

October 30, 2015

# Mathematical Aspects of Self-Organized Dynamics

Consensus, Emergence of Leaders, and Social Hydrodynamics

By [Eitan Tadmor](#)

Simulation of agent-based Vicsek dynamics and the corresponding macroscopic density/velocity fields.

from Adam Cohan

01:20



[Simulation of agent-based Vicsek dynamics and the corresponding macroscopic density/velocity fields.](#) Courtesy of Sebastien Motsch.

Left: The agent-based Vicsek dynamics [2]. Right: The corresponding macroscopic density/velocity fields. These should be compared with the hydrodynamic description of the Vicsek model in P. Degond and S. Motsch, (2008), Continuum limit of self-driven particles with orientation interaction, Math. Mod. Meth. Appl. Sci., 18 1193-1215.”

Nature and human societies offer many examples of self-organized behavior. Ants form colonies, birds fly in flocks, mobile networks coordinate a rendezvous, and human opinions evolve into parties. These are simple examples of collective dynamics that tend to self-organize into large-scale clusters of colonies, flocks, parties, etc. A standard framework for such collective dynamics is based on environmental averaging

$$\mathbf{p}_i(t + \Delta t) = \sum_{j=1}^N a_{ij} \mathbf{p}_j(t) \quad \sum_{j=1}^N a_{ij} = 1.$$

(1)

The "environment" in this case consists of  $N$  agents, identified by their time-dependent vector of features

$p_i(t)$ . Each agent modifies its features by a weighted average of features from the neighboring agents  $\mathbf{p}_j(t)$ . These weighted averages are quantified (1) in terms of non-negative weights,  $a_{ij} \geq 0$ . Two prototypical examples are the Hegselmann-Krause model for opinion dynamics [1] in which  $\mathbf{p}_i(t)$  encodes vectors of "opinions," and the Vicsek model for flocking [2] where each  $\mathbf{p}_i(t) = (\mathbf{x}_i(t), \omega_i(t))$  encodes the location and orientation. We can express the dynamics in a more general form as a process of alignment,

$$p_i(t + \Delta t) = p_i(t) + \alpha \Delta t \sum_{j \neq i}^N a_{ij} (p_j - p_i), \quad \sum_{j=1}^N a_{ij} = 1. \quad (2)$$

Here, agent  $\mathbf{p}_i(t)$  aligns its features at a rate proportional to a weighted average of the differences relative to its "neighbors"  $\{\mathbf{p}_j - \mathbf{p}_i\}$ . The positive parameter quantifies the frequency of alignment; the specific case  $\alpha = 1/\Delta t$  recovers environmental averaging. The Cucker-Smale alignment model for flocking [3] where  $\mathbf{p}_i(t) = (\mathbf{x}_i(t), \mathbf{v}_i(t))$  encodes the location and velocity, serves as an example.

Different models distinguish themselves with different weights,  $a_{ij}$ , which quantify the communication between every pair of agents. Typically, they involve an influence function,  $\phi$ ,

$$a_{ij} = \frac{1}{\text{deg}_i(t)} \phi(|x_j(t) - x_i(t)|), \quad \text{deg}_i(t) := \sum_{k=1}^N \phi(|x_k(t) - x_i(t)|). \quad (3)$$

The Hegselmann-Krause model is local; it is driven by compactly-supported influence functions so that each agent interacts only with those that have similar opinions. Similarly, the Vicsek model involves local influence functions, but with additional stochastic noise. Cucker-Smale models involve global  $\phi$ 's, where  $\text{deg}_i \equiv N$ , or local ones advocated in [4].

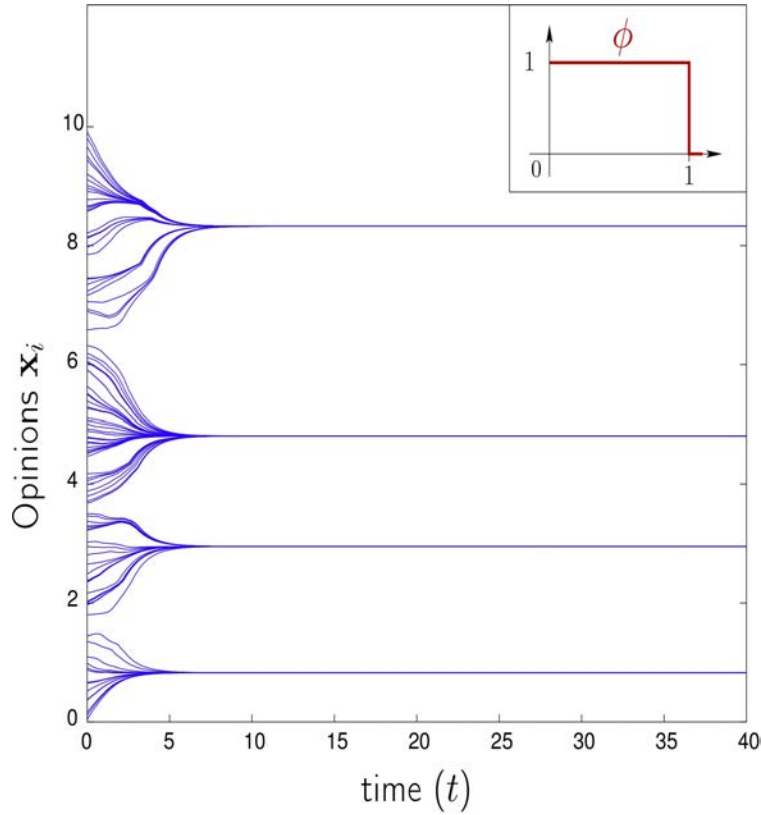
We note in passing the tacit assumption that the alignment in (3) depends on the notion of geometric distance. But this need not apply in general cases, for example, when measuring distances between vectors of opinions, or when the alignment dynamics is dictated by topological neighborhoods. Indeed, the latter is motivated by the detailed observations carried out in the STARFLAG project (2008), indicating that birds communicate with a fixed number of neighbors rather than a fixed geometric neighborhood.

These examples are part of a larger paradigm advocated by I. Aoki (1982) and C. Reynolds (1987), in which different rules for repulsion (in the "immediate vicinity" of each agent) and "far-field" attraction (cohesion) augment the mid-range alignment (2). Models based on this paradigm are found in different disciplines, including aggregation of bacteria and swarming in biology, complex networks which arise as a result of human interactions in social sciences (opinion dynamics and traffic networks, for example), and production lines and robotic networks in engineering. Different disciplines utilize different approaches to study such systems. Biologists inquire whether the observed self-organized patterns are system specific, while physicists seek analogies between different near-equilibrium patterns. Computer scientists trace their graph dynamics while engineers may ask how to control such systems.

The rapidly-growing mathematical literature devoted to such systems addresses several natural questions which arise in this context. We focus on two of them, associated with the important limits of  $t \rightarrow \infty$  and  $N \rightarrow \infty$ .

(i) What is the large-time behavior of (2),(3) and what are the more general classes of alignment models as  $t \rightarrow \infty$ ? In particular, what types of "rules of engagement" lead to the emergence of clusters and other more complex large-scale patterns? When the dynamics is global in the sense that every agent is able to communicate with every other agent ( $a_{ij} > 0$ ), the agents will approach one cluster. Thus, the large-time behavior of global alignment leads to consensus. In more realistic scenarios, however, the communication is local, that is, the influence function  $\phi$  vanishes when two agents with

features far apart attempt to communicate. In these cases, the important question of reaching a consensus is more subtle because it depends on the propagation of connectivity of the underlying graph associated with the time-dependent matrix  $a_{ij} \equiv a_{ij}(\mathbf{p}(t))$ . The propagation of connectivity is intimately related to the specific “rules of interactions.” A particularly intriguing aspect of this issue was observed in [4]: if the influence function  $\phi$  is heterophilious in the sense that it is increasing over its finite support, then it is likely to produce a few/fewer clusters, and eventually will evolve to a consensus shown in Figure 1. This counter-intuitive consequence of heterophilious dynamics is of potential importance in applications. Why is it only the “likely” behavior of heterophilious dynamics? This is due to lack of stability for general agent-based dynamics with a fixed (small) number of agents.



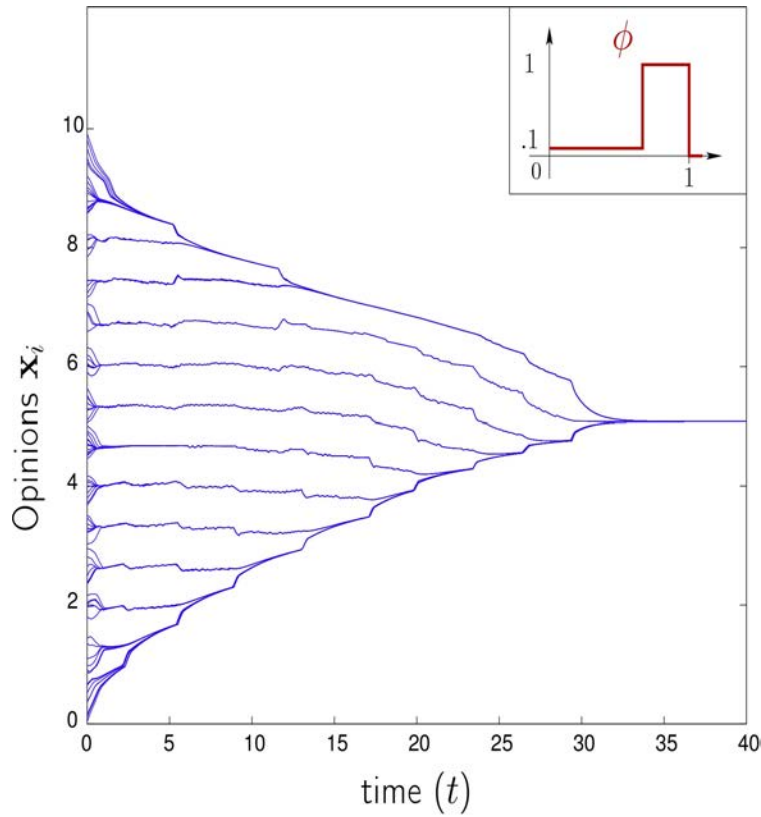


Figure 1. (Heterogeneous dynamics). Large-time behavior of Hegselmann-Krause model with 100 uniformly distributed opinions [4]. Left: Influence function  $\phi(r) := \mathbf{1}_{\{0 < r < 1\}}$  yields four parties. Right: Emergence of consensus with increasing  $\phi(r) := 0.1_{\{r \leq \frac{1}{\sqrt{2}}\}} + 1_{\{\frac{1}{\sqrt{2}} \leq r < 1\}}$ .

There are many other related aspects involved in different “rules of engagement” of collective dynamics, of which we mention three.

An important aspect in the self-organization of many mechanical systems is synchronization. The prototype is the Kuramoto model [5], which encodes the orientation of coupled oscillators,  $p_j(t) = e^{i\theta_j(t)}$ , governed by the coupling function  $\phi(\theta) = \sin\theta/\theta$ . It is also important to note that distances measured in realistic scenarios of “living” agents can only be estimated. This is the source for stochastic noise as in the Vicsek model, which leads to phase transition [2]. Finally, the underlying assumption in (2),(3) is that agents align with their neighbors along the radius vector of their respective positions, through vision (flocking), lasers (robots), etc. Unlike physical particles, however, “living” agents do not necessarily act along their relative radius vectors. Consider for example pheromones rather than vision, or the way humans follow the influential ideas of those who are “ahead.” In these cases, the interaction among agents may take place along projections on their relative trails. “Living” agents take into account only those neighbors who are “moving ahead” in a forward cone, which leads to the emergence of leaders illustrated in Figure 2.

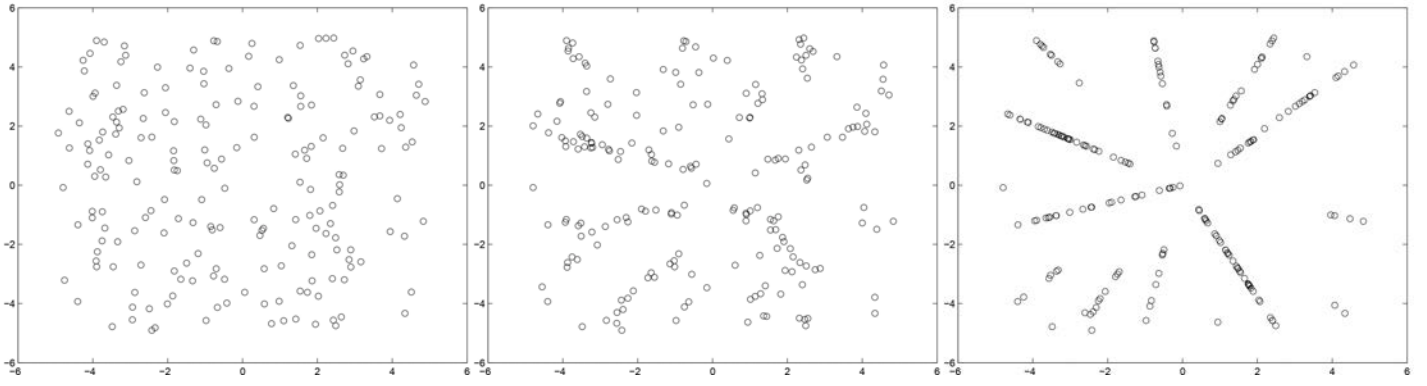


Figure 2. (Emergence of Leaders) from random initial conditions at  $t=0$  (left), followed by snapshots at  $t=0.5$  and  $t=5$ .

A key issue in systems with a large number of agents is understanding their group behavior rather than tracing the dynamics of each of their agents. This brings us to the second question. (ii) What is the qualitative behavior of self-organized dynamics for very large groups ( $N \rightarrow \infty$ )? Agent-based models like (2),(3) lend themselves to standard kinetic and hydrodynamics descriptions.

For a kinetic description, consider an ensemble of a large number of agents with time-dependent distribution  $\mathbf{f}(t, \mathbf{x}, \mathbf{v})$  which realizes the (assumed large  $N$ -limit of the empirical distribution of agents,  $1/N \sum \delta_{x_{i(t)}}(\mathbf{x}) \otimes \delta_{v_{i(t)}}(\mathbf{v})$ ). Expressed in terms of its macroscopic density,  $\rho(t, \mathbf{x}) := \int \mathbf{f}(t, \mathbf{x}, v) d\mathbf{v}$ , and momentum,  $\mathbf{m}(t, \mathbf{x}) := \int \mathbf{v} \mathbf{f}(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$ , the dynamics of such an ensemble is governed by the Vlasov-Fokker-Planck equation

$$f_t + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot ((\mathbf{u} - \mathbf{v}) f) = \sigma \Delta_v f \quad (4)$$

The third term on the left represents alignment towards the mean velocity  $\mathbf{u} = \frac{\phi * \mathbf{m}}{\phi * \rho}$ , while the term on the right represents diffusion due to possible noise. Studying the stability around global Maxwellians associated with (4) addresses the difficulties in analyzing the stability of agent-based dynamics with a fixed number of agents.

A further simplification is obtained with the macroscopic description of self-organized dynamics. It is governed by conservation of mass,  $\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$ , coupled with the balance equation

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} + \left(\frac{1}{\rho}\right) \nabla_x P = \frac{1}{\text{deg}} \int \phi(|\mathbf{x} - \mathbf{y}|) (\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})) \rho(\mathbf{y}) d\mathbf{y}$$

$$\text{deg}(t, \mathbf{x}) := \phi * \rho. \quad (5)$$

The pressure term on the left encodes the closure of (4) and the expression on the right of (5) is the alignment towards the shifted means. Both terms are model-dependent. When  $\phi$  is singular, it can be viewed as fractional-order Laplacian; when  $\phi$  is global, it is reminiscent of the nonlocal means found in image processing. In the present context of social hydrodynamics, (5) involves local smooth  $\phi$ 's. We know that if smooth solutions of (5) exist, they must flock. But alignment-based models reflect the

competition on resources; left unchecked, it may lead to finite-time “blow-up.” This is where the closure with additional repulsion forces in the form of compressible (or incompressible) pressure comes into play. Current work includes analytical and computational methods to study the stability of such systems, specifically, whether the regularity of their solutions persists in time and what large-scale structures emerge from the social hydrodynamics governed by (5).

More on the current work including open questions concerning the modeling, analysis, and computation of collective dynamics can be found in [4] and the references therein.

---

Acknowledgments: Research was supported in part by NSF grants DMS10-08397 and RNMS11-07444 (KI-Net) and by ONR grant 000141512094.

References:

- [1] Hegselmann, R., & Krause, U. (2002). Opinion dynamics and bounded confidence: models, analysis and simulation. *J. Artificial Societies and Social Simulation*, 5 (3).
- [2] Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I., & Shochet, O. (1995). Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.* 75, 1226 – 1229.
- [3] Cucker, F., & Smale, S. (2007). Emergent behavior in flocks. *IEEE Trans. Aut. Control*, 52(5), 852-862.
- [4] Motsch S., & Tadmor, E. (2014). Heterophilious dynamics enhances consensus. *SIAM Review* 56(4), 577–621.
- [5] Kuramoto, Y. International symposium on mathematical problems in mathematical physics. *Lecture Notes in Physics* 39, (1975), 420-422.

Eitan Tadmor is the recipient of the 2015 Peter Henrici Prize. This article is based, in part, on the associated prize lecture that he delivered at the 8th International Congress on Industrial and Applied Mathematics held in Beijing earlier this year.



SIAM President Pam Cook presents the 2015 Peter Henrici Prize to Eitan Tadmor.

Tadmor is a Distinguished University Professor at the University of Maryland, College Park, and Director of its Center for Scientific Computation and Mathematical Modeling. He is a faculty member at CSCAMM, the Department of Mathematics, and IPST.