, \dots$ . Finally, we let  $C_1, C_2, \dots$  denote related parameters which arise from computations with these constants.

THEOREM 2.1. Consider the multi-species system (1.1) with two or more species. Let  $(\rho_{\alpha}, \mathbf{u}_{\alpha}, \mathbb{P}_{\alpha})$  be a non-vacuous strong solution<sup>2</sup> of (1.1), subject to compactly supported initial data  $(\rho_{\alpha 0}, \mathbf{u}_{\alpha 0}, \mathbb{P}_{\alpha 0})$  with  $D(0) < \infty$ , and boundary conditions (H3). Assume that  $K(r)$  has heavy-tailed connectivity of order  $\eta \geq 0$ , namely, there exist  $C_K, R > 0$  such that

$$\lambda_2(\Delta_{\#}K(r)) \geq C_K(1+r)^{-\eta}, \quad r \geq R. \tag{2.5}$$

Moreover, assume that the crowd disperses at the rate of order  $\gamma \geq 0$ , namely, there exists  $C_D > 0$  such that for all  $t \geq 0$ ,

$$D(t) \leq C_D(1+t)^{\gamma}, \quad \gamma \geq 0. \tag{2.6}$$

If the heavy-tail condition holds in the sense that

$$\eta\gamma < 1, \tag{2.7}$$

then there is a large time flocking behavior with fractional exponential decay rate

$$\delta\mathcal{E}(t) \leq C_R \exp\{-C_{\zeta}t^{(1-\eta\gamma)}\} \delta\mathcal{E}(0), \quad C_{\zeta} := 2\zeta C_K C_D^{-\eta}, \quad \zeta := 1 - \frac{\max_{\alpha} M_{\alpha}}{M} > 0. \tag{2.8}$$

Theorem 2.1 extends the mono-kinetic, pressure-less case [HT2021, Theorem 4.1]. It applies to general entropic pressure laws (1.6), and general symmetric communication protocol satisfying (2.5): the kernels  $\phi_{\alpha\beta}$  need not be metric nor upper-bounded. At the

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<sup>2</sup>That is, (H1) holds for  $(\rho_{\alpha}(t, \cdot), \mathbf{u}_{\alpha}(t, \cdot), \mathbb{P}_{\alpha}(t, \cdot)) \in (L^{\infty} \cap L^1_+(\mathbb{R}^d)) \times W^{1,\infty}(\mathbb{R}^d) \times W^{1,\infty}(\mathbb{R}^d)$ ,  $\alpha \in \mathcal{I}$ .

same time, it extends the heavy-tail condition for flocking of a single species asserted in [Tad2022, corollary 4.2]. The decay estimate (2.8) reflects a competition between the possible dispersion of the crowd as its diameter  $D(t)$  may grow in time and the decay rate in the strength of communication strength,  $\lambda_2(\Delta_{\mathcal{M}}K(r))$ , as the ‘edge of the crowd’ may grow with  $r$ . Theorem 2.1 tells us that if their composition has a non-integrable tail so that

$$\int_0^t \lambda_2(\Delta_{\mathcal{M}}K(D(\tau))d\tau \leq C_R e^{-C_\zeta 1t^{(1-\eta\gamma)}} \xrightarrow{t \rightarrow \infty} 0, \quad \eta\gamma < 1,$$

then the different species flock towards the mean velocity  $\bar{\mathbf{u}}_\infty$

$$\sum_{\alpha \in \mathcal{I}} \int |\mathbf{u}_\alpha(t, \mathbf{x}) - \bar{\mathbf{u}}_\infty|^2 \rho_\alpha(t, \mathbf{x}) d\mathbf{x} \lesssim e^{-C_\zeta 1t^{(1-\eta\gamma)}} \delta \mathcal{E}(0);$$

moreover, there is a (fractional) exponential decay of internal fluctuations,

$$\sum_\alpha \int \|\mathbb{P}_\alpha(t, \mathbf{x})\|^2 d\mathbf{x} = \sum_{\alpha \in \mathcal{I}} \int |\mathbf{v} - \mathbf{u}_\alpha(t, \mathbf{x})|^2 f_\alpha(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} d\mathbf{x} \lesssim e^{-C_\zeta t^{(1-\eta\gamma)}} \delta \mathcal{E}(0).$$

2.2. *The example of “pressure-less” equations.* A key aspect of theorem 2.1 is a dispersion bound which controls the spatial diameter,  $D(t) \lesssim (1+t)^\gamma$ . As a prototypical example we consider the mono-kinetic “pressure-less” closure,  $\mathbb{P}_\alpha = 0$  [HT2021]. In this case, the alignment dynamics (1.1a)<sub>2</sub> decouples into scalar transport equations for the components of  $\mathbf{u}_\alpha = (u_\alpha^1, \dots, u_\alpha^d)$ ,

$$\partial_t u_\alpha^i + \mathbf{u}_\alpha \cdot \nabla_{\mathbf{x}} u_\alpha^i = \sum_\beta \int_{S_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (u_\beta^i(t, \mathbf{x}') - u_\alpha^i(t, \mathbf{x})) \rho'_\beta d\mathbf{x}'.$$

Assume  $\eta < 1$ , then a maximum principle of the scalar velocity components eventually leads to the uniform bound  $D(t) \leq D_+ < \infty$ , i.e., theorem 2.1 applies with  $\gamma = 0$ , leading to exponential decay  $\delta \mathcal{E}(t) \leq C_R e^{-C_\zeta t} \delta \mathcal{E}(0)$ . In fact, there is exponential decay of velocity fluctuations in the *uniform* norm [HT2021, step #3 in the proof of theorem 1.1]

$$\max_\alpha \max_{\mathbf{x} \in S_\alpha(t)} |\mathbf{u}_\alpha(t, \mathbf{x}) - \bar{\mathbf{u}}| \leq C_R e^{-C_\zeta t} \max_\alpha \max_{\mathbf{x} \in S_\alpha(t)} |\mathbf{u}_{\alpha 0}(\mathbf{x}) - \bar{\mathbf{u}}|.$$

We conclude that flocking of pressure-less dynamics is dictated for any heavy-tailed connectivity of order  $\eta < 1$ , (2.5)

$$\lambda_2(\Delta_{\mathcal{M}}K(r)) \geq C_K(1+r)^{-\eta}, \quad \eta < 1, \quad r \geq R.$$

**3. Self-interactions based on fractional  $p$ -alignment.** We now turn our attention to the main aspect of this work — multi-species alignment with pressure. In this case, one does not have access to pointwise bounds on the velocities  $\mathbf{u}_\alpha$ , which in turn imply the desired pointwise bound on the diameters,  $D_\alpha(t)$ , propagating with these velocities. Instead, we follow the single-species arguments of [Tad2022, §6], in order to secure direct bounds on the dispersion of  $D_\alpha(t)$ . To this end, observe that the heavy-tailed flocking scenario in theorem 2.1 is quantified in terms of the spectral gap (2.4) which is *independent of* self-interactions,  $\{\phi_{\alpha\alpha}\}$ . The desired dispersion bound will be obtained when we consider enhanced self-interactions; specifically, we consider self-interactions



based on *singular communication kernels*,  $\phi_{\alpha\alpha}(\mathbf{x}, \mathbf{x}') = |\mathbf{x} - \mathbf{x}'|^{d+2sp}$ ,  $0 < s < 1$ ,  $p \geq 1$ . Such kernels greatly emphasize the alignment with immediate neighbors over far away neighbors, leading to

$$\left\{ \begin{aligned} & \partial_t(\rho_\alpha \mathbf{u}_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + \mathbb{P}_\alpha) \\ & = \int_{S_\alpha} \frac{|\mathbf{u}'_\alpha - \mathbf{u}_\alpha|^{2p-2} (\mathbf{u}'_\alpha - \mathbf{u}_\alpha)}{|\mathbf{x}' - \mathbf{x}|^{d+2sp}} \rho_\alpha \rho'_\alpha \, d\mathbf{x}' \\ & \quad + \sum_{\beta \neq \alpha} \int_{S_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (\mathbf{u}'_\beta - \mathbf{u}_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \end{aligned} \right. \tag{3.1}$$

Self-interactions in this case amount to *weighted fractional 2p-Laplacians*; more precisely, the first integrand on the right of (3.1) is the subdifferential of the weighted Gagliardo fractional energy (suppressing the time dependence) [DPV2012]

$$\mathcal{J}_{2p,s}(\mathbf{u}_\alpha) = \iint \frac{|\mathbf{u}_\alpha(\mathbf{x}') - \mathbf{u}_\alpha(\mathbf{x})|^{2p}}{|\mathbf{x}' - \mathbf{x}|^{d+2sp}} \rho_\alpha \rho'_\alpha \, d\mathbf{x} \, d\mathbf{x}', \quad 0 < s < 1, \, p \geq 1.$$

Interactions based on  $p$ -alignment,  $p > 1$  in the context of a single species, were introduced in [HKK2014, CCH2014] and further developed in [Tad2022]. We note that (3.1) corresponds to the multi-species agent-based description with self-interactions based on  $p$ -alignment

$$\begin{aligned} \frac{d}{dt} \mathbf{v}_i^\alpha &= \frac{1}{N_\alpha} \sum_{j=1}^{N_\alpha} \phi_{\alpha\alpha}(\mathbf{x}_j^\alpha, \mathbf{x}_i^\alpha) |\mathbf{v}_j^\alpha - \mathbf{v}_i^\alpha|^{2p-2} (\mathbf{v}_j^\alpha - \mathbf{v}_i^\alpha) \\ & \quad + \sum_{\beta \neq \alpha} \frac{1}{N_\beta} \sum_{j=1}^{N_\beta} \phi_{\alpha\beta}(\mathbf{x}_j^\beta, \mathbf{x}_i^\alpha) (\mathbf{v}_j^\beta - \mathbf{v}_i^\alpha), \end{aligned} \tag{3.2}$$

with singular kernels  $\phi_{\alpha\alpha}(\mathbf{x}, \mathbf{x}') = |\mathbf{x} - \mathbf{x}'|^{-(d+2sp)}$ . The passage from (3.2) to (3.1) can be justified only in the case of bounded (or at least integrable)  $\phi_{\alpha\alpha}$  and remains formal in the singular case. We close this section noting that since the flocking bound in (2.8) is independent of self-interactions, the main theorem 2.1 still applies to the case of self-interactions based on fractional  $p$ -alignment in (3.1).

3.1. *Energy dissipation in entropic p-alignment.* The notion of ‘entropic pressure’ in (1.4) requires an adjustment for  $p$ -alignment. Following [Tad2022, remark 6.1], we refer to  $\mathbb{P}_\alpha$  as an *entropic pressure* tensor associated with species  $\alpha$  in (3.1) with  $C^1$  ‘heat-flux’  $\mathbf{q}_\alpha$  if its non-negative trace  $\rho_\alpha e_\alpha := \frac{1}{2} \text{trace}(\mathbb{P}_\alpha) \geq 0$  satisfies

$$\begin{aligned} & \partial_t(\rho_\alpha e_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha e_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha) + \text{trace}(\mathbb{P}_\alpha \nabla \mathbf{u}_\alpha) \\ & \leq -\frac{1}{2} D_\alpha^{-(d+2sp)}(t) \int_{S_\alpha} ((2e_\alpha)^p + (2e'_\alpha)^p) \rho_\alpha \rho'_\alpha \, d\mathbf{x}' \\ & \quad - 2 \sum_{\beta \neq \alpha} \int_{S_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') e_\alpha \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \end{aligned} \tag{3.3}$$

The self-interaction terms in (3.3) and (1.4) for ‘pure’ alignment  $p = 1$  end with the same energy dissipation statement. Specifically, manipulating (1.1a)<sub>1</sub> and (3.1) yields, corresponding to (1.8),

$$\begin{aligned} \partial_t \left( \frac{\rho_\alpha}{2} |\mathbf{u}_\alpha|^2 \right) + \nabla_{\mathbf{x}} \cdot \left( \frac{\rho_\alpha}{2} |\mathbf{u}_\alpha|^2 \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha \right) - \text{trace}(\mathbb{P}_\alpha \nabla \mathbf{u}_\alpha) \\ = - \int_{\mathcal{S}_\alpha} \frac{|\mathbf{u}'_\alpha - \mathbf{u}_\alpha|^{2p-2} \mathbf{u}_\alpha \cdot (\mathbf{u}_\alpha - \mathbf{u}'_\alpha)}{|\mathbf{x}' - \mathbf{x}|^{d+2ps}} \, d\mathbf{x} \\ - \sum_{\beta \neq \alpha} \int_{\mathcal{S}_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \end{aligned}$$

Adding the entropic description of the pressure postulated in (3.3) we find, arguing along the lines of (2.3),

$$\begin{aligned} \frac{d}{dt} \delta \mathcal{E}(t) \leq -\frac{1}{2} \sum_{\alpha} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\alpha(t)} \left( \frac{|\mathbf{u}'_\alpha - \mathbf{u}_\alpha|^{2p}}{|\mathbf{x}' - \mathbf{x}|^{d+2sp}} + D_\alpha^{d+2sp}(t) ((2e_\alpha)^p + (2e'_\alpha)^p) \right) \rho_\alpha \rho'_\alpha \, d\mathbf{x} \, d\mathbf{x}' \\ - \frac{1}{2} \sum_{\beta \neq \alpha} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}'_\beta - \mathbf{u}_\alpha|^2 + 2e_\alpha + 2e'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}'. \end{aligned}$$

In particular, ignoring the negative contributions coming from internal energy and from cross interactions terms yields

$$\frac{d}{dt} \delta \mathcal{E}(t) \leq -\frac{1}{2} \sum_{\alpha} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\alpha(t)} \frac{|\mathbf{u}'_\alpha - \mathbf{u}_\alpha|^{2p}}{|\mathbf{x}' - \mathbf{x}|^{d+2sp}} \rho_\alpha \rho'_\alpha \, d\mathbf{x} \, d\mathbf{x}'. \tag{3.4}$$

Thus, the contribution coming from self-interactions based on singular  $p$ -alignment implies that the velocities  $\mathbf{u}_\alpha$  are bounded in the (homogeneous) Sobolev spaces  $\dot{W}^{s,2p}(\mathcal{S}_\alpha)$ . Specifically, taking into account the non-vacuous bound assumed in (H1) then integration of (3.4) yields

$$\begin{aligned} \int_0^t \sum_{\alpha} \|\mathbf{u}_\alpha(\tau, \mathbf{x})\|_{\dot{W}^{s,2p}(\mathcal{S}_\alpha)}^{2p} \, d\tau \\ \leq C_\rho^2 \int_0^t \sum_{\alpha} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\alpha(t)} \frac{|\mathbf{u}_\alpha(t, \mathbf{x}') - u_\alpha(t, \mathbf{x})|^{2p}}{|\mathbf{x}' - \mathbf{x}|^{d+2sp}} \rho_\alpha \rho'_\alpha \, d\mathbf{x} \, d\mathbf{x}' \, d\tau \\ \leq C_\rho^2 C_0^2, \quad C_0^2 := 2 \sum_{\alpha} \int_{\mathcal{S}_\alpha(0)} \rho_{\alpha 0} E_{\alpha 0} \, d\mathbf{x}, \quad C_\rho := \frac{1}{\rho_-}. \end{aligned} \tag{3.5}$$

**3.2. Multi-species with entropic pressure and fractional  $p$ -alignment.** The enstrophy bound (3.5) implies a dispersion bound sought in (2.6). We follow the argument in [Tad2022]. The mass propagation by (1.1a)<sub>1</sub> implies

$$\frac{d}{dt} D_\alpha(t) \leq \delta \mathbf{u}_\alpha(t), \quad \delta \mathbf{u}_\alpha(t) := \max_{\mathbf{x}, \mathbf{x}' \in \mathcal{S}_\alpha(t)} |\mathbf{u}(t, \mathbf{x}) - \mathbf{u}(t, \mathbf{x}')|.$$

Gagliardo-Nirenberg inequality implies for all  $d/2p < s < 1$  there holds<sup>3</sup> [DPV2012, MRR2013]

$$|\mathbf{u}(t, \mathbf{x}) - \mathbf{u}(t, \mathbf{x}')| \leq C_s \|\mathbf{u}_\alpha(t, \cdot)\|_{\dot{W}^{s, 2p}(\mathcal{S}_\alpha)} |\mathbf{x} - \mathbf{x}'|^{s-\theta}, \quad \theta := \frac{d}{2p} < s < 1,$$

and hence  $\frac{d}{dt} D_\alpha(t) \leq C_s \|\mathbf{u}_\alpha(t, \cdot)\|_{\dot{W}^{s, 2p}(\mathcal{S}_\alpha)} D_\alpha^{s-\theta}(t)$ . It follows that

$$\frac{d}{dt} \sum_\alpha D_\alpha^{1+\theta-s}(t) \leq C'_s \sum_\alpha \|\mathbf{u}_\alpha(t, \cdot)\|_{\dot{W}^{s, 2p}(\mathcal{S}_\alpha)}, \quad C'_s = (1 + \theta - s)C_s. \quad (3.6)$$

Now, since  $1 + \theta - s < 1$ , then  $D_\alpha^{1+\theta-s} \leq \sum_\alpha D_\alpha^{1+\theta-s}$ , and integration of (3.6) yields

$$\begin{aligned} D^{1+\theta-s}(t) &\leq \sum_\alpha D_{\alpha 0}^{1+\theta-s} + \left( \int_0^t \sum_\alpha \|\mathbf{u}_\alpha(\tau, \cdot)\|_{\dot{W}^{s, 2p}(\mathcal{S}_\alpha)}^{2p} d\tau \right)^{\frac{1}{2p}} \left( \int_0^t 1 d\tau \right)^{\frac{1}{(2p)'}} \\ &\leq \sum_\alpha D_{\alpha 0}^{1+\theta-s} + C'_s (C_\rho C_0)^{\frac{1}{p}} t^{\frac{1}{(2p)'}}. \end{aligned}$$

We conclude that multi-species crowd driven by self-interaction of  $p$ -alignment dynamics (3.1) can be dispersed at a rate no faster than

$$D(t) \leq C_D (1+t)^{\gamma_p}, \quad \gamma_p = \frac{2p-1}{2p(1+\theta-s)}, \quad \theta = \frac{d}{2p} < s < 1. \quad (3.7)$$

This bound can be improved: in appendix C we use a bootstrap argument to show a slower rate of order

$$D(t) \leq C'_D (1+t)^{\gamma_*}, \quad \gamma_* = \frac{2p-1}{2p(1+\theta-s)+\eta}, \quad \theta = \frac{d}{2p} < s < 1.$$

Theorem 2.1 applies, leading to flocking behavior of order  $\lesssim \exp\{-t^{1-\eta\gamma_*}\}$  which we summarize in the following.

**THEOREM 3.1.** Let  $(\rho_\alpha, \mathbf{u}_\alpha, \mathbb{P}_\alpha)$  be a non-vacuous strong solution of (1.1a)<sub>1</sub>, (3.1) satisfying (H1)–(H3), with cross interactions,  $\Phi(\mathbf{x}, \mathbf{x}') \succeq K(|\mathbf{x} - \mathbf{x}'|)$ , and self-interactions based on  $p$ -alignment of order  $p > \frac{d}{2}$ . Assume that  $K(r)$  has tail connectivity of order  $\eta \geq 0$ , (2.5)

$$\lambda_2(\Delta_{\#} K(r)) \geq C_K (1+r)^{-\eta}, \quad r \geq R.$$

If the heavy-tail condition (2.7) holds,

$$\eta\gamma_p < 1, \quad \gamma_p := \frac{2p-1}{2p(1+\theta-s)}, \quad \theta = \frac{d}{2p} < s < 1,$$

then there is a large time flocking behavior with fractional exponential decay rate

$$\delta \mathcal{E}(t) \leq C_R \exp\{-C_\zeta t^\mu\} \delta \mathcal{E}(0), \quad \mu = \frac{2p(1+\theta-s) - 2(p-1)\eta}{2p(1+\theta-s) + \eta} > 0, \quad (3.8)$$

with constant  $C_\zeta = 2\zeta C_K (C'_D)^{-\eta}$ .

<sup>3</sup>It is here that we use the assumed smoothness of the boundaries of  $\mathcal{S}_\alpha$  in (H2).

**REMARK 3.2 (Lack of exponential decay bound).** We leave open the question of a uniform dispersion bound,  $D(t) \leq D_+ < \infty$  corresponding to  $\gamma_* = 0$ , which in turn would imply the exponential decay  $\delta \mathcal{E}(t) \leq C_R e^{-C_\zeta t}$ . This will require an improved bootstrap argument in appendix C, along the lines of [Tad2022, Appendix E].

**3.3. Multi-species with entropic pressure in one-dimension.** The methodology leading to theorem 3.1 consists of two main parts: (i) an  $\eta$ -tailed array of cross interactions which enforces flocking of multi-species dynamics; and (ii) self-interactions based on  $p$ -alignment with singular head which dictate the dispersion rate  $\gamma_p$ . Observe that this line of argument requires  $d/2p < 1$ , and therefore flocking of self-interactions based on ‘pure’ alignment,  $p = 1$ , is restricted to the  $d = 1$ -case,

$$\left\{ \begin{aligned} & \partial_t(\rho_\alpha u_\alpha) + \partial_x(\rho_\alpha u_\alpha^2 + \mathbb{P}_\alpha) \\ & = \int_{\mathcal{S}_\alpha} \frac{(u'_\alpha - u_\alpha)}{|x' - x|^{1+2s}} \rho_\alpha \rho'_\alpha dx' + \sum_{\beta \neq \alpha} \int_{\mathcal{S}_\beta} \phi_{\alpha\beta}(x, x')(u'_\beta - u_\alpha) \rho_\alpha \rho'_\beta dx', \end{aligned} \right. \quad (t, x) \in (\mathbb{R}_t, \mathbb{R}) \tag{3.9}$$

with scalar entropic pressures,  $\mathbb{P}_\alpha$ , satisfying (assuming no heat flux  $q_\alpha = 0$ )

$$\partial_t \mathbb{P}_\alpha + \partial_x(\mathbb{P}_\alpha u) + 2\mathbb{P}_\alpha \partial_x u \leq -2\mathbb{P}_\alpha D_\alpha^{1+2s}(t)M. \tag{3.10}$$

Theorem 3.1 applies with  $\gamma_1 = \frac{1}{3 - 2s}$ .

**COROLLARY 3.3 (Multi-species in one-dimension).** Consider the one-dimensional multi-species system (1.1a)<sub>1</sub>, (3.9) with entropic pressure (3.10) and satisfying (H1), (H3). If the heavy-tail connectivity condition holds

$$\eta + 2s < 3, \quad \frac{1}{2} < s < 1,$$

then there is a large time flocking behavior with fractional exponential rate

$$\delta \mathcal{E}(t) \leq \exp\left\{-2C_\zeta t^\mu\right\} \delta \mathcal{E}(0), \quad \mu = \frac{3 - 2s}{3 - 2s + \eta} > 0. \tag{3.11}$$

Singular interactions of a single species in one dimension with mono-kinetic closure were extensively studied in [ST2017a, ST2017b, DKRT2018, ST2018a, ST2020b] and we refer to the review [MMPZ2019] and the additional references therein. Corollary 3.3 extends these flocking results to multi-species in one dimension with entropic pressure laws. Going beyond the one-dimensional corollary 3.3 clarifies the motivation for our discussion of self-interactions based on fractional  $p$ -alignment,  $p > 1$ , which extend the discussion to higher dimensions.

**Appendix A. From agent-based to hydrodynamic description.** We begin with the derivation of the multi-species hydrodynamic description (1.1a) from the agent-based dynamics (1.5).

The large crowd dynamics of the different species can be encoded in terms of their empirical distribution  $f_\alpha(t, \mathbf{x}, \mathbf{v}) := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} \delta_{\mathbf{x}_i^\alpha(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_i^\alpha(t)}(\mathbf{v})$ , which are governed by

the kinetic Valsov equation in state variables  $(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$ , e.g., [HT2021],

$$\partial_t f_\alpha + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\alpha + \nabla_{\mathbf{v}} \cdot Q_\alpha(f_\alpha, \mathcal{F}) = 0, \quad \mathcal{F} = \{f_\beta\}, \tag{A.1}$$

where different species are interconnected through pairwise communication protocol on the right (we abbreviate  $f_\alpha = f_\alpha(t, \mathbf{x}, \mathbf{v})$ ,  $f'_\beta = f_\beta(t, \mathbf{x}', \mathbf{v}')$  and likewise  $\square = \square(t, \mathbf{x})$ ,  $\square' = \square(t, \mathbf{x}')$  etc.)

$$Q_\alpha(f_\alpha, \mathcal{F}) := \sum_{\beta} \int_{S_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (\mathbf{v}' - \mathbf{v}) f_\alpha f'_\beta \, d\mathbf{v}' \, d\mathbf{x}'.$$

The large crowd dynamics of  $f_\alpha$ 's is captured by their first two moments which we assume to exist — the density  $\rho_\alpha(t, \mathbf{x}) := \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}$  and the momentum

$$\rho_\alpha \mathbf{u}_\alpha(t, \mathbf{x}) := \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} \mathbf{v} f_\alpha(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}. \tag{1.1a}_1$$

Integration of (A.1) yields the mass equation

$$\partial_t \rho_\alpha + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha) = 0.$$

The first  $\mathbf{v}$ -moment of (A.1) yields

$$\partial_t \int_{\mathbb{R}^d} \mathbf{v} f_\alpha \, d\mathbf{v} = -\nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \mathbf{v} \mathbf{v}^\top f_\alpha \, d\mathbf{v} + \int_{\mathbb{R}^d} Q_\alpha(f_\alpha, \mathcal{F}) \, d\mathbf{v}. \tag{A.2}$$

For the first term on the right of (A.2)  $\mathbf{v} \mathbf{v}^\top \equiv -\mathbf{u}_\alpha \mathbf{u}_\alpha^\top + (\mathbf{v} \mathbf{u}_\alpha^\top + \mathbf{u}_\alpha \mathbf{v}^\top) + (\mathbf{v} - \mathbf{u}_\alpha)(\mathbf{v} - \mathbf{u}_\alpha)^\top$ , where the first two moments of  $f_\alpha$  add up to  $\mathbf{u}_\alpha (\rho \mathbf{u})_\alpha^\top = \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha$ , and the third yields the pressure tensor (1.6),

$$\int_{\mathbb{R}^d} \mathbf{v} \mathbf{v}^\top f_\alpha \, d\mathbf{v} = \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + \mathbb{P}_\alpha, \quad \mathbb{P}_\alpha = \int_{\mathbb{R}^d} (\mathbf{v} - \mathbf{u}_\alpha)(\mathbf{v} - \mathbf{u}_\alpha)^\top f_\alpha \, d\mathbf{v};$$

the second term on the right of (A.2) yields

$$\int_{\mathbb{R}^d} Q_\alpha(f_\alpha, \mathcal{F}) \, d\mathbf{v} = \sum_{\beta} \int_{S_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') ((\rho \mathbf{u})'_\beta \rho_\alpha - (\rho \mathbf{u})_\alpha \rho'_\beta) \, d\mathbf{x}' = \mathbf{A}_\alpha(\rho, \mathbf{u}),$$

and we recover the momentum equation (1.1a)<sub>2</sub>

$$\partial_t (\rho_\alpha \mathbf{u}_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha + \mathbb{P}_\alpha) = \mathbf{A}_\alpha(\rho, \mathbf{u}).$$

Observe that the system (1.1) is not a purely hydrodynamic description since the pressure in (1.6) still requires a *closure* of the  $\mathbf{v}$ -dependent second-order moments of  $f_N$ . This is our point of departure from the flocking analysis in [HT2021]: the hydrodynamic description of alignment in (1.1) is left open. Following [Tad2022], we will trace the decay of energy fluctuations, showing that it applies to general entropic pressure stress tensors (1.6).

A.1. *Energy balance.* We derive the energy balance as preparation for studying the long-time behavior of hydrodynamics (1.1). The total energy is given by the second moment which is assumed to exist

$$\rho_\alpha E_\alpha(t, \mathbf{x}) = \lim_{N_\alpha \rightarrow \infty} \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} f_\alpha(\mathbf{x}, \mathbf{v}, t) \, d\mathbf{v},$$

and is decomposed into kinetic and internal energy corresponding to the decomposition  $\frac{1}{2}|\mathbf{v}|^2 = \frac{1}{2}|\mathbf{u}_\alpha|^2 + \frac{1}{2}|\mathbf{v} - \mathbf{u}_\alpha|^2 + \mathbf{u}_\alpha \cdot (\mathbf{v} - \mathbf{u}_\alpha)$ . Noticing that  $\int_{\mathbb{R}^d} (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha \, d\mathbf{v} = 0$ ,

$$\rho_\alpha E_\alpha = \frac{\rho_\alpha}{2} |\mathbf{u}_\alpha|^2 + \rho_\alpha e_\alpha, \quad \rho_\alpha e_\alpha := \frac{1}{2} \int_{\mathbb{R}^d} |\mathbf{v} - \mathbf{u}_\alpha|^2 f_\alpha \, d\mathbf{v}.$$

The balance of energy can be obtained by integrating (A.1) against  $\frac{|\mathbf{v}|^2}{2}$ , obtaining

$$\partial_t(\rho_\alpha E_\alpha) + I_\alpha = II_\alpha.$$

Here  $I_\alpha$  is the transport-based term which we express as

$$\begin{aligned} I_\alpha &= \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} (\mathbf{v} \cdot \nabla_{\mathbf{x}} f_\alpha) \, d\mathbf{v} \\ &\equiv \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} \mathbf{u}_\alpha f_\alpha \, d\mathbf{v} + \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha \, d\mathbf{v} \\ &\equiv \nabla_{\mathbf{x}} \cdot \left( \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} f_\alpha \, d\mathbf{v} \right) \mathbf{u}_\alpha \\ &\quad + \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \left[ \frac{|\mathbf{u}_\alpha|^2}{2} + (\mathbf{v} - \mathbf{u}_\alpha) \cdot \mathbf{u}_\alpha + \frac{|\mathbf{v} - \mathbf{u}_\alpha|^2}{2} \right] (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha \, d\mathbf{v} \\ &= \nabla_{\mathbf{x}} \cdot (\rho_\alpha E_\alpha \mathbf{u}_\alpha) + \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} (\mathbf{v} - \mathbf{u}_\alpha) (\mathbf{v} - \mathbf{u}_\alpha)^\top \mathbf{u}_\alpha f_\alpha \, d\mathbf{v} \\ &\quad + \nabla_{\mathbf{x}} \cdot \int_{\mathbb{R}^d} \frac{|\mathbf{v} - \mathbf{u}_\alpha|^2}{2} (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha \, d\mathbf{v} \\ &= \nabla_{\mathbf{x}} \cdot (\rho_\alpha E_\alpha \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha), \end{aligned}$$

involving the pressure tensor  $\mathbb{P}_\alpha$ , (1.6), and a heat-flux vector,  $\mathbf{q}_\alpha$ ,

$$\mathbf{q}_\alpha := \frac{1}{2} \int_{\mathbb{R}^d} |\mathbf{v} - \mathbf{u}_\alpha|^2 (\mathbf{v} - \mathbf{u}_\alpha) f_\alpha \, d\mathbf{v}, \tag{A.3}$$

and  $II_\alpha$  is the alignment-based enstrophy term given by

$$\begin{aligned}
 II_\alpha &= \int_{\mathbb{R}^d} \mathbf{v} \cdot Q_\alpha(f_\alpha, \mathcal{F}) \, d\mathbf{v} = - \sum_{\beta \in \mathcal{I}} \int_{\mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{v} \cdot (\mathbf{v} - \mathbf{v}') f_\alpha f'_\beta \, d\mathbf{v} \, d\mathbf{v}' \right) \, d\mathbf{x}' \\
 &= - \sum_{\beta \in \mathcal{I}} \int_{\mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (2E_\alpha - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x}'.
 \end{aligned}$$

Combining, we formally end up with the energy balance

$$\begin{aligned}
 \partial_t(\rho_\alpha E_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha E_\alpha \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha) \\
 = - \sum_{\beta \in \mathcal{I}} \int_{\mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta + e_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \tag{A.4}
 \end{aligned}$$

Thus, the energy equality which arises from a kinetic description is viewed here as a special case of the inequality (1.10) associated with the general class of entropic pressures,

$$\begin{aligned}
 \partial_t(\rho_\alpha E_\alpha) + \nabla_{\mathbf{x}} \cdot (\rho_\alpha E_\alpha \mathbf{u}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha + \mathbf{q}_\alpha) \\
 = - \sum_{\beta \in \mathcal{I}} \int_{\mathcal{S}_\beta} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta + e_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x}'. \tag{A.5}
 \end{aligned}$$

A.2. *Energy fluctuations.* Integrating (A.5) and summing over  $\alpha \in \mathcal{I}$  we find

$$\begin{aligned}
 \frac{d}{dt} \sum_{\alpha \in \mathcal{I}} \int_{\mathcal{S}_\alpha(t)} \rho_\alpha E_\alpha \, d\mathbf{x} \\
 \leq - \sum_{\alpha} \int_{\partial \mathcal{S}_\alpha(t)} (\rho_\alpha E_\alpha \mathbf{u}_\alpha \cdot \mathbf{n}_\alpha + \mathbb{P}_\alpha \mathbf{u}_\alpha \cdot \mathbf{n}_\alpha + \mathbf{q}_\alpha \cdot \mathbf{n}_\alpha) \, dS \\
 - \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') (|\mathbf{u}_\alpha|^2 - \mathbf{u}_\alpha \cdot \mathbf{u}'_\beta + e_\alpha) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\
 = - \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \left( \frac{1}{2} |\mathbf{u}_\alpha - \mathbf{u}'_\beta|^2 + e_\alpha + e'_\beta \right) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}'. \tag{A.6}
 \end{aligned}$$

The boundary conditions assumed in (H3) imply there is no energy flux and hence the boundary integrals on the right vanish,<sup>4</sup> while the symmetrization assumed in (1.1c) implies, upon change of variables  $(\alpha, \mathbf{x}) \leftrightarrow (\beta, \mathbf{x}')$ , that the second term admits the symmetric form of the integrals on the right.

The inequality (A.6) quantifies the energy dissipation in terms of the negative total enstrophy on the right. This is better expressed in an equivalent symmetric form, in terms of the *energy fluctuations*

$$\delta \mathcal{E}(t) = \frac{1}{2M} \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \left\{ \frac{1}{2} |\mathbf{u}_\alpha(t, \mathbf{x}) - \mathbf{u}_\beta(t, \mathbf{x}')|^2 + e_\alpha(t, \mathbf{x}) + e_\beta(t, \mathbf{x}') \right\} \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}'. \tag{A.7}$$

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<sup>4</sup>In fact, here one can consider a larger class of *energy dissipative* boundary condition.

REMARK A.1. Observe that the definition of energy fluctuation in (A.7) coincides with the one we had in (2.1). Indeed, since the total mass in (1.2) and momentum in (1.3) are conserved in time, then the mean velocity is invariant in time,

$$\bar{\mathbf{u}} = \frac{\sum_{\alpha} \mathbf{m}_{\alpha}}{\sum_{\alpha} M_{\alpha}} = \frac{\sum_{\alpha} \mathbf{m}_{\alpha 0}}{\sum_{\alpha} M_{\alpha 0}},$$

and the macroscopic portion of the energy fluctuations (A.7) can be expressed as fluctuations around that mean velocity,

$$\begin{aligned} & \sum_{\alpha, \beta} \iint_{\mathcal{S}_{\alpha}(t) \times \mathcal{S}_{\beta}(t)} \left( \frac{1}{2} |\mathbf{u}_{\alpha}(t, \mathbf{x}) - \mathbf{u}_{\beta}(t, \mathbf{x}')|^2 \right) \rho_{\alpha} \rho'_{\beta} \, d\mathbf{x} \, d\mathbf{x}' \\ & \sum_{\alpha, \beta} \iint_{\mathcal{S}_{\alpha}(t) \times \mathcal{S}_{\beta}(t)} \left( \frac{1}{2} |\mathbf{u}_{\alpha}(t, \mathbf{x}) - \bar{\mathbf{u}}|^2 + (\mathbf{u}_{\alpha} - \bar{\mathbf{u}}) \cdot (\bar{\mathbf{u}} - \mathbf{u}'_{\beta}) + \frac{1}{2} |\mathbf{u}_{\beta}(t, \mathbf{x}') - \bar{\mathbf{u}}|^2 \right) \rho_{\alpha} \rho'_{\beta} \, d\mathbf{x} \, d\mathbf{x}' \\ & = M \sum_{\alpha} \int_{\mathcal{S}_{\alpha}(t)} |\mathbf{u}_{\alpha}(t, \mathbf{x}) - \bar{\mathbf{u}}|^2 \rho_{\alpha}(t, \mathbf{x}) \, d\mathbf{x}. \end{aligned}$$

Hence, the energy fluctuation (A.7) coincides with its equivalent definition (2.1) stated in theorem 2.1

$$\delta \mathcal{E}(t) = \sum_{\alpha} \int_{\mathcal{S}_{\alpha}(t)} |\mathbf{u}_{\alpha}(t, \mathbf{x}) - \bar{\mathbf{u}}|^2 \rho_{\alpha}(t, \mathbf{x}) \, d\mathbf{x}.$$

Noting that

$$\delta \mathcal{E}(t) = \sum_{\alpha} \int \left\{ \frac{1}{2} \rho_{\alpha} |\mathbf{u}_{\alpha}|^2 + \rho_{\alpha} e_{\alpha} \right\} \, d\mathbf{x} - \frac{1}{2M} \left| \sum_{\alpha} \mathbf{m}_{\alpha} \right|^2, \quad \mathbf{m}_{\alpha} := \int \mathbf{u}_{\alpha} \rho_{\alpha} \, d\mathbf{x},$$

with a total mass,  $M := \sum_{\alpha} M_{\alpha}$ , and total momentum,  $\sum_{\alpha} \mathbf{m}_{\alpha}$ , which are conserved in time we end up with the symmetric version of the dissipation statement (A.6), expressed in terms of energy fluctuations,

$$\begin{aligned} & \frac{d}{dt} \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_{\alpha}(t) \times \mathcal{S}_{\beta}(t)} \left\{ \frac{1}{2} |\mathbf{u}_{\alpha} - \mathbf{u}'_{\beta}|^2 + e_{\alpha} + e'_{\beta} \right\} \rho_{\alpha} \rho'_{\beta} \, d\mathbf{x} \, d\mathbf{x}' \\ & = \frac{d}{dt} \sum_{\alpha \in \mathcal{I}} \int_{\mathcal{S}_{\alpha}(t)} \rho_{\alpha} E_{\alpha} \, d\mathbf{x} \tag{A.8} \\ & \leq - \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_{\alpha}(t) \times \mathcal{S}_{\beta}(t)} \phi_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \left\{ \frac{1}{2} |\mathbf{u}_{\alpha} - \mathbf{u}'_{\beta}|^2 + e_{\alpha} + e'_{\beta} \right\} \rho_{\alpha} \rho'_{\beta} \, d\mathbf{x} \, d\mathbf{x}'. \end{aligned}$$



**Appendix B. Flocking of strong solutions — proof of theorem 2.1.** Recall that the metric kernels  $k_{\alpha\beta}(r)$  are assumed to decrease with the distance  $r$  and hence (A.8) implies

$$\begin{aligned} \frac{d}{dt} \delta \mathcal{E}(t) &\leq - \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} k_{\alpha\beta}(|\mathbf{x} - \mathbf{x}'|) \left( \frac{1}{2} |\mathbf{u}_\alpha - \mathbf{u}'_\beta|^2 + e_\alpha + e'_\beta \right) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\ &\leq - \sum_{\alpha, \beta \in \mathcal{I}} k_{\alpha\beta}(D(t)) \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \left( \frac{1}{2} |\mathbf{u}_\alpha - \mathbf{u}'_\beta|^2 + e_\alpha + e'_\beta \right) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \end{aligned}$$

For the first term on the right we use the weighted Poincaré inequality [HT2021, Lemma 3.2]

$$\begin{aligned} &\sum_{\alpha, \beta \in \mathcal{I}} k_{\alpha\beta}(D(t)) \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \frac{1}{2} |\mathbf{u}_\alpha - \mathbf{u}'_\beta|^2 \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\ &\geq \frac{\zeta}{M} \lambda_2(\Delta_{\mathcal{M}} K(D(t))) \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \frac{1}{2} |\mathbf{u}_\alpha - \mathbf{u}'_\beta|^2 \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}', \quad \zeta = 1 - \frac{\max_\alpha M_\alpha}{M}. \end{aligned}$$

For the remaining terms

$$\begin{aligned} &\sum_{\alpha, \beta \in \mathcal{I}} k_{\alpha\beta}(D(t)) \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} (e_\alpha + e'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\ &= \sum_\beta \text{deg}_\beta(K) \sum_\alpha \int_{\mathcal{S}_\alpha(t)} e_\alpha \rho_\alpha \, d\mathbf{x} + \sum_\alpha \text{deg}_\alpha(K) \sum_\beta \int_{\mathcal{S}_\beta(t)} e'_\beta \rho'_\beta \, d\mathbf{x}', \end{aligned}$$

where  $\text{deg}_\alpha$ , the degree of connectivity of species  $\alpha$ , has the lower bound [HT2021, eq. (3.10)]

$$\text{deg}_\alpha(K) := \sum_{\gamma \neq \alpha} k_{\alpha\gamma}(D(t)) M_\gamma \geq \zeta \lambda_2(\Delta_{\mathcal{M}} K(D(t))).$$

We end up with

$$\begin{aligned} &\sum_{\alpha, \beta \in \mathcal{I}} k_{\alpha\beta}(D(t)) \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} (e_\alpha + e'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\ &\geq \frac{\zeta}{M} \lambda_2(\Delta_{\mathcal{M}} K(D(t))) \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} (e_\alpha + e'_\beta) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}'. \end{aligned}$$

Adding the last two inequalities we conclude the dissipation statement of energy fluctuations

$$\begin{aligned} & \frac{d}{dt} \delta \mathcal{E}(t) \\ & \leq -2\zeta \lambda_2(\Delta_{\mathcal{M}} K(D(t))) \times \frac{1}{2M} \sum_{\alpha, \beta \in \mathcal{I}} \iint_{\mathcal{S}_\alpha(t) \times \mathcal{S}_\beta(t)} \left( \frac{1}{2} |\mathbf{u}_\alpha - \mathbf{u}'_\beta|^2 + e_\alpha + e'_\beta \right) \rho_\alpha \rho'_\beta \, d\mathbf{x} \, d\mathbf{x}' \\ & \quad - 2\zeta \lambda_2(\Delta_{\mathcal{M}} K(D(t))) \times \delta \mathcal{E}(t). \end{aligned} \tag{B.1}$$

We now turn to address the main flocking bound in (2.8).

*Proof of theorem 2.1.* Integrating the decay of energy fluctuations (B.1),

$$\frac{d}{dt} \delta \mathcal{E}(t) \leq -2\zeta \lambda_2(\Delta_{\mathcal{M}} K(D(t))) \times \delta \mathcal{E}(t),$$

combined with the assumed bounds (2.5) and (2.6),

$$\lambda_2(\Delta_{\mathcal{M}} K(D(t))) \geq C_K (1 + C_D(1+t)^\gamma)^{-\eta}, \quad r \geq R,$$

implies the desired bound

$$\delta \mathcal{E}(t) \leq C_R \exp\{-2\zeta C_K C_D^{-\eta} t^{1-\eta\gamma}\} \delta \mathcal{E}(0), \quad \eta\gamma < 1,$$

with a constant  $C_R > 0$ . □

**Appendix C. An improved dispersion bound.** Assume that we secured the dispersion bound  $D(t) \leq C_D(1+t)^\gamma$ . Then, theorem 2.1 applies, leading to flocking behavior with fractional exponential decay rate which we rewrite as

$$\delta \mathcal{E}(t) \leq C_R \frac{1}{\chi(t)} \delta \mathcal{E}(0), \quad \chi(t) := \exp\{C_\zeta t^{(1-\eta\gamma)}\}. \tag{C.1}$$

This bound, which was shown to hold with  $\gamma = \gamma_p = \frac{2p-1}{2p(1+\theta-s)}$ , can be improved. To this end, we rewrite (3.4), (C.1) in the form

$$\begin{aligned} \frac{d}{dt} \chi(t) \delta \mathcal{E}(t) & \leq -\frac{\rho^2}{2} \chi(t) \sum_\alpha \|\mathbf{u}_\alpha(t, \cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}_\alpha)}^{2p} + C_R \frac{\dot{\chi}(t)}{\chi(t)} \delta \mathcal{E}(0) \\ & \leq -\frac{\rho^2}{2} \chi(t) \sum_\alpha \|\mathbf{u}_\alpha(t, \cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}_\alpha)}^{2p} + C_2 t^{-\eta\gamma} \delta \mathcal{E}(0), \quad C_2 = 2C_\zeta(1-\eta\gamma)C_R. \end{aligned}$$

This implies

$$\int_0^t \chi(\tau) \sum_\alpha \|\mathbf{u}_\alpha(\tau, \cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}_\alpha)}^{2p} \, d\tau \leq 2C_\rho^2 \delta \mathcal{E}(0) + C_3 t^{1-\eta\gamma} \delta \mathcal{E}(0), \quad C_3 = 2C_\rho^2 C_2 \frac{1}{1-\eta\gamma}.$$

Now we revisit (3.6) with the last weighted bound, obtaining

$$\begin{aligned}
 D^{1+\theta-s}(t) &\leq \sum_{\alpha} D_{\alpha 0}^{1+\theta-s} \\
 &\quad + \left( \int_0^t \sum_{\alpha} \chi(t) \|\mathbf{u}_{\alpha}(\tau, \cdot)\|_{\dot{W}^{s,2p}(\mathcal{S}_{\alpha})}^{2p} d\tau \right)^{\frac{1}{2p}} \left( \int_0^t \left( \frac{1}{\chi^{1/2p}(\tau)} \right)^{(2p)'} d\tau \right)^{\frac{1}{(2p)'}} \\
 &\leq \sum_{\alpha} D_{\alpha 0}^{1+\theta-s} + C_4 t^{\frac{1-\eta\gamma}{2p}}, \quad C_4 = (C_3 \delta \mathcal{E}'(0))^{\frac{1}{2p}} \int_0^{\infty} \chi^{-\frac{1}{2p-1}}(\tau) d\tau.
 \end{aligned}$$

Thus, we end up with the improved dispersion bound

$$D(t) \leq C'_D (1+t)^{\gamma'}, \quad \gamma' = \frac{1-\eta\gamma}{2p(1+\theta-s)}.$$

This argument can be repeated: since  $\eta\gamma_p < 1$  then  $\frac{\eta}{2p(1+\theta-s)} = \frac{\eta\gamma_p}{2p-1} < 1$  and hence the iterations  $\gamma \mapsto \gamma'$  converge to  $\gamma_{\infty} = \frac{1}{2p(1+\theta-s) + \eta}$ . In particular, since  $2p-1 > 1$  then after finitely many iterations we will reach the improved rate,  $\gamma_* = (2p-1)\gamma_{\infty}$ ,

$$D(t) \leq C'_D (1+t)^{\gamma_*}, \quad \gamma_* = \frac{2p-1}{2p(1+\theta-s) + \eta}.$$

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