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Flocking Hydrodynamics with External Potentials

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Abstract

We study the large-time behavior of a hydrodynamic model which describes the collective behavior of continuum of agents, driven by pairwise alignment interactions with additional external potential forcing. The external force tends to compete with the alignment which makes the large time behavior very different from the original Cucker–Smale (CS) alignment model, and far more interesting. Here we focus on uniformly convex potentials. In the particular case of *quadratic* potentials, we are able to treat a large class of admissible interaction kernels, $\phi(r) \gtrsim (1+r^2)^{-\beta}$ with 'thin' tails $\beta \leqslant 1$—thinner than the usual 'fat-tail' kernels encountered in CS flocking $\beta \leqslant 1/2$; we discover unconditional flocking with exponential convergence of velocities *and* positions towards a Dirac mass traveling as harmonic oscillator. For general convex potentials, we impose a stability condition, requiring a large enough alignment kernel to avoid crowd scattering. We then prove, by hypocoercivity arguments, that both the velocities *and* positions of a smooth solution must flock. We also prove the existence of global smooth solutions for one and two space dimensions, subject to critical thresholds in initial configuration space. It is interesting to observe that global smoothness can be guaranteed for sub-critical initial data, independently of the apriori knowledge of large time flocking behavior.

Contents

1. Introduction

We are concerned with the hydrodynamic alignment model with external potential forcing:

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \displaystyle\int \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y},t) - \mathbf{u}(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y} - \nabla U(\mathbf{x}). \end{cases} \tag{1.1}$$

Here $(\rho(\mathbf{x},t), \mathbf{u}(\mathbf{x},t))$ are the local density and velocity field of a continuum of agents, depending on the spatial variables $\mathbf{x} \in \Omega = \mathbb{R}^d$ or $\mathbb{T}^d$ and time $t \in \mathbb{R}_{\geqslant 0}$. The integral term on the right represents the alignment between agents, quantified in terms of the pairwise interaction kernel $\phi = \phi(r) \geqslant 0$. In many realistic scenarios, agents driven by alignment are also subject to other forces—external forces from environment, pairwise attractive-repulsive forces, etc. Such forces may *compete* with alignment, which makes the large time behavior very different from the original potential-free model and far more interesting. One of the simplest types of external forces is *potential force*, given by the fixed external potential $U(\mathbf{x})$ on the right of (1.1). This is the main topic on the current work.

The system (1.1) is a realization of the large-crowd dynamics of the agent-based system in which $N \gg 1$ agents identified with their position and velocity pair, $(\mathbf{x}_i(t), \mathbf{v}_i(t)) \in (\Omega \times \mathbb{R}^d)$, are driven by Cucker–Smale (CS) alignment [4,5], with additional external potential force

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \dfrac{1}{N} \displaystyle\sum_{j \neq i} \phi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{v}_j - \mathbf{v}_i) - \nabla U(\mathbf{x}_i) \end{cases} \quad i = 1, \ldots, N. \tag{1.2}$$

In the absence of any other forcing terms, both the agent-based system (1.2) and its large crowd description (1.1) have been studied intensively in the last decade. The most important feature of the potential-free CS model, (1.2) with $U \equiv 0$, is its *flocking* behavior: for a large class of interaction kernels satisfying the 'fat tail' condition,

$$\int_0^\infty \phi(r)\,\mathrm{d}r = \infty, \tag{1.3}$$

and the *global* alignment of velocities follows [7,8], $|\mathbf{v}_i(t) - \mathbf{v}_j(t)| \stackrel{t\to\infty}{\longrightarrow} 0$. The corresponding potential-free continuum system, (1.1) with $U \equiv 0$, was studied in [2,7,8,11]; the large time behavior of its smooth solutions is captured by flocking, $|\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{y},t)|\rho(\mathbf{x})\rho(\mathbf{y}) \stackrel{t\to\infty}{\longrightarrow} 0$, similar to the underlying discrete system. Moreover, the existence of one- and two-dimensional global smooth solutions was proved for a large class of initial configurations which satisfy certain critical threshold condition, [1,9,13,14,16] and general multiD problems with nearly aligned initial data [6,12].

In this paper we study the alignment dynamics in the d-dimensional continuum system (1.1). The dynamics of (1.1) is driven by the decay of its *total energy*,

$$E(t) := \int \left(\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2 + U(\mathbf{x})\right)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x}. \tag{1.4}$$

The fundamental bookkeeping of (1.1) is the L^2-energy decay

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\frac{1}{2}\iint \phi(|\mathbf{x}-\mathbf{y}|)|\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)|^2\rho(\mathbf{x},t)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \tag{1.5}$$

which quantifies the decay *rate* of the energy in terms of *energy fluctuations* on the right. A parallel study of the discrete alignment dynamics (1.2), which we omit, can be carried out in terms of the corresponding discrete energy bookkeeping

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N}\sum_i\left(\frac{1}{2}|\mathbf{v}_i|^2 + U(\mathbf{x}_i)\right) = -\frac{1}{2N^2}\sum_{i,j}\phi(|\mathbf{x}_i-\mathbf{x}_j|)|\mathbf{v}_j-\mathbf{v}_i|^2.$$

We focus on the following two key aspects of the continuum alignment dynamics (1.1):

• *The Flocking Phenomena of Global Smooth Solutions*, if they exist. Such results are well known in the absence of external potential—smooth solutions subject to pure alignment must flock [8,9,16], but the presence of external potential has a confining effect which competes with alignment. Here we explore the flocking phenomena in the presence of *uniformly convex* potentials

$$aI_{d\times d} \leqslant \nabla^2 U(\mathbf{x}) \leqslant AI_{d\times d}, \qquad 0 < a < A. \tag{1.6}$$

The upper-bound on the right is *necessary* for existence of 1D global smooth solutions, consult Theorems 4.2–4.3 below; the uniform convexity on the left is necessary for the flocking behavior. We discover, in Sect. 3, that both the velocities *and* positions of smooth solution must flock at algebraic rate under a linear stability condition (3.10), $m_0\phi(0) > \frac{A}{\sqrt{a}}$. The necessity of a precise stability condition, at least in the general convex case, remains open. We can be much more precise in the special case of *quadratic potentials*,

$$U(\mathbf{x}) = \frac{a}{2}|\mathbf{x}|^2, \qquad a > 0. \tag{1.7}$$

Here, in Sect. 2, we discover unconditional flocking of velocities and positions with *exponential* convergence to a Dirac mass traveling as a harmonic oscillator. Moreover, the confining effect of the quadratic potential applies to interaction kernels, $\phi(r) \gtrsim (1+r^2)^{-\beta}$ which allow for 'thin' tails $\beta \leqslant 1$—thinner than the usual 'fat-tail' kernels encountered in CS flocking (1.3).

• *Existence of Smooth Solutions for all Time.* In the absence of external force, the existence of global smooth solutions of the one- and respectively two-dimensional (1.1) was proved in [1,16] and respectively [9], provided the initial data is 'below' certain critical thresholds, expressed in terms of the initial data ρ_0, $\nabla\cdot\mathbf{u}_0$, and the spectral gap of $\nabla_S\mathbf{u}_0$. We mention in passing that in case of singular kernel ϕ, then smooth solutions exist independent of an initial threshold [13]). In the presence of additional convex potential, (1.6), we discover that the critical thresholds still exist, though they are tamed by the presence of U (consult [17]). In particular, they limit the maximal velocity $\max_{t\geqslant 0,\mathbf{x}\in\Omega}|\mathbf{u}(\mathbf{x},t)|$. The case of quadratic potential (1.7) is a special case: when $U(\mathbf{x})=\frac{a}{2}|\mathbf{x}|^2$, it can be shown to have a limited affect on the dynamics of the spectral gap of $\nabla_S\mathbf{u}$ (which is a crucial step of the regularity result in [9]). Consequently, it further simplifies the quadratic critical threshold for global regularity, requiring $\nabla\cdot\mathbf{u}_0+\phi*\rho_0\geqslant 0$, in agreement with the threshold of the external-free case. These results are summarized in Sect. 4.

2. Statement of Main Results: Flocking with Quadratic Potentials

We focus attention to *quadratic potentials*, $U(\mathbf{x})=\dfrac{a}{2}|\mathbf{x}|^2$, where (1.1) reads as

$$\begin{cases}\partial_t\rho+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u})=0,\\ \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u}=\displaystyle\int\phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y},t)-\mathbf{u}(\mathbf{x},t))\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{y}-a\mathbf{x}.\end{cases}\tag{2.1}$$

2.1. General Considerations

We begin by recording general observations on system (1.1) which is subject to sufficiently smooth data $(\rho_0,\mathbf{u}_0)$, such that $\rho_0\geqslant 0$ is compactly supported. Denote the total mass

$$m_0:=\int\rho_0(\mathbf{x})\,\mathrm{d}\mathbf{x}>0.$$

• *Interaction Kernels* We assume that the system (1.1) is driven by an interaction kernel from a general class of *admissible kernels*.

Assumption 2.1. (*Admissible kernels*) We consider (1.1) with interaction kernel ϕ such that

(i) $\phi(r)$ is positive, decreasing and bounded : $0<\phi(r)\leqslant\phi(0):=\phi_+<\infty;$ (2.2a)

(ii) $\phi(r)$ decays slow enough at infinity in the sense that $\displaystyle\int^{\infty}r\phi(r)\,\mathrm{d}r=\infty.$ (2.2b)

Note that (2.2b) allows for a larger admissible class of ϕ's with *thinner* tails than the usual 'fat-tail' assumption (1.3) which characterizes unconditional flocking of potential-free alignment, e.g., the original choice of Cucker–Smale, $\phi(r)=(1+r^2)^{-\beta}$, $\beta\leqslant 1/2$ is now admissible for the improved range $\beta\leqslant 1$.

• *Vanishing Means* The distinctive feature of the alignment dynamics with quadratic potential (2.1), is its Galilean invariance w.r.t. the dynamics of means. Thus, if we let $(\mathbf{x}_c, \mathbf{u}_c)$ denote the mean position and the mean velocity

$$\begin{cases} \mathbf{x}_c(t) := \dfrac{1}{m_0}\displaystyle\int \mathbf{x}\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\ \mathbf{u}_c(t) := \dfrac{1}{m_0}\displaystyle\int \mathbf{u}(\mathbf{x},t)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x}, \end{cases} \tag{2.3}$$

then the translated quantities centered around the means, $\widehat{\rho}(\mathbf{x},t) := \rho(\mathbf{x}_c(t)+\mathbf{x},t)$ and $\widehat{\mathbf{u}}(\mathbf{x},t) := \mathbf{u}(\mathbf{x}_c(t)+\mathbf{x},t) - \mathbf{u}_c(t)$, satisfy the same system (2.1) with vanishing mean location and mean velocity. We can therefore assume without loss of generality, after re-labeling $(\widehat{\rho}, \widehat{\mathbf{u}}) \rightsquigarrow (\rho, \mathbf{u})$, that the solution of (2.1) satisfies

$$\int \mathbf{x}\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \equiv 0, \quad \int \mathbf{u}(\mathbf{x},t)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \equiv 0, \quad \text{for all } t \geqslant 0. \tag{2.4}$$

Remark that the same Galilean invariance is intimately related to the fact that quadratic external forcing can be interpreted as *pairwise interactions*. Thus, in the context of the discrete dynamics (1.2) with $U = \frac{1}{2}a|\mathbf{x}|^2$ for example, we end up with

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \dfrac{1}{N}\displaystyle\sum_{j\neq i}\phi(|\mathbf{x}_i-\mathbf{x}_j|)(\mathbf{v}_j-\mathbf{v}_i) - \dfrac{a}{N}\sum_{j\neq i}(\mathbf{x}_i-\mathbf{x}_j). \end{cases} \tag{2.5}$$

Indeed, the means of (1.2) with this quadratic potential—the center of mass $\mathbf{x}_c(t) := {}^1/_N \sum_i \mathbf{x}_i$ and mean velocity $\mathbf{u}_c(t) := {}^1/_N \sum_i \mathbf{v}_i$, satisfy (2.16) below; subtracting the means, we find that the translated quantities $\mathbf{x}_i \mapsto \mathbf{x}_i - \mathbf{x}_c(t)$, $\mathbf{v}_i \mapsto \mathbf{v}_i - \mathbf{u}_c(t)$ satisfy (2.5).

2.2. *Bounded Support*

A priori estimates for the growth rate of the support of ρ is the key for proving flocking results for admissible kernels ϕ with proper decay at infinity. For the case without external potential, it is straightforward to show that the velocity variation $\max\limits_{t\geqslant 0,\, \mathbf{x},\mathbf{y}\in \text{supp}\,\rho(\cdot,t)} |\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)|$ is non-increasing, which implies the linear growth, $\text{diam}(\text{supp}\,\rho(\cdot,t)) = \mathcal{O}(t)$ which in turn yields the 'fat-tail' condition (1.3). Here we show that confining effect of the external potential enforces the support of $\rho(\cdot,t)$ to remain *uniformly bounded*.

To this end, define the maximal particle energy

$$P(t) := \max_{\mathbf{x}\in \text{supp}\,\rho(\cdot,t)} \Big(\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2 + U(\mathbf{x})\Big). \tag{2.6}$$

The confinement effect of the external potential shows that this L^∞-particle energy remains uniformly bounded in time. We then 'pair' the quadratic growth of $U(\mathbf{x})$ with the admissibility of thin-tails assumed in (2.2b), to show that supp $\rho(\cdot,t)$ remains uniformly bounded.

Lemma 2.2. *(***Uniform bounds on particle energy***) Let* $(\rho, \mathbf{u})$ *be a smooth solution to* (2.1) *with an admissible interaction kernel (2.2). Then the particle energy and hence the support of* $\rho(\cdot, t)$ *remain uniformly bounded*

$$\frac{a}{8}D^2(t) \leqslant P(t) \leqslant R_0, \qquad D(t) := diam(supp\ \rho(\cdot, t)). \tag{2.7}$$

Here, the spatial scale $R_0 = R_0(\phi_+, m_0, a, E_0, P_0)$ *is dictated by* (2.11)*, below.*

For the proof, follow the particle energy $F(\mathbf{x}, t) := \frac{1}{2}|\mathbf{u}(\mathbf{x}, t)|^2 + U(\mathbf{x})$ along characteristics

$$\begin{aligned}
F' &= \partial_t F + \mathbf{u} \cdot \nabla F \\
&= \mathbf{u} \cdot \left(-\mathbf{u} \cdot \nabla \mathbf{u} + \int \phi(\mathbf{x} - \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} - \nabla U(\mathbf{x}) \right) \\
&\quad + \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \mathbf{u} \cdot \nabla U(\mathbf{x}) \\
&= \mathbf{u} \cdot \left(\int \phi(\mathbf{x} - \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \right) \\
&= \int \phi(\mathbf{x} - \mathbf{y})(\mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{y}) - |\mathbf{u}(\mathbf{x})|^2)\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&= \int \phi(\mathbf{x} - \mathbf{y})\Big(- \frac{1}{4}|\mathbf{u}(\mathbf{y})|^2 + \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{y}) - |\mathbf{u}(\mathbf{x})|^2 \Big)\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&\quad + \int \phi(\mathbf{x} - \mathbf{y})\frac{1}{4}|\mathbf{u}(\mathbf{y})|^2\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&= - \int \phi(\mathbf{x} - \mathbf{y})\Big|\mathbf{u}(\mathbf{x}) - \frac{1}{2}\mathbf{u}(\mathbf{y})\Big|^2 \rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \\
&\quad + \frac{1}{4} \int \phi(\mathbf{x} - \mathbf{y})|\mathbf{u}(\mathbf{y})|^2\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} \leqslant \frac{\phi_+}{2} E_k(t),
\end{aligned}$$

where $E_k(t)$ denotes the *kinetic energy*

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) \leqslant \frac{\phi_+}{2}E_k(t), \qquad E_k(t) := \frac{1}{2}\int |\mathbf{u}(\mathbf{x}, t)|^2\rho(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}. \tag{2.8}$$

We emphasize that the bound (2.8) applies to general symmetric kernels ϕ and is otherwise independent of the fine structure of the potential U. Recalling the diameter $D(t) = \mathrm{diam}(\mathrm{supp}\ \rho(\cdot, t))$, then L^2-energy decay (1.5) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant -\frac{1}{2}\phi(D(t)) \iint |\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{y}, t)|^2 \rho(\mathbf{x}, t)\rho(\mathbf{y}, t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y};$$

in view of the vanishing means assumed in the quadratic case, (2.4), this decay rate can be formulated in terms of the kinetic energy

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant -2m_0\phi(D(t))E_k(t). \tag{2.9}$$

Further more, the support of $\rho(\cdot, t)$ can be bounded in terms of the particle energy, so we have

$$P(t) \geqslant U(x) = \frac{a}{2} \max_{\text{supp } \rho(\cdot,t)} |\mathbf{x}|^2 \geqslant \frac{a}{8} D^2(t), \qquad D(t) = \text{diam}(\text{supp } \rho(\cdot, t)). \tag{2.10}$$

Finally, by the fat-tail assumption (2.2b), $\int^{\infty} \phi(\sqrt{8r/a})\, \mathrm{d}r = \frac{a}{4} \int^{\infty} r\phi(r)\, \mathrm{d}r = \infty$, there exists a finite spatial scale $R_0 > P_0$ such that

$$\int_{P_0}^{R_0} \phi(\sqrt{8r/a})\, \mathrm{d}r > \frac{\phi_+}{4m_0} E_0. \tag{2.11}$$

We now consider the functional $Q(t) := E(t) + \frac{4m_0}{\phi_+} \int_{R_0}^{P(t)} \phi(\sqrt{8r/a})\, \mathrm{d}r$ which we claim is non-positive: indeed, by (2.11), $Q(0) \leqslant 0$ and in view of (2.8)–(2.10), $Q(t)$ decreasing in time

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(t) \leqslant -2m_0\phi(D(t))E_k(t) + \frac{4m_0}{\phi_+}\frac{\phi_+}{2} E_k(t) \times \phi(\sqrt{8P(t)/a}) \leqslant 0.$$

It follows that the particle energy remains uniformly bounded:

$$\frac{4m_0}{\phi_+} \int_{R_0}^{P(t)} \phi(\sqrt{8r/a})\, \mathrm{d}r \leqslant Q(t) \leqslant 0,$$

hence $P(t)$ remain bounded, $P(t) \leqslant R_0$, and the uniform bound on $D(t)$ stated in (2.7) follows from (2.10). □

For the typical example of $\phi(r) = c_0(1 + r^2)^{-\beta}$, we find that (2.11) holds, with

$$R_0 \geqslant \frac{a}{8}\left[\left(\left(1 + \frac{8}{a} P_0\right)^{1-\beta} + \frac{2(1-\beta)\phi_+}{ac_0m_0} E_0\right)^{\frac{1}{1-\beta}} - 1\right].$$

2.3. Flocking of Smooth Solutions with Exponential Rate

The *uniform-in-time* bound on the supp $\rho(\cdot, t)$in (2.7) shows that the values $\phi(r)$ with $r > \sqrt{8R_0/a}$ play no role in the solution of (2.1). We can therefore assume without loss of generality that our admissible ϕ's are uniformly bounded from below:

$$\phi(r) \geqslant \phi(D(t)) \geqslant \phi_- > 0, \qquad \phi_- := \phi\Big(\frac{\sqrt{8R_0}}{\sqrt{a}}\Big). \tag{2.12}$$

This enables us prove our main statement of flocking with exponential decay.

Theorem 2.3. (Flocking with L^2-exponential decay) *Let* $(\rho, \mathbf{u})$ *be a global smooth solution of* (2.1)*, subject to compactly supported* ρ_0*. Then there holds the flocking estimate at exponential rate in both velocity and position:*

$$\delta E(t) := \iint (|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}(\mathbf{y}, t)|^2 + a|\mathbf{x} - \mathbf{y}|^2)\rho(\mathbf{x}, t)\rho(\mathbf{y}, t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \leqslant 2 \cdot \delta E_0 \cdot e^{-\lambda t}. \tag{2.13}$$

Here $\lambda = \lambda(a, \phi_-, \phi_+, m_0) > 0$.

Remark 2.4. In fact, one could develop a small-data result, where the exponential flocking asserted in Theorem 2.3 is extended to U's close to quadratic potential provided under appropriate smallness condition on the initial data.

From the proof of Theorem 2.3, one can take the decay rate

$$\lambda = \lambda(a) := \frac{1}{2}\min\left\{\frac{m_0\phi_-}{m_0^2\phi_+^2/a + 3/2}, \frac{\sqrt{a}}{2}\right\} \tag{2.14}$$

If one fixes m_0, ϕ_+, ϕ_- and considers the asymptotic behavior for $a \to 0$, then the decay rate $\lambda = \mathcal{O}(a)$. For $a \to \infty$, the decay rate $\lambda = \mathcal{O}(1)$. This shows that *the strength of external potential force* may have significant influence on the rate of flocking, and a weak potential tends to give a slower decay. One could interpret this as follows: to achieve an equilibrium, both velocity and position have to align; if the potential force is weak, then the alignment of position happens on a slower time scale, since the potential-free Cucker–Smale interaction does not provide position alignment.
Next, we turn to improve the L^2-flocking estimate in Theorem 2.3 into an L^∞ estimate.

Theorem 2.5. *(***Flocking with uniform exponential decay***) Let* $(\rho, \mathbf{u})$ *be a global smooth solution of* (2.1)*, subject to compactly supported* ρ_0*. Then*

$$\delta P(t) := \max_{\mathbf{x},\mathbf{y}\in supp\ \rho(\cdot,t)} (|\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{y},t)|^2 + a|\mathbf{x}-\mathbf{y}|^2) \leqslant C_\infty \cdot \delta P_0 \cdot e^{-\lambda t/2}, \quad \forall t \geqslant 0 \tag{2.15}$$

where the decay rate $\lambda = \lambda(a) > 0$ *given by* (2.14) *and* C_∞ *is a positive constant given by*

$$C_\infty = 4\Big(1 + \phi_+^2 m_0^2\Big(\frac{2}{m_0\phi_-\lambda(a)} + \frac{4}{a}\Big)\Big).$$

Note that since $\delta E \leqslant m_0^2 \cdot \delta P$, the L^∞-version of flocking stated in Theorem 2.5 is an improvement of Theorem 2.3. This improvement can be useful in studying the existence of global smooth solution for two-dimensional systems asserted in Theorem 4.4 below.

Remark 2.6. (*blow-up as* $a \ll 1$) We note in passing that (2.15) does not recover the velocity alignment in the potential-free case due to the blow-up of $C_\infty = \mathcal{O}(1/a)$ as $a \to 0$. The growing bound is due to the proof in which we estimate the momentum $\phi * (\rho\mathbf{u})$ as a source term by using L^2 exponential decay in Theorem 2.3: yet, the L^2-decay rate $\lambda(a)$ deteriorates as $a \to 0$, and the effect of an increasing source term leads to the blow-up of C_∞. Indeed, it is known that the unconditional velocity alignment in the potential-free case is restricted to the 'fat-tails' (1.3), hence our approach for the thinner tails (2.2) cannot apply uniformly in $1/a$.

2.4. Convergence to Harmonic Oscillator

Recall that $(\mathbf{x}_c, \mathbf{u}_c)$ denote the mean position and the mean velocity of the dynamics (2.3). A distinctive feature of alignment dynamics with quadratic potential (2.1) is that these means are governed by the harmonic oscillator

$$\begin{cases} \dot{\mathbf{x}}_c = \mathbf{u}_c \\ \dot{\mathbf{u}}_c = -a\mathbf{x}_c \end{cases} \rightsquigarrow \begin{bmatrix} \mathbf{x}_c(t) \\ \frac{1}{\sqrt{a}}\mathbf{u}_c(t) \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{a}t) & \sin(\sqrt{a}t) \\ -\sin(\sqrt{a}t) & \cos(\sqrt{a}t) \end{bmatrix} \begin{bmatrix} \mathbf{x}_c(0) \\ \frac{1}{\sqrt{a}}\mathbf{u}_c(0) \end{bmatrix}. \tag{2.16}$$

The L^2-flocking statement (3.13) implies that the dynamics concentrates along this harmonic oscillator. Indeed, since $\displaystyle\int |\mathbf{x} - \mathbf{x}_c|^2 \rho(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} \lesssim m_0^{-1} e^{-\lambda t}$, it follows that for arbitrary test function $\chi \in W_c^{1,\infty}$,

$$\begin{aligned} \Big| \int \rho(\mathbf{x}, t)\chi(\mathbf{x})\,\mathrm{d}\mathbf{x} - m_0\chi(\mathbf{x}_c)\Big| &\leqslant \int \rho(\mathbf{x}, t)\big|(\chi(\mathbf{x}) - \chi(\mathbf{x}_c)\big|\,\mathrm{d}\mathbf{x} \\ &\leqslant m_0^{1/2}\Big(\int |\mathbf{x} - \mathbf{x}_c|^2\rho\,\mathrm{d}\mathbf{x}\Big)^{1/2} |\nabla\chi(\xi)|_\infty \lesssim \|\nabla\chi\|_\infty e^{-\lambda t/2}, \end{aligned}$$

and

$$\begin{aligned} \Big| \int \rho\mathbf{u}(\mathbf{x},t)\chi(\mathbf{x})\,\mathrm{d}\mathbf{x} - m_0\mathbf{u}_c(t)\chi(\mathbf{x}_c)\Big| &\leqslant \int \rho|\mathbf{u}|\cdot\big|(\chi(\mathbf{x}) - \chi(\mathbf{x}_c)\big|\,\mathrm{d}\mathbf{x} \\ &\leqslant m_0^{1/2}\Big(\int \rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\Big)^{1/2}\Big(\int |\mathbf{x} - \mathbf{x}_c|^2\rho\,\mathrm{d}\mathbf{x}\Big)^{1/2} |\nabla\chi(\xi)|_\infty \lesssim \sqrt{E_0}\cdot\|\nabla\chi\|_\infty e^{-\lambda t/2}. \end{aligned}$$

We conclude that the smooth solutions of (2.1) converges exponentially to the dynamics of harmonic oscillator (2.16)

$$\begin{cases} \rho(\mathbf{x}, t) - m_0\delta(\mathbf{x} - \mathbf{x}_c(t)) \stackrel{t\to\infty}{\longrightarrow} 0, \\ \rho\mathbf{u}(\mathbf{x}, t) - m_0\mathbf{u}_c(t)\delta(\mathbf{x} - \mathbf{x}_c(t)) \stackrel{t\to\infty}{\longrightarrow} 0. \end{cases} \tag{2.17}$$

The convergence is interpreted weakly in $\big(W^{1,\infty}\big)'$. An even stronger notion of convergence follows from the L^∞-flocking estimate (2.15) which in turn implies the uniform decay of the support $|\mathbf{x} - \mathbf{x}_c|\mathbb{1}_{\mathrm{supp}\,\rho(\mathbf{x},t)} \lesssim e^{-\lambda t/2}$. It follows that the concentration bounds above apply to arbitrary test function $\chi \in C_c$ and convergence to the harmonic oscillator (2.17) follows in the sense of measures.

The last conclusion could be examined in light of the self-propelled dynamics studied in [3]. Their dynamics—driven by a different competition between confining potential and self-acceleration/deceleration, led to interesting (double-)milling patterns, and it was believed that such milling patterns will not to survive by either velocity alignment or repulsive interactions [3, p. 36]. The convergence to harmonic oscillator asserted in (2.17) rebukes this belief, at least when confining potential competes with velocity alignment.

3. Statement of Main Results: Flocking with General Convex Potentials

3.1. General Considerations

We now turn our attention to alignment dynamics (1.1) with more general strictly convex potentials, (1.6). The flocking results are more restricted. We begin with specifying the smaller class of admissible interaction kernels.

Assumption 3.1. (**Admissible kernels**) We consider (1.1) with interaction kernel ϕ such that

(i) $\phi(r)$ is positive, decreasing and bounded : $0 < \phi(r) \leqslant \phi(0) := \phi_+ < \infty$;

$$\tag{3.1a}$$

(ii) $\phi(r)$ decays slow enough at infinity in the sense that $\limsup_{r\to\infty} r\phi(r) = \infty$.

$$\tag{3.1b}$$

Notice that (3.1b) is only slightly more restrictive than the usual 'fat-tail' assumption $\int_0^\infty \phi(r)\,\mathrm{d}r = \infty$, which characterize unconditional flocking in the case of potential-free alignment [7,8].

We begin noting that the basic bookkeeping of energy decay (1.4) still holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\frac{1}{2}\iint \phi(|\mathbf{x}-\mathbf{y}|)|\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)|^2\rho(\mathbf{x},t)\rho(\mathbf{y},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.$$

• **Uniform Bounds** A necessary main ingredient in the analysis of (1.1) is the uniform bound of diam(supp $\rho(\cdot,t)$), and the amplitude of velocity $\max_{\mathbf{x}\in \text{supp}\,\rho}|\mathbf{u}(\mathbf{x},t)|$. Our next lemma shows that whenever one has a uniform bound of $|\mathbf{u}(\mathbf{x},t)|+|\mathbf{x}|$ for the *restricted* class of lower-bounded ϕ's which scales like $\mathcal{O}(1/\min\phi)$, then it implies a uniform bound of $|\mathbf{u}(\mathbf{x},t)|+|\mathbf{x}|$ for the general class of admissible ϕ's (2.2).

Lemma 3.2. *(***The reduction to lower-bounded** ϕ**'s***) Consider* (1.1) *with a with the* <u>*restricted*</u> *class of lower-bounded* ϕ*'s:*

$$0 < \phi_- \leqslant \phi(r) \leqslant \phi_+ < \infty. \tag{3.2}$$

Assume that the solutions $(\widetilde{\rho},\widetilde{\mathbf{u}})$ *associated with the restricted* (1.1),(3.2)*, satisfy the uniform bound (with constants* $C_\pm$ *depending on* U,ϕ_+,m_0 *and* E_0*)*

$$\max_{t\geqslant 0,\,\mathbf{x}\in supp\,\widetilde{\rho}(\cdot,t)}(|\widetilde{\mathbf{u}}(\mathbf{x},t)|+|\mathbf{x}|) \leqslant \max\left\{C_+\cdot \max_{\mathbf{x}\in supp\,\widetilde{\rho}_0}(|\widetilde{\mathbf{u}}_0(\mathbf{x})|+|\mathbf{x}|),\,\frac{C_-}{\phi_-}\right\}. \tag{3.3}$$

Then the following holds for solutions associated with a <u>*general*</u> *admissible kernel* ϕ *(3.1): if* $(\rho,\mathbf{u})$ *is a smooth solution of* (1.1)*, then there exists* $\alpha > 0$ *(depending on the initial data* $(\rho_0,\mathbf{u}_0)$*), such that* $(\rho,\mathbf{u})$ *coincides with the solution,* $(\widetilde{\rho}_\alpha,\widetilde{\mathbf{u}}_\alpha)$*, associated with the lower-bounded* $\phi_\alpha(r) := \max\{\phi(r),\alpha\}$.

This means that if ϕ belongs to the general class of admissible kernels (3.1), then we can assume, without loss of generality, that ϕ coincides with the lower bound ϕ_α and hence the uniform bound (3.3) holds with $\phi_- = \alpha$. The justification of this reduction step is outlined below.

Proof of Lemma 3.2. By the condition (2.2b), there exists r_0 such that $r_0\phi(r_0) \geqslant 2C_-$, and one could take large enough r_0 such that

$$r_0 \geqslant 2C_+ \cdot \max_{\mathbf{x}\in\operatorname{supp}\rho_0}(|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|). \tag{3.4}$$

Let $\alpha = \phi(r_0)$. By assumption, (3.3) holds for the lower-bounded ϕ_α, so that

$$\max_{t\geqslant 0,\,\mathbf{x}\in\operatorname{supp}\rho_\alpha(\cdot,t)}(|\mathbf{u}_\alpha(\mathbf{x},t)| + |\mathbf{x}|) \leqslant \max\left\{C_+ \cdot \max_{\mathbf{x}\in\operatorname{supp}\rho_0}(|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|), \frac{C_-}{\alpha}\right\} \tag{3.5}$$

where $(\rho_\alpha, \mathbf{u}_\alpha)$ is the smooth solution of (1.1) with interaction kernel ϕ_α, which we assume to exist. Therefore, for any $t \geqslant 0$ and any $\mathbf{x}, \mathbf{y} \in \operatorname{supp}\rho_\alpha(\cdot,t)$, we have

$$|\mathbf{x}-\mathbf{y}| \leqslant |\mathbf{x}| + |\mathbf{y}| \leqslant 2\max\left\{C_+ \cdot \max_{\mathbf{x}\in\operatorname{supp}\rho_0}(|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|), \frac{C_-}{\alpha}\right\}. \tag{3.6}$$

By definition,

$$\frac{C_-}{\alpha} = \frac{C_-}{\phi(r_0)} \leqslant \frac{r_0}{2}. \tag{3.7}$$

Together with (3.4), we obtain that $|\mathbf{x}-\mathbf{y}| \leqslant r_0$, for which, by the monotonicity of ϕ, $\phi(|\mathbf{x}-\mathbf{y}|) \geqslant \phi(r_0) = \alpha$, but for this, $\mathbf{x}, \mathbf{y}$, which persist with a ball of diameter r_0, we have $\phi(|\mathbf{x}-\mathbf{y}|) = \phi_\alpha(|\mathbf{x}-\mathbf{y}|)$, so the dynamics of $(\rho_\alpha, \mathbf{u}_\alpha)$ coincides with $(\rho, \mathbf{u})$. □

Remark 3.3. For the special case $\phi(r) = \dfrac{\phi_+}{(1+r^2)^{\beta/2}}$ with $\beta < 1$, the proof of Corollary 3.2 shows that one could take

$$\alpha = \phi(r_0), \quad r_0 = \max\left\{4\left(\frac{C_-}{\phi_+}\right)^{\frac{1}{1-\beta}}, 2C_+ \cdot \max_{\mathbf{x}\in\operatorname{supp}\rho_0}(|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|)\right\}. \tag{3.8}$$

Therefore, the lower cut-off at α, which depends on β, m_0, ϕ_+ and the initial data, gets smaller when β approaches 1.

The following proposition asserts the uniform bounds (3.3) exist for the *restrictive* class of kernels bounded from below, under very mild conditions on U:

Proposition 3.4. *Assume that the potential U satisfies*

$$\frac{a}{2}|\mathbf{x}|^2 \leqslant U(\mathbf{x}) \leqslant \frac{A}{2}|\mathbf{x}|^2, \quad a|\mathbf{x}| \leqslant |\nabla U(\mathbf{x})| \leqslant A|\mathbf{x}|, \quad \forall \mathbf{x} \in \Omega, \quad 0 < a \leqslant A. \tag{3.9}$$

Consider the alignment system (1.1),(3.9) *with an interaction kernel which is assumed to be bounded from below,* (3.2)*. Then there exist constants $C_{\pm}$, depending on U, ϕ_+, m_0 and E_0, such that* (3.3) *holds.*

Remark 3.5. We note in passing that if U is strictly convex potential satisfying (1.6) then (3.9) follows. Indeed, assuming without loss of generality, that U has a global minimum at the origin so that $U(0) = \nabla U(0) = 0$, and expressing $\nabla U(\mathbf{x}) = \int_0^1 \nabla^2 U(s\mathbf{x})\mathbf{x}\,\mathrm{d}s$ we find $|\nabla U(\mathbf{x})| \leqslant \int_0^1 A|\mathbf{x}|\,\mathrm{d}s = A|\mathbf{x}|$ while strict convexity implies

$$\mathbf{x} \cdot \nabla U(\mathbf{x}) = \int_0^1 \mathbf{x}^\top \nabla^2 U(s\mathbf{x})\mathbf{x}\,\mathrm{d}s \geqslant a|\mathbf{x}|^2 \quad \rightsquigarrow \quad |\nabla U(\mathbf{x})| \geqslant a|\mathbf{x}|;$$

moreover, expressing $U(\mathbf{x}) = \int_0^1 \nabla U(s\mathbf{x}) \cdot \mathbf{x}\,\mathrm{d}s$, we find

$$\frac{a}{2}|\mathbf{x}|^2 = \int_0^1 \frac{1}{s} a|s\mathbf{x}|^2\,\mathrm{d}s \leqslant \int_0^1 \frac{1}{s}\nabla U(s\mathbf{x}) \cdot s\mathbf{x}\,\mathrm{d}s \leqslant U(\mathbf{x}) \leqslant \int_0^1 A|s\mathbf{x}| \cdot |\mathbf{x}|\,\mathrm{d}s = \frac{A}{2}|\mathbf{x}|^2.$$

Thus, the assumed bounds (3.9) follow from (1.6). In fact, (3.9) allows for more general scenarios than uniform convexity including, notably, more complicated topography involving than one local minima. It is straightforward to generalize Proposition 3.4 to the case when (3.9) only holds for sufficiently large $|\mathbf{x}|$. We omit the details.

3.2. Flocking of Smooth Solutions with Convex Potentials

From now on we will restrict attention to uniformly lower bounded kernels, so that ϕ satisfies (3.2), $0 < \phi_- \leqslant \phi(\mathbf{x}) \leqslant \phi_+$. The reduction Lemma 3.2 tells us that the results will automatically apply to the class of all admissible kernels which satisfy (2.2). We develop a hypocoercivity argument, different from the one used in the quadratic case, which gives the following L^2-flocking estimate with algebraic decay rate:

Theorem 3.6. (Flocking with L^2-algebraic decay) *Consider the system* (1.1) *with uniformly convex potential* (1.6), $0 < aI_{d\times d} \leqslant \nabla^2 U(\mathbf{x}) \leqslant AI_{d\times d}$ *and with a C^1 admissible interaction kernel ϕ, (3.1). Assume, in addition, that ϕ satisfies the linear stability condition*

$$m_0\phi(0) > \frac{A}{\sqrt{a}}. \tag{3.10}$$

Let $(\rho, \mathbf{u})$ *be a global smooth solution subject to compactly support* ρ_0*. Then there holds flocking at algebraic rate in both velocity and position, namely, there exist a constant C (with increasing dependence on* $|\phi'|_\infty$*) such that*

$$\delta E(t) := \iint (|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2 + a|\mathbf{x} - \mathbf{y}|^2)\rho(\mathbf{x})\rho(\mathbf{y})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \leqslant \frac{C}{\sqrt{1+t}}\delta E_0. \tag{3.11}$$

The proof of Theorem 3.6 involves three ingredients. First, from the total energy estimate, we show that when t is large enough, most of the agents almost concentrate as a Dirac mass, traveling at almost the same velocity. Second, for such a concentrated state, one can replace ϕ by the *constant* kernel $\phi(0)$ without affecting the dynamics too much, which in turn implies that the agents near the Dirac mass will be attracted to it, consult Theorem 3.7 below. Third, this gives some monotonicity of the energy dissipation rate, which in turn gives (3.11).

The L^∞ counterpart of Theorem 3.6 is still open. If one could obtain an L^∞ flocking estimate, then it might be possible to have flocking estimates for ϕ with thinner tails, similar to what was done in Sect. 2.

The origin of the stability condition (3.10) can be traced to the case of a *constant kernel*, ϕ, where the algebraic convergence stated in Theorem 3.6 is in fact improved to exponential rate.

Theorem 3.7. *(***Flocking with** L^2**-exponential decay– constant** ϕ*) Let* $(\rho, \mathbf{u})$ *subject to compactly supported* ρ_0 *be a global smooth solution of* (1.1) *with uniformly convex potential* (1.6)*,* $0 < aI_{d\times d} \leqslant \nabla^2 U(\mathbf{x}) \leqslant AI_{d\times d}$*, and assume that the interaction kernel* ϕ *is constant satisfying*

$$m_0\phi > \frac{A}{\sqrt{a}}. \tag{3.12}$$

Then it undergoes unconditional flocking at exponential rate in both velocity and position: there exist $\lambda > 0$ *and* $C > 0$ *depend on* $a, A, m_0\phi$ *such that*

$$\delta E(t) \leqslant C \cdot \delta E_0 \cdot e^{-\lambda t}. \tag{3.13}$$

Remark 3.8. One may wonder about the necessity of the stability condition (3.10). In fact, already in the simplest case of a constant ϕ where the Cucker–Smale (1.2) is reduced to

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \phi \cdot (\bar{\mathbf{v}} - \mathbf{v}_i) - \nabla U(\mathbf{x}_i) \end{cases} \qquad \bar{\mathbf{v}} := \frac{1}{N}\sum_j \mathbf{v}_j, \tag{3.14}$$

one may encounter 'orbital instability', where arbitrarily small initial fluctuations $|\mathbf{x}_i(0) - \mathbf{x}_j(0)| + |\mathbf{v}_i(0) - \mathbf{v}_j(0)|$ subject to 1d non-convex potential may grow to be $\mathcal{O}(1)$ at some time. The stability condition (3.10) guarantees, in the case of convex potentials, strong enough alignment that prevents scattering and eventual flocking. The question of the precise necessary stability condition vis a vis convexity remains open.

4. Existence of Global Smooth Solutions

The proof of the existence of smooth solutions proceeds in two parallel tracks. We seek smooth solutions of (1.1), $\big(\rho, \mathbf{u}\big)(\mathbf{x}, t)$, restricted to $\mathbf{x} \in \text{supp}\, \rho(\cdot, t)$. This notion of a *restricted solution* ignores the possibly non-trivial velocity field induced by the non-local alignment term in $(1.1)_2$ in the vacuous region $\mathbf{x} \notin \text{supp}\, \{\rho(\cdot, t)\}$. To this end, (1.1) is interpreted in its Lagrangian formulation e.g., [10, §2], where $\text{supp}\, \rho(\cdot, t) = X(t, \text{supp}\, \rho_0)$ is the pushfoward of the initial compact support, $\text{supp}\, \rho_0 \subset \Omega$, by the characteristic flow

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t, \alpha) = \mathbf{u}(X(t, \alpha), t), \quad X(0, \alpha) = \alpha. \tag{4.1}$$

Expressed in terms of the Lagrangian velocity $\mathbf{v}(t, \alpha) := \mathbf{u}(X(t, \alpha), t)$, (1.1) recasts into the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t, \alpha) = \mathbf{v}(t, \alpha), \quad \alpha \in \text{supp}\, \rho_0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}(t, \alpha) = \displaystyle\int \phi(|X(t, \alpha) - X(t, \beta)|)(\mathbf{v}(t, \alpha) - \mathbf{v}(t, \beta))\rho_0(\beta)\, \mathrm{d}\beta - \nabla U(X(t, \alpha)). \end{cases} \tag{4.2}$$

Observe that smooth restricted solutions are unique as long as the flow is well posted. Thus, making sense of such restricted solutions requires a globally defined flow map, $X(t, \alpha)$ for $\alpha \in \Omega = \mathbb{R}^d$ or $= \mathbb{T}^d$, in order to guarantee that a well-defined $\text{supp}\, \{\rho(\cdot, t)\}$ has a global-in-time life-span, i.e., that the evolution of $\text{supp}\, \{\rho(\cdot, t)\}$ is secured away from self-intersections. This global flow is attached to a second notion of a solution, simply called *global solution*, where (1.1) is satisfied throughout space, $(\mathbf{x}, t) \in (\Omega \times \mathbb{R}_{\geqslant 0})$.

Clearly, a globally defined solution yields a restricted solution, simply by restricting $(\rho, \mathbf{u})$ on $\text{supp}\, \rho(t, \cdot)$. The converse, however, requires a proper process of extension which needs not hold in general. It was worked in the 1D potential-free case in [10]. Here we shall work out the extension process of multi-dimensional extension in the presence of external potential. The following extension lemma is at the heart of matter :

Lemma 4.1. (Extension procedure) *Assume that* (1.1) *satisfies the a priori bound*

$$\max_{\mathbf{x} \in supp\ \rho(\cdot, t)} |\mathbf{x}| \leqslant C_1, \quad \forall t \geqslant 0. \tag{4.3}$$

Fix $\mathsf{R}_\infty > 2C_1$ *and set the cut-off kernel*

$$\widetilde{\phi}(r) := \begin{cases} \phi(r), & r \leqslant \mathsf{R}_\infty \\ \phi(\mathsf{R}_\infty), & r > \mathsf{R}_\infty. \end{cases} \tag{4.4}$$

Denote by $\widetilde{(1.1)}$ *the dynamics of* (1.1) *with* ϕ *replaced by* $\widetilde{\phi}$ *and assume it admits a global smooth solution on* $\Omega \times [0, T)$. *Then a smooth restricted solution of* (1.1), $(\rho, \mathbf{u})$ *defined on* $\text{supp}\, \{\rho(\cdot, t)\}, t \in [0, T)$, *admits a global smooth extension,* $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ *on* Ω, *which is a solution of* $\widetilde{(1.1)}$. *In particular, therefore, if* $\widetilde{(1.1)}$ *admits global smooth solutions for all time, then the lifespan of smooth restricted solutions is infinite.*

Lemma 4.1 tells us that it suffices to focus on the question of existence of global smooth solutions for 'uniformly aligning' model $\widetilde{(1.1)}$ with lower-bounded kernel $\phi_- = \phi(\mathsf{R}_\infty) > 0$. In fact, since $\widetilde{\phi}(r)$ remains constant for $r > \mathsf{R}_\infty > 2C_1$, it dictates a 'finite horizon' of $\widetilde{(1.1)}$ alluded to in [9, §1.3]: beyond this finite horizon, $\widetilde{(1.1)}_2$ is reduced to relaxation

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \phi_- \cdot \Big(\int (\rho\mathbf{u})(\mathbf{y},t)\,\mathrm{d}\mathbf{y} - m_0\mathbf{u}(\mathbf{x},t)\Big) - \nabla U(\mathbf{x}), \quad \mathrm{dist}\{\mathbf{x}, \mathrm{supp}\,\{\rho(\cdot,t)\}\} > \mathsf{R}_\infty.$$

Proof. Let $(\rho, \mathbf{u})(\cdot, t)$ be a restricted smooth solution of (1.1) and let $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ be a global smooth solution of $\widetilde{(1.1)}$ subject to given initial conditions $(\widetilde{\rho}_0, \widetilde{\mathbf{u}}_0)$ which coincide with $(\rho_0, \mathbf{u}_0)$ on supp ρ_0. The alignment kernel on the right of $(1.1)_2$, $\phi(|\mathbf{x}-\mathbf{y}|)$, engages $\mathbf{x}, \mathbf{y} \in \mathrm{supp}\,\rho(\cdot,t)$ and by (4.3), $r = |\mathbf{x}-\mathbf{y}| \leqslant 2C_1$. However, $\widetilde{\phi}(r) = \phi(r)$ for $r \leqslant 2C_1$, hence $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ is also a solution of (1.1) on supp $\{\rho(\cdot,t)\}$, and therefore, its restriction on supp $\{\rho(\cdot,t)\}$ coincides with $(\rho, \mathbf{u})$.
Now assume that $\widetilde{(1.1)}$ admits smooth solutions for all time but a smooth restricted solution of (1.1), $(\rho, \mathbf{u})$, has a finite maximal existence time $T < \infty$. Then, since its extension $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ has a smooth continuation to $t = T$, this leads to a contradiction with the finite lifespan T. Specifically, take $\delta > 0$ small enough. By (4.3) $\max_{\mathbf{x}\in\mathrm{supp}\,\widetilde{\rho}(\cdot,T-\delta)} |\mathbf{x}| \leqslant C_1$, and since $\widetilde{\mathbf{u}}$—the propagation speed of supp $\widetilde{\rho}$, is clearly uniformly bounded, then $\max_{\mathbf{x}\in\mathrm{supp}\,\widetilde{\rho}(\cdot,T)} |\mathbf{x}| \leqslant C_1 + \epsilon/2$. Therefore $(\widetilde{\rho}, \widetilde{\mathbf{u}})$ solves (1.1) on $[0, T]$, which contradicts the maximality of T. □

Smooth solutions of alignment models are secured under certain *critical threshold* conditions on the set of initial configurations, e.g., [1,9,13,15,16]. As noted in the 1D study of [10], these initial threshold conditions should be imposed *throughout* space to guarantee existence of a global smooth flow map (4.1). Below we derive threshold conditions for existence of global smooth solutions of $\widetilde{(1.1)}$. By the extension lemma, this implies existence of restricted smooth solutions for all time.

4.1. Existence of 1D Solutions with General Convex Potentials

We begin with one-dimension (for which $\mathbf{u}, \mathbf{x}$ are scalars, written as u, x). The 1D setup is covered in the next two theorems, where we
(i) guarantee the existence of global smooth solution for a class of sub-critical initial configurations; and
(ii) guarantee a finite time blow-up for a class of super-critical initial configurations.

Theorem 4.2. *(***Global smooth solutions—1D problem***) Consider the one-dimensional hydrodynamic alignment* (1.1) *with sub-quadratic potential U so that*

$$a \leqslant U''(x) \leqslant A < \frac{(m_0\phi_-)^2}{4}, \quad \forall x \in \Omega, \tag{4.5}$$

and subject to sub-critical initial configurations, (ρ_0, u_0)*, such that*

$$\partial_x u_0(x) + (\phi * \rho_0)(x) > \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - A} \quad \textit{for all} \quad x \in \Omega. \tag{4.6}$$

Then (1.1) *admits a restricted smooth solution,* $(\rho, u)(x,t) : (\textit{supp}\,\{\rho(\cdot,t)\}, \mathbb{R}_{\geqslant 0}) \to (L^1_+, W^{1,\infty})$.

Observe that the statement of Theorem 4.2 is independent of the lower-bound a, whether positive of negative: its only role enters in the upper-bound of

$$\max u_x(\cdot,t) \lesssim \max\left\{c_0(\max_x u_0', m_0, \phi_+), \sqrt{\max\{0, -2a\}}\right\}.$$

Theorem 4.3. *(***Finite-time blow-up—1D problem***) Assume* $U''(x) \geqslant a$, $\forall x \in \Omega$. *The 1D problem* (1.1) *admits finite-time blow-up under the following circumstances.*
(i) *If* a *is large enough so that*

$$a > \frac{(m_0\phi_+)^2}{4}, \tag{4.7}$$

then there is unconditional blowup: $\partial_x u$ *blows up to* $-\infty$ *in finite time for any initial data.*
Otherwise, blow-up occurs if the initial data is super-critical in one of the following two configurations:
(ii) *If* $a > 0$ *is not large enough for* (4.7) *to hold*[1]*, then* $\partial_x u$ *blows up to* $-\infty$ *in finite time if there exists* $x \in \Omega$ *such that*

$$\partial_x u_0(x) + (\phi * \rho_0)(x) < \frac{m_0\phi_+}{2} - \sqrt{\frac{(m_0\phi_+)^2}{4} - a}. \tag{4.8}$$

(iii) *If* $a \leqslant 0$*, then* $\partial_x u$ *blows up to* $-\infty$ *in finite time if there exists* $x \in \Omega$ *such that*[2]

$$\partial_x u_0(x) + (\phi * \rho_0)(x) < \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - a}. \tag{4.9}$$

Note that in the potential-free case $U = 0$, Theorems 4.2 and 4.3 amount to the sharp threshold condition $\partial_x u_0(x) + (\phi * \rho_0)(x) \geqslant 0$ which is necessary and sufficient for global 1D regularity, see [1,13]. When the external potential U is added, these theorems indicate that convex U enhances the scenario of blowup in (1.1), while concave U's makes more restrictive scenarios for possible blow up. In other words, *the size of* U'' determines the influence of the external potential on the threshold for the existence of global smooth solution.

[1] Notice that in this condition the RHS in (4.8) is positive.
[2] Notice that in this condition the RHS of (4.9) is negative.

4.2. Existence of 2D solutions with general convex potentials

For the existence of global smooth solution for general external potentials, one faces the difficulty of lack of exponential decaying flocking estimate in the finite horizon region, and the contribution from (the variations of) $\mathbf{u}$ to the dynamics, though bounded, may accumulate over time. Interestingly, we discover that the issue can be resolved by strengthening the critical threshold, requiring that $\max_{\mathbf{x}} |\mathbf{u}(\mathbf{x}, t)|$ remains uniformly bounded in time and the quantity $\nabla \cdot \mathbf{u}_0 + \phi * \rho_0$ remains above certain positive threshold. In the special case of quadratic potential, this lower bound is further relaxed to zero, recovering a similar threshold as in the potential-free case.

Theorem 4.4. (Global 2D smooth solutions with convex potential) *Consider the two-dimensional system* (1.1) *with sub-quadratic potential* U*,*

$$aI_{2\times 2} \leqslant \nabla^2 U(\mathbf{x}) \leqslant AI_{2\times 2}, \tag{4.10}$$

and subject to initial data $(\rho_0, \mathbf{u}_0)$.[3] *Assume the velocity field satisfies the uniform bound*

$$|\mathbf{u}(\mathbf{x}, t)| \leqslant u_{\max} < \infty \textit{ for all } \mathbf{x} \textit{ such that } dist\{\mathbf{x}, supp\ \{\rho(\cdot, t)\}\} < \mathsf{R}_\infty, \tag{4.11}$$

so that

$$C_{max} := 4m_0|\phi'|_\infty u_{max} + A - a < \frac{m_0^2\phi_-^2}{2} - 2A =: C_A. \tag{4.12}$$

If the initial data $(\rho_0, \mathbf{u}_0)$ *are sub-critical in the sense that the following two conditions hold*

$$\nabla \cdot \mathbf{u}_0(\mathbf{x}) + (\phi * \rho_0)(\mathbf{x}) \geqslant \sqrt{C_A - \sqrt{C_A^2 - C_{max}^2}} \textit{ for all } \mathbf{x} \in \Omega, \tag{4.13a}$$

and the initial spectral gap $(\eta_S)_0$*, is not 'too large' so that,*

$$\max_{\mathbf{x}\in\Omega} |(\eta_S)_0(\mathbf{x})| \leqslant \sqrt{C_A + \sqrt{C_A^2 - C_{max}^2}}, \tag{4.13b}$$

then (1.1) *admits restricted smooth solution* $(\rho, \mathbf{u})(\mathbf{x}, t) : (supp\ \{\rho(\cdot, t)\}, \mathbb{R}_{\geqslant 0}) \to (L_+^1, W^{1,\infty})$.[4]

Notice that Proposition 3.4 already gives an a priori estimate

$$\max_{t\geqslant 0,\, \mathbf{x}\in \text{supp } \rho(\cdot,t)} |\mathbf{u}(\mathbf{x}, t)| = \max\left\{C_+ \cdot \max_{\mathbf{x}\in\text{supp } \rho_0} (|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|), \frac{C_-}{\phi_-}\right\} \tag{4.14}$$

for a general class of external potentials, including those satisfying (1.6) (with the further assumption that the unique global minimum of U is $U(0) = 0$, without

[3] Note that U need not be convex, i.e., we allow negative a's.

[4] η_S denotes the difference of the two eigenvalues of $\nabla_S \mathbf{u} := 1/2\big(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\big)$.

loss of generality). To upgrade it to a uniform bound sought in (4.11), requires the extended solution, $|\widetilde{\mathbf{u}}(\cdot,t)|$, governed by the uniformly aligning model outlined in lemma 4.1, remains uniformly bounded in the bounded horizon region

$$\max_{t\geqslant 0}\big\{|\mathbf{u}(\mathbf{x},t)|:\ 0<\text{dist}\{\mathbf{x},\text{supp}\,\{\rho(\cdot,t)\}\}<\mathsf{R}_\infty\big\}\leqslant u_{\max}.$$

Remark 4.5. (*Global smooth solutions with 2D quadratic potentials*) The case of quadratic potential carries out further simplification of the critical threshold conditions (4.13). Specifically, the first term in C_{max} originates with the bound of the residual term, see (6.8) below

$$|R_{ij}|\leqslant 2m_0|\phi'|_\infty u_{\max},\qquad R_{ij}=\int\partial_j\phi(|\mathbf{x}-\mathbf{y}|)\big(u_i(\mathbf{x},t)-u_i(\mathbf{y},t)\big)\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

In the quadratic case, we have the exponential flocking towards the zero mean velocity (2.4), $\max_{\mathbf{y}\in\text{supp}\,\{\rho(\cdot,t)\}}|\mathbf{u}(\mathbf{y},t)|\lesssim e^{-\lambda t/4}$. Moreover, it can shown (we omit the details) that in this case, the velocity field along particle path remains uniformly integrable in time

$$\int_{t=0}^{\infty}|R_{ij}(X(t,\alpha),t)|\,\mathrm{d}t\leqslant|\phi'|_\infty C_R,$$

which implies the uniform-in-time bound on the spectral gap:

$$\max_{\mathbf{x}\in\Omega}|\eta_S(\mathbf{x},t)|\leqslant\max_{\mathbf{x}\in\Omega}|(\eta_S)_0|+2|\phi'|_\infty C_R. \tag{4.15}$$

Consequently, in the quadratic case we can discard of the R_{ij} bound and are left with $C_{max}=A-a=0$ at the expense of having modified C_a. We end up with the simplified critical threshold (see remark 6.1 below)

$$\nabla\cdot\mathbf{u}_0(\mathbf{x})+(\phi*\rho_0)(\mathbf{x})\geqslant 0\quad\text{for all}\ \ \mathbf{x}\in\Omega, \tag{4.16a}$$

$$\max_{\mathbf{x}\in\Omega}|(\eta_S)_0(\mathbf{x})|+2|\phi'|_\infty C_R\leqslant\sqrt{2C_a},\qquad C_a=\frac{m_0^2\phi_-^2}{2}-2a>0. \tag{4.16b}$$

5. Proof of Main Results: Hypocoercivity Bounds

5.1. Quadartic Potentials

We prove Theorems 2.3 and 2.5, making use of the uniform lower-bound of $\phi(r)\geqslant\phi_-$ in (2.12).

Proof of Theorem 2.3. Since the fluctuations functional $\delta E(\rho,\mathbf{u})$ in (3.13) satisfies $\delta E(\rho,\mathbf{u})=\delta E(\hat\rho,\hat{\mathbf{u}})$, it suffices to study (2.1) with $(\mathbf{x}_c(0)=0,\mathbf{u}_c(0))=(0,0)\rightsquigarrow(\mathbf{x}_c(t),\mathbf{u}_c(t))\equiv(0,0)$, for which the fluctuations coincide with (multiple of) the energy

$$\delta E(t)=4m_0\int\Big(\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2+\frac{a}{2}|\mathbf{x}|^2\Big)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x}. \tag{5.1}$$

As before, the energy decay is dictated by the minimal value $\min_{\mathbf{x},\mathbf{y}\in\text{supp}\,\{\rho(\cdot,t)\}} \phi(|\mathbf{x}-\mathbf{y}|) \geqslant \phi_- := \phi(\sqrt{8R_0/a})$,

$$
\begin{aligned}
\partial_t \int \Big(\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2 + \frac{a}{2}|\mathbf{x}|^2\Big)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} &= -\frac{1}{2}\iint \phi(\mathbf{x}-\mathbf{y})|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leqslant -\frac{\phi_-}{2}\iint |\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} = -m_0\phi_- \int |\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{5.2}
$$

Then we compute the cross term

$$
\begin{aligned}
&\partial_t \int \mathbf{u}(\mathbf{x},t)\cdot\mathbf{x}\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&= -\int (\mathbf{u}(\mathbf{x},t)\cdot\mathbf{x})\nabla\cdot(\rho\mathbf{u})\,\mathrm{d}\mathbf{x} + \int \mathbf{x}\cdot\Big(-\mathbf{u}\cdot\nabla\mathbf{u} + \int \phi(\mathbf{x}-\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} - a\mathbf{x}\Big)\rho\,\mathrm{d}\mathbf{x} \\
&= -a\int |\mathbf{x}|^2\rho\,\mathrm{d}\mathbf{x} + \int |\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{x} + \iint \phi(\mathbf{x}-\mathbf{y})\mathbf{x}\cdot(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leqslant -a\int |\mathbf{x}|^2\rho\,\mathrm{d}\mathbf{x} + \int |\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{x} + \frac{\phi_+}{2}\iint \Big(\frac{a}{m_0\phi_+}|\mathbf{x}|^2 + \frac{m_0\phi_+}{a}|\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x})|^2\Big)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= -\frac{a}{2}\int |\mathbf{x}|^2\rho\,\mathrm{d}\mathbf{x} + \Big(1+\frac{m_0^2\phi_+^2}{a}\Big)\int |\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

Adding a λ-multiple of this cross term—λ is yet to be determined, we conclude that

$$
\begin{aligned}
&\partial_t \int \Big(\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2 + \frac{a}{2}|\mathbf{x}|^2 + 2\lambda\mathbf{u}(\mathbf{x},t)\cdot\mathbf{x}\Big)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\
&\qquad \leqslant -\Big(m_0\phi_- - 2\lambda\Big(1+\frac{m_0^2\phi_+^2}{a}\Big)\Big)\int |\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{x} - 2\lambda\int \frac{a}{2}|\mathbf{x}|^2\rho\,\mathrm{d}\mathbf{x},
\end{aligned} \tag{5.3}
$$

which means the LHS is a Lyapunov functional if $\lambda > 0$ is small enough; in fact, we set

$$
\lambda = \frac{1}{2}\min\left\{\frac{m_0\phi_-}{(1+\frac{m_0^2\phi_+^2}{a})+\frac{1}{2}}, \frac{\sqrt{a}}{2}\right\}, \tag{5.4}
$$

to conclude that the Lyapunov functional

$$
V(t) := \int \Big(\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2 + \frac{a}{2}|\mathbf{x}|^2 + 2\lambda\mathbf{u}(\mathbf{x},t)\cdot\mathbf{x}\Big)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \tag{5.5}
$$

admits the decay bound $\frac{\mathrm{d}}{\mathrm{d}t}V(t) \leqslant -\lambda\int (|\mathbf{u}|^2 + a|\mathbf{x}|^2)\rho\,\mathrm{d}\mathbf{x}$. Noting that this modified Lyapunov functional is comparable to the energy functional (recall $2\lambda \leqslant \sqrt{a}/2$)

$$
\frac{\delta E}{4m_0} = \frac{1}{2}\int (|\mathbf{u}|^2 + a|\mathbf{x}|^2)\rho\,\mathrm{d}\mathbf{x} \leqslant V(t) \leqslant \int (|\mathbf{u}|^2 + a|\mathbf{x}|^2)\rho\,\mathrm{d}\mathbf{x} = \frac{\delta E}{2m_0},
$$

we conclude its *dissipativity* $V'(t) \leqslant -\lambda V(t)$, which in turn proves the L^2-flocking bound (3.13), $\frac{\delta E(t)}{4m_0} \leqslant V(t) \leqslant \frac{\delta E_0}{2m_0}e^{-\lambda t}$. □

Proof of Theorem 2.5. We define the perturbed energy functional

$$F_1(\mathbf{x},t) := \frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2 + \frac{a}{2}|\mathbf{x}|^2 + 2\lambda_1 \mathbf{u}(\mathbf{x},t)\cdot\mathbf{x} \tag{5.6}$$

where $\lambda_1 > 0$ is yet to be determined. Then we compute the derivative of F_1 along characteristics:

$$\begin{aligned} F_1' &= \partial_t F_1 + \mathbf{u}\cdot\nabla F_1 \\ &= (\mathbf{u}+2\lambda_1\mathbf{x})\cdot\left(-\mathbf{u}\cdot\nabla\mathbf{u} + \int \phi(\mathbf{x}-\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} - a\mathbf{x}\right) \\ &\quad + \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + a\mathbf{u}\cdot\mathbf{x} + 2\lambda_1|\mathbf{u}|^2 + 2\lambda_1\mathbf{x}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \\ &= -2\lambda_1 a|\mathbf{x}|^2 + (\mathbf{u}+2\lambda_1\mathbf{x})\cdot\left(\int \phi(\mathbf{x}-\mathbf{y})(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\right) + 2\lambda_1|\mathbf{u}|^2 \\ &= -2\lambda_1 a|\mathbf{x}|^2 - (\phi*\rho)|\mathbf{u}|^2 + \mathbf{u}\cdot(\phi*(\rho\mathbf{u})) + 2\lambda_1\mathbf{x}\cdot((\phi*(\rho\mathbf{u})) - (\phi*\rho)\mathbf{u}) + 2\lambda_1|\mathbf{u}|^2. \end{aligned} \tag{5.7}$$

We bound the convolution terms of the right of (5.7): by (2.7) we have $m_0\phi_- \leqslant (\phi*\rho)(\mathbf{x}) \leqslant m_0\phi_+$; further, by (5.1), $\delta E(t) > 4m_0E_k(t)$, and the exponential decay of the L^2-Lyapunov functional, (3.13), we have that

$$\begin{aligned} |(\phi*(\rho\mathbf{u}))(\mathbf{x})| &= \left|\int \phi(\mathbf{x}-\mathbf{y})\mathbf{u}(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\right| \\ &\leqslant \phi_+\int|\mathbf{u}(\mathbf{y})|\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \leqslant \phi_+\sqrt{m_0}\left(\int|\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{y}\right)^{1/2} \leqslant \phi_+\sqrt{m_0}\frac{\sqrt{2\delta E_0}}{\sqrt{2m_0}}e^{-\lambda t/2}. \end{aligned}$$

We conclude that the perturbed energy functional F_1 does not exceed

$$\begin{aligned} F_1' &\leqslant -2\lambda_1 a|\mathbf{x}|^2 - m_0\phi_-|\mathbf{u}|^2 + \left(\frac{m_0\phi_-}{2}|\mathbf{u}|^2 + \frac{\phi_+^2}{2m_0\phi_-}\delta E_0\cdot e^{-\lambda t}\right) \\ &\quad + \left(\frac{\lambda_1 a}{2}|\mathbf{x}|^2 + \frac{2\lambda_1\phi_+^2}{a}\delta E_0\cdot e^{-\lambda t}\right) + 2\lambda_1 m_0\phi_+\left(\frac{a}{4m_0\phi_+}|\mathbf{x}|^2 + \frac{m_0\phi_+}{a}|\mathbf{u}|^2\right) + 2\lambda_1|\mathbf{u}|^2 \\ &\leqslant -\lambda_1 a|\mathbf{x}|^2 - \left(\frac{m_0\phi_-}{2} - 2\lambda_1\left(1+\frac{m_0^2\phi_+^2}{a}\right)\right)|\mathbf{u}|^2 + C_0\cdot\delta E_0\cdot e^{-\lambda t} \end{aligned}$$

with

$$C_0 = \left(\frac{1}{2m_0\phi_-} + \frac{2\lambda_1}{a}\right)\phi_+^2. \tag{5.8}$$

Therefore, by choosing λ_1 as

$$\lambda_1 := \frac{1}{4}\min\left\{\frac{m_0\phi_-}{(1+\frac{m_0^2\phi_+^2+1}{a})+\frac{1}{4}}, \frac{\sqrt{a}}{2}\right\} \geqslant \frac{\lambda}{2}, \tag{5.9}$$

one has

$$F_1'(t) \leqslant -\frac{\lambda}{2}(a|\mathbf{x}|^2 + |\mathbf{u}|^2) + C_0\cdot\delta E_0\cdot e^{-\lambda t} \leqslant -\frac{\lambda}{2}F_1(t) + C_0\cdot\delta E_0\cdot e^{-\lambda t},$$

with the explicit bound $F_1(t) \leqslant e^{-\lambda t/2}(F_1(0) + 2C_0\cdot\delta E_0/\lambda)$. Finally, since $\max\limits_{\mathbf{x}\in\operatorname{supp}\{\rho(\cdot,t)\}} F_1(\mathbf{x},t)$ is comparable with δP, namely $\frac{1}{8}\delta P \leqslant F_1 \leqslant \frac{1}{2}\delta P$ and $\delta E \leqslant m_0^2\cdot\delta P$, the result (2.15) follows with $C_\infty = 4(1+4C_0m_0^2/\lambda)$. □

5.2. General Convex Potentials

We begin with the proof of Proposition 3.4, which confirms the the uniform bound $|\mathbf{u}| + |\mathbf{x}|$ in terms of $\mathcal{O}(1/\phi_-)$. The main idea is to study the evolution of the particle energy $\frac{1}{2}|\mathbf{u}(\mathbf{x}, t)|^2 + U(\mathbf{x})$ along characteristics, and conduct hypocoercivity arguments to handle the possible increment of the particle energy due to the Cucker–Smale interaction.

Proof of Proposition 3.4. We define

$$F(\mathbf{x}, t) = \frac{1}{2}|\mathbf{u}(\mathbf{x}, t)|^2 + U(\mathbf{x}) + c\mathbf{u}(\mathbf{x}, t) \cdot \nabla U(\mathbf{x}), \tag{5.10}$$

with $c > 0$ being small, to be chosen. Then it follows from the assumptions on U that

$$\begin{aligned} F - \frac{1}{4}|\mathbf{u}|^2 - \frac{a}{4}|\mathbf{x}|^2 &= \frac{1}{4}|\mathbf{u}|^2 + (U(\mathbf{x}) - \frac{a}{4}|\mathbf{x}|^2) + c\mathbf{u}(\mathbf{x}, t) \cdot \nabla U(\mathbf{x}) \\ &\geqslant \frac{1}{4}|\mathbf{u}|^2 + \frac{a}{4}|\mathbf{x}|^2 - \frac{c}{2}(\frac{1}{4c}|\mathbf{u}|^2 + 4c|\nabla U(\mathbf{x})|^2) \\ &\geqslant \frac{1}{8}|\mathbf{u}|^2 + \frac{a}{4}|\mathbf{x}|^2 - 2c^2A^2|\mathbf{x}|^2 \geqslant 0. \end{aligned} \tag{5.11}$$

Now fix $c \leqslant \sqrt{\dfrac{a}{8A^2}}$. Then we compute the derivative of F along characteristics:

$$\begin{aligned} F' &= \partial_t F + \mathbf{u} \cdot \nabla F \\ &= (\mathbf{u} + c\nabla U(\mathbf{x})) \cdot \Big(-\mathbf{u} \cdot \nabla \mathbf{u} + \int \phi(\mathbf{x} - \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} - \nabla U(\mathbf{x})\Big) \\ &\quad + \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \mathbf{u} \cdot \nabla U(\mathbf{x}) + c\mathbf{u}^\top \nabla^2 U(\mathbf{x})\mathbf{u} + c\nabla U(\mathbf{x}) \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \\ &= -c|\nabla U(\mathbf{x})|^2 + (\mathbf{u} + c\nabla U(\mathbf{x})) \cdot \Big(\int \phi(\mathbf{x} - \mathbf{y})(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big) + c\mathbf{u}^\top \nabla^2 U(\mathbf{x})\mathbf{u} \\ &= -c|\nabla U(\mathbf{x})|^2 - (\phi * \rho)|\mathbf{u}|^2 + \mathbf{u} \cdot (\phi * (\rho\mathbf{u})) + c\nabla U(\mathbf{x}) \cdot ((\phi * (\rho\mathbf{u})) - (\phi * \rho)\mathbf{u}) \\ &\quad + c\mathbf{u}^\top \nabla^2 U(\mathbf{x})\mathbf{u}. \end{aligned} \tag{5.12}$$

Noticing that $m_0\phi_- \leqslant (\phi * \rho)(\mathbf{x}) \leqslant m_0\phi_+$, the convolution term on the right of (5.12) can be upper-bounded in terms of the dissipating energy $E(t)$ in (1.4)

$$\begin{aligned} |(\phi * (\rho\mathbf{u}))(\mathbf{x})| &= \left|\int \phi(\mathbf{x} - \mathbf{y})\mathbf{u}(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}\right| \leqslant \phi_+ \int |\mathbf{u}(\mathbf{y})|\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &\leqslant \phi_+ \int |\mathbf{u}(\mathbf{y})|\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \leqslant \phi_+ m_0^{1/2}\Big(\int |\mathbf{u}|^2\rho\,\mathrm{d}\mathbf{y}\Big)^{1/2} \leqslant 2\phi_+ m_0^{1/2}E^{1/2}(0), \quad \forall \mathbf{x}. \end{aligned}$$

Therefore

$$\begin{aligned} F' &\leqslant -c|\nabla U(\mathbf{x})|^2 - m_0\phi_-|\mathbf{u}|^2 + (\frac{m_0\phi_-}{2}|\mathbf{u}|^2 + \frac{2}{m_0\phi_-}\phi_+^2 m_0 E_0) \\ &\quad + (\frac{c}{4}|\nabla U(\mathbf{x})|^2 + 4c\phi_+^2 m_0 E_0) + (cm_0\phi_+)(\frac{1}{4m_0\phi_+}|\nabla U(\mathbf{x})|^2 + m_0\phi_+|\mathbf{u}|^2) + cA|\mathbf{u}|^2 \\ &\leqslant -\frac{c}{2}|\nabla U(\mathbf{x})|^2 - \big(\frac{m_0\phi_-}{2} - c(A + m_0^2\phi_+^2)\big)|\mathbf{u}|^2 + C_0 \end{aligned}$$

with

$$C_0 = \Big(\frac{2}{m_0\phi_-} + 4c\Big)\phi_+^2 m_0 E_0. \tag{5.13}$$

Therefore, by choosing

$$c = \min\left\{\frac{m_0\phi_-}{A + 2(A + m_0^2\phi_+^2)}, \sqrt{\frac{a}{8A^2}}\right\}, \tag{5.14}$$

one has

$$F' \leqslant -\frac{c}{2}(|\nabla U(\mathbf{x})|^2 + A|\mathbf{u}|^2) + C_0. \tag{5.15}$$

Next we notice that

$$\begin{aligned} F &\leqslant \frac{1}{2}|\mathbf{u}|^2 + \frac{A}{2}|\mathbf{x}|^2 + \frac{c}{2}(\frac{1}{c}|\mathbf{u}|^2 + cA^2|\mathbf{x}|^2) \\ &\leqslant \max\left\{1, \frac{1+c^2A}{2}\right\}(|\mathbf{u}|^2 + A|\mathbf{x}|^2) = |\mathbf{u}|^2 + A|\mathbf{x}|^2 \end{aligned}$$

and

$$|\nabla U(\mathbf{x})|^2 + A|\mathbf{u}|^2 \geqslant \min\left\{A, \frac{a^2}{A}\right\}(|\mathbf{u}|^2 + A|\mathbf{x}|^2) = \frac{a^2}{A}(|\mathbf{u}|^2 + A|\mathbf{x}|^2).$$

This means that if

$$F(\mathbf{x}, t) \geqslant \frac{2AC_0}{a^2 c} := C_F, \tag{5.16}$$

then $F' \leqslant 0$. Thus F cannot further increase (along characteristics) if it is larger than C_F. It is clear that $c = \mathcal{O}(\phi_-)$ and $C_0 = \mathcal{O}(1/\phi_-)$ for small ϕ_-. Therefore $C_F = \mathcal{O}(1/\phi_-^2)$.

Therefore, by (5.11) we get

$$\begin{aligned} |\mathbf{u}| + |\mathbf{x}| &\leqslant 2\Big(1 + \frac{1}{\sqrt{a}}\Big)\sqrt{F} \leqslant 2(1 + \frac{1}{\sqrt{a}})\sqrt{\max\{C_F, \max_{\mathbf{x}\in \text{supp}\,\rho_0} F(\mathbf{x}, 0)\}} \\ &\leqslant 2\Big(1 + \frac{1}{\sqrt{a}}\Big)\sqrt{\max\{C_F, \max_{\mathbf{x}\in \text{supp}\,\rho_0} |\mathbf{u}_0(\mathbf{x})|^2 + A|\mathbf{x}|^2\}} \\ &\leqslant \max\left\{C_+ \cdot \max_{\mathbf{x}\in \text{supp}\,\rho_0} (|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|), 2(1 + \frac{1}{\sqrt{a}})\sqrt{C_F},\right\}, \quad C_+ := 2\sqrt{A}\left(1 + \frac{1}{\sqrt{a}}\right), \end{aligned}$$

and the term $2(1 + \frac{1}{\sqrt{a}})\sqrt{C_F}$ scales like $\mathcal{O}(1/\phi_-)$ for small ϕ_-. □

When dealing with convex potential $U(\mathbf{x}) = \frac{a}{2}|\mathbf{x}|^2$ we used the fact that the mean location $\mathbf{x}_c$ and mean velocity $\mathbf{u}_c$ satisfies the closed system, (2.16), which enabled us to convert the measure of L^2-fluctuations into an energy-based functional. In case of general convex potentials, however, the mean location $\mathbf{x}_c$ and mean velocity $\mathbf{u}_c$ do not satisfy a closed system and therefore one cannot reduce the problem with $\mathbf{x}_c = \mathbf{u}_c = 0$, for which δE is equivalent to the total energy. Therefore one cannot using hypocoercivity on the energy estimate to obtain the decay of δE. Instead, we will construct a Lyapunov functional which is equivalent to δE directly. We begin with the case of a constant interaction kernel.

Proof of Theorem 3.7. Recall that we assumed ϕ is constant. Denote $K := m_0\phi$ so that the convolution terms with ϕ amount to simple averaging, $(\phi * f)(\mathbf{x}) = K\int f\,\mathrm{d}\mathbf{x}$. We will use the ρ-weighted quantities

$$\langle f(\mathbf{x},\mathbf{y}), g(\mathbf{x},\mathbf{y})\rangle_\rho := \iint f(\mathbf{x},\mathbf{y})\cdot g(\mathbf{x},\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},\quad |f(\mathbf{x},\mathbf{y})|_\rho^2 := \langle f(\mathbf{x},\mathbf{y}), f(\mathbf{x},\mathbf{y})\rangle$$

for any scalar or vector functions f, g, where we suppress its dependence on t.

We compute the time derivative of the following quantity (where $\beta > 0$ to be determined):

$$F(t) = \frac{K}{2}|\mathbf{x}-\mathbf{y}|_\rho^2 + \langle \mathbf{x}-\mathbf{y}, \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle_\rho + \frac{\beta}{2}|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|_\rho^2 \tag{5.17}$$

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}t} = \iint \Big[&\Big(\frac{K}{2}|\mathbf{x}-\mathbf{y}|^2 + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})) + \frac{\beta}{2}|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\Big)\times \\ &\Big(-\nabla_\mathbf{x}\cdot(\rho(\mathbf{x})\mathbf{u}(\mathbf{x}))\rho(\mathbf{y}) - \nabla_\mathbf{y}\cdot(\rho(\mathbf{y})\mathbf{u}(\mathbf{y}))\rho(\mathbf{x})\Big) \\ &+\Big(\mathbf{x}-\mathbf{y}+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\Big)\cdot\Big(-\mathbf{u}(\mathbf{x})\cdot\nabla_\mathbf{x}\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{y})\cdot\nabla_\mathbf{y}\mathbf{u}(\mathbf{y}) \\ &-K\mathbf{u}(\mathbf{x}) + K\mathbf{u}(\mathbf{y}) - \nabla U(\mathbf{x}) + \nabla U(\mathbf{y})\Big)\rho(\mathbf{x})\rho(\mathbf{y})\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\end{aligned} \tag{5.18}$$

$$\begin{aligned} = \iint \Big[&\Big(K(\mathbf{x}-\mathbf{y}) + \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}) + \nabla_\mathbf{x}\mathbf{u}(\mathbf{x})(\mathbf{x}-\mathbf{y}+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})))\Big)\cdot\mathbf{u}(\mathbf{x}) \\ &+\Big(-K(\mathbf{x}-\mathbf{y}) - (\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})) - \nabla_\mathbf{y}\mathbf{u}(\mathbf{y})(\mathbf{x}-\mathbf{y}+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})))\Big)\cdot\mathbf{u}(\mathbf{y}) \\ &+\Big(\mathbf{x}-\mathbf{y}+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\Big)\cdot\Big(-\mathbf{u}(\mathbf{x})\cdot\nabla_\mathbf{x}\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{y})\cdot\nabla_\mathbf{y}\mathbf{u}(\mathbf{y}) \\ &-K\mathbf{u}(\mathbf{x}) + K\mathbf{u}(\mathbf{y}) - \nabla U(\mathbf{x}) + \nabla U(\mathbf{y})\Big)\Big]\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\ = \iint \Big[&-(K\beta-1)|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2 - (\mathbf{x}-\mathbf{y})\cdot(\nabla U(\mathbf{x})-\nabla U(\mathbf{y})) \\ &-\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot(\nabla U(\mathbf{x})-\nabla U(\mathbf{y}))\Big]\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}.\end{aligned} \tag{5.19}$$

Notice that

$$(\mathbf{x}-\mathbf{y})\cdot(\nabla U(\mathbf{x})-\nabla U(\mathbf{y})) = \int_0^1 (\mathbf{x}-\mathbf{y})^\top\nabla^2 U((1-\theta)\mathbf{y}+\theta\mathbf{x})(\mathbf{x}-\mathbf{y})\,\mathrm{d}\theta \geqslant a|\mathbf{x}-\mathbf{y}|^2, \tag{5.20}$$

and similarly that

$$|(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot(\nabla U(\mathbf{x})-\nabla U(\mathbf{y}))| \leqslant A|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|\cdot|\mathbf{x}-\mathbf{y}|. \tag{5.21}$$

Then we obtain

$$(5.19) \leqslant -(K\beta-1)|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|_\rho^2 - a|\mathbf{x}-\mathbf{y}|_\rho^2 + A\beta|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|_\rho\cdot|\mathbf{x}-\mathbf{y}|_\rho. \tag{5.22}$$

We want to choose a β such that the RHS of (5.22), as a quadratic form, is negative-definite, i.e., its discriminant is

$$A^2\beta^2 - 4a(K\beta-1) = A^2\beta^2 - 4aK\beta + 4a < 0. \tag{5.23}$$

This is possible, since by (3.12) $(4aK)^2 - 16A^2a = 16a(aK^2 - A^2) > 0$, and we can take

$$\beta := \frac{2aK}{A^2}, \tag{5.24}$$

and then

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leqslant -\mu_1\big(|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|_\rho^2 + a|\mathbf{x} - \mathbf{y}|_\rho^2\big) = -\mu_1\delta E, \tag{5.25}$$

for some $\mu_1 > 0$ (whose explicit form will be given in Remark 5.2). With this choice of β, the discriminant of the LHS of (5.19) is

$$1^2 - 4\frac{K}{2}\frac{\beta}{2} = 1 - \frac{2aK^2}{A^2} < 1 - \frac{2aA^2}{aA^2} = -1,$$

and thus it is positive definite. One can estimate F above and below by $\mu_3\delta E \leqslant F \leqslant \mu_2\delta E$ for some $\mu_2 > \mu_3 > 0$. Therefore $F(t) \leqslant F(0)e^{-\frac{\mu_1}{\mu_2}}$ and then

$$\delta E(t) \leqslant \frac{1}{\mu_3}F(t) \leqslant \frac{1}{\mu_3}F(0)e^{-\frac{\mu_1}{\mu_2}} \leqslant \frac{\mu_2}{\mu_3}\delta E(0)e^{-\frac{\mu_1}{\mu_2}}.$$

□

Remark 5.1. The key idea of the proof is the cancellation of the term $K(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))$ in (5.19). For large K, this term is $O(K)$, while the two good terms are $\mathcal{O}(K)$ and $\mathcal{O}(1)$ respectively. If this term was not cancelled, then it could not be absorbed by the good terms.

In fact, the positive/negative $K(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y}))$ terms are given by the time derivative of $\frac{K}{2}|\mathbf{x} - \mathbf{y}|_\rho^2$ and $\langle \mathbf{x} - \mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\rangle_\rho$, respectively. Therefore, in the Lyapunov functional, one cannot change the coefficient ratio between a square term $|\mathbf{x} - \mathbf{y}|_\rho^2$ and the cross term $\langle \mathbf{x} - \mathbf{y}, \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\rangle_\rho$. This is an essential difference from the standard hypocoercivity theory (for which the cross term can be arbitrarily small).

Remark 5.2. One can obtain the explicit expression of μ_1 from (5.22) by letting the good terms absorb the bad term exactly, i.e., solving the quadratic equation

$$(K\beta - 1 - \mu_1)(a - a\mu_1) = \frac{A^2\beta^2}{4}$$

yields $\mu_1 = \frac{aK^2}{A^2} - \sqrt{\frac{a^2K^4}{A^4} - \frac{aK^2}{A^2} + 1} > 0$; similarly, one obtains $\mu_{2,3}$ as

$$\mu_{2,3} = \frac{1}{2a}\left(\frac{a^2K}{A^2} + \frac{K}{2} \pm \sqrt{\Big(\frac{a^2K}{A^2} + \frac{K}{2}\Big)^2 - 4a\Big(\frac{aK^2}{2A^2} - \frac{1}{4}\Big)}\right) > 0.$$

To handle the case with non-constant ϕ, we start with the following lemma:

Lemma 5.3. *With the same assumptions as Theorem 3.6, further assume the apriori uniform bound on the velocity field*

$$\max_{t\geqslant 0,\, \mathbf{x}\in supp\, \{\rho(\cdot,t)\}} (|\mathbf{u}(\mathbf{x},t)| + |\mathbf{x}|) \leqslant u_{\max} < \infty. \tag{5.26}$$

Fix any ϵ_1 small enough. Assume that at time t_0, one can write supp $\rho(\cdot, t_0)$ into the disjoint union of two subsets:

$$supp\ \rho(\cdot, t_0) = S_1 \cup S_2, \quad S_1 \cap S_2 = \emptyset, \tag{5.27}$$

which satisfies

$$\int_{S_2} \rho(\mathbf{x}, t_0)\, \mathrm{d}\mathbf{x} \leqslant \eta\epsilon_1, \tag{5.28}$$

with $\eta > 0$ depending on ϕ, U, $u_{\max}$ but independent of ϵ_1, and

$$\delta P(t_0; S_1) := \sup_{\mathbf{x},\mathbf{y}\in S_1} (|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2 + a|\mathbf{x}-\mathbf{y}|^2) \leqslant \epsilon_1. \tag{5.29}$$

Let $S_1(t)$, $S_2(t)$ be the image of S_1, S_2 under the characteristic flow map from t_0 to t. Then

$$\delta P(t; S_1(t)) \leqslant \epsilon_1, \quad \forall t \geqslant t_0 \tag{5.30}$$

In this lemma, S_1 consists of the particles which are almost concentrated as a Dirac mass, and S_2 the other particles, which can be far away from the Dirac mass, but whose total mass is small. The lemma claims that the Dirac mass will not scatter around for all time. This can be viewed as a perturbative extension of the constant ϕ case, applied to the Dirac mass S_1.

Also notice that (3.3) gives (5.26) with $u_{\max}$ being the RHS of (3.3).

Proof. Define

$$F(\mathbf{x},\mathbf{y},t) := \frac{K}{2}|\mathbf{x}-\mathbf{y}|^2 + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)) + \frac{\beta}{2}|\mathbf{u}(\mathbf{x},t)-\mathbf{u}(\mathbf{y},t)|^2,$$
$$F_\infty(t; S) = \max_{\mathbf{x},\mathbf{y}\in S} F(\mathbf{x},\mathbf{y},t)$$

where $K = m_0\phi(0)$, and the choice of β is the same as the proof of Theorem 3.7, so that F is a positive-definite quadratic form. Fix two characteristics $\mathbf{x}(t)$ and $\mathbf{y}(t)$ with $\mathbf{x}(t_0), \mathbf{y}(t_0) \in S_1$, and we compute the time derivative of F along characteristics:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}F(\mathbf{x}(t),\mathbf{y}(t),t)\\
&\quad=\partial_t F+\mathbf{u}(\mathbf{x})\cdot\nabla_{\mathbf{x}}F+\mathbf{u}(\mathbf{y})\cdot\nabla_{\mathbf{y}}F\\
&\quad=((\mathbf{x}-\mathbf{y})+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})))\cdot\Big(-\mathbf{u}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x})+\mathbf{u}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{y})\\
&\quad+\int\phi(\mathbf{x}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}-\int\phi(\mathbf{y}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{y}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\Big)\\
&\quad+\mathbf{u}(\mathbf{x})\cdot(K(\mathbf{x}-\mathbf{y})+(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))+(\mathbf{x}-\mathbf{y})\cdot\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x})+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x}))\\
&\quad-\mathbf{u}(\mathbf{y})\cdot\big(K(\mathbf{x}-\mathbf{y})+(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))+(\mathbf{x}-\mathbf{y})\cdot\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y})+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y})\big)\\
&\quad=-(K\beta-1)|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\\
&\quad-(\mathbf{x}-\mathbf{y})\cdot(\nabla U(\mathbf{x})-\nabla U(\mathbf{y}))-\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot(\nabla U(\mathbf{x})-\nabla U(\mathbf{y}))\\
&\quad+\big((\mathbf{x}-\mathbf{y})+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\big)\cdot\Big(\int(\phi(\mathbf{x}-\mathbf{z})-\phi(0))(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\\
&\quad-\int(\phi(\mathbf{y}-\mathbf{z})-\phi(0))(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{y}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\Big).
\end{aligned}$$

The first three terms are less than a negative definite quadratic form, as in the proof of Theorem 3.7. Now we handle the last term, which results from the fact that ϕ is not constant.

By the definition of $S_1(t)$, one has $\mathbf{x}(t),\mathbf{y}(t)\in S_1(t)$ for all $t\geqslant t_0$. If $\mathbf{z}\in S_1(t)$, then $|\mathbf{x}-\mathbf{z}|\leqslant\sqrt{\delta P(t;S_1(t))/a}\leqslant C_1\sqrt{F_\infty(t;S_1(t))}$ for some constant C_1, since F is comparable with $|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2+a|\mathbf{x}-\mathbf{y}|^2$. Therefore

$$|\phi(\mathbf{x}-\mathbf{z})-\phi(0)|\leqslant|\phi'|_\infty C_1\sqrt{F_\infty(t;S_1(t))}.\tag{5.31}$$

It follows that

$$\left|((\mathbf{x}-\mathbf{y})+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})))\cdot\int_{S_1(t)}(\phi(\mathbf{x}-\mathbf{z})-\phi(0))(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\right|\leqslant C_2F_\infty(t;S_1(t))^{3/2},$$

with $C_2=(1/\sqrt{a}+\beta)m_0|\phi'|_\infty C_1^3$.

If $\mathbf{z}\in S_2(t)$, then we use the uniform bound (5.26) to estimate $\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x})$, and obtain

$$\left|((\mathbf{x}-\mathbf{y})+\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})))\cdot\int_{S_2(t)}(\phi(\mathbf{x}-\mathbf{z})-\phi(0))(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\right|\leqslant C_3\eta\epsilon_1F_\infty(t;S_1(t))^{1/2},$$

with $C_3=(1/\sqrt{a}+\beta)C_1\cdot 2\phi_+\cdot 2u_{\max}$. Similar conclusions hold with $\mathbf{x}$ and $\mathbf{y}$ exchanged.

Therefore we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\mathbf{x}(t),\mathbf{y}(t),t)\leqslant-\mu F(\mathbf{x}(t),\mathbf{y}(t),t)+C_2F_\infty(t;S_1(t))^{3/2}+C_3\eta\epsilon_1F_\infty(t;S_1(t))^{1/2},$$

with $\mu>0$ a constant. Taking $\mathbf{x}(t),\mathbf{y}(t)$ as the characteristics where $\max_{\mathbf{x},\mathbf{y}\in S_1(t)}F(\mathbf{x},\mathbf{y},t)$ is achieved, we obtain

$$\frac{\mathrm{d}f}{\mathrm{d}t}\leqslant-\mu f+C_2f^{3/2}+C_3\eta\epsilon_1f^{1/2},\quad f(t)=F_\infty(t;S_1(t)).$$

Now set $\eta = \frac{C_3}{C_2}$ and assume $\epsilon_1 \leqslant \frac{\mu^2}{16C_2^2}$, then $\frac{\mathrm{d}f}{\mathrm{d}t} < 0$ whenever $f(t) = \epsilon_1$, and hence the bound $f(t) < \epsilon_1$ persists in time. The conclusion of the theorem follows from the fact that f and $\delta P(t; S_1(t))$ are comparable (up to adjust the upper bound ϵ_1 by constant multiple). □

The next lemma guarantees the existence of a partition satisfying the assumptions of Lemma 5.3, in case the L^2 variation of velocity and location is small.

Lemma 5.4. *With the same assumptions as in Theorem 3.6, for any* $\epsilon_1 > 0$*,*

$$\delta E(t_0) < \frac{m_0 \eta \epsilon_1^2}{2} \tag{5.32}$$

implies the existence of a partition satisfying (5.28) *and* (5.29).

Proof. Recall that $(\mathbf{x}_c(t), \mathbf{u}_c(t))$ denote the mean location and velocity (2.3). Then

$$\begin{aligned}
&\iint (|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2 + a|\mathbf{x} - \mathbf{y}|^2)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= \iint (|(\mathbf{u}(\mathbf{x}) - \mathbf{u}_c) - (\mathbf{u}(\mathbf{y}) - \mathbf{u}_c)|^2 + a|(\mathbf{x} - \mathbf{x}_c) - (\mathbf{y} - \mathbf{x}_c)|^2)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&= 2m_0 \int (|\mathbf{u}(\mathbf{x}) - \mathbf{u}_c|^2 + a|\mathbf{x} - \mathbf{x}_c|^2)\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{5.33}$$

Thus, at time t_0,

$$\int_{|\mathbf{u}(\mathbf{x})-\mathbf{u}_c|^2 + a|\mathbf{x}-\mathbf{x}_c|^2 \geqslant \frac{\epsilon_1}{4}} \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{4}{\epsilon_1}\int (|\mathbf{u}(\mathbf{x}) - \mathbf{u}_c|^2 + a|\mathbf{x} - \mathbf{x}_c|^2)\rho(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{4}{\epsilon_1}\frac{1}{2m_0}\frac{m_0\eta\epsilon_1^2}{2} = \eta\epsilon_1.$$

Therefore, we can take $S_2 := \{\mathbf{x} : |\mathbf{u}(\mathbf{x}) - \mathbf{u}_c|^2 + a|\mathbf{x} - \mathbf{x}_c|^2 \geqslant \epsilon_1/4\}$, and (5.28) is satisfied. Then for any $\mathbf{x}, \mathbf{y} \in S_1 := \operatorname{supp} \rho \backslash S_2$, one has

$$\begin{aligned}
|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2 + a|\mathbf{x} - \mathbf{y}|^2 &\leqslant |(\mathbf{u}(\mathbf{x}) - \mathbf{u}_c) - (\mathbf{u}(\mathbf{y}) - \mathbf{u}_c)|^2 + a|(\mathbf{x} - \mathbf{x}_c) - (\mathbf{y} - \mathbf{x}_c)|^2 \\
&\leqslant 2(|\mathbf{u}(\mathbf{x}) - \mathbf{u}_c|^2 + a|\mathbf{x} - \mathbf{x}_c|^2 + |\mathbf{u}(\mathbf{y}) - \mathbf{u}_c|^2 + a|\mathbf{y} - \mathbf{x}_c|^2) \\
&\leqslant 4\frac{\epsilon_1}{4} = \epsilon_1,
\end{aligned}$$

which means (5.29) is also satisfied. □

Proof of Theorem 3.6. We start by a hypocoercivity argument on the energy estimate. Using the notation in the proof of Theorem 3.7,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}-\mathbf{y}, \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle \\
&\quad= \iint \Big[(\mathbf{x}-\mathbf{y})\cdot(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))(-\nabla_{\mathbf{x}}\cdot(\rho(\mathbf{x})\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})-\nabla_{\mathbf{y}}\cdot(\rho(\mathbf{y})\mathbf{u}(\mathbf{y}))\rho(\mathbf{x})) \\
&\quad+(\mathbf{x}-\mathbf{y})\cdot\Big(-\mathbf{u}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x})+\int\phi(\mathbf{x}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}-\nabla U(\mathbf{x}) \\
&\quad-\mathbf{u}(\mathbf{y})\cdot\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y})+\int\phi(\mathbf{y}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{y}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}-\nabla U(\mathbf{y})\Big)\rho(\mathbf{x})\rho(\mathbf{y})\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\quad= |\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2+\iint(\mathbf{x}-\mathbf{y})\cdot\Big(\int\phi(\mathbf{x}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z} \\
&\quad+\int\phi(\mathbf{y}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{y}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\Big)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}-\langle\mathbf{x}-\mathbf{y},\nabla U(\mathbf{x})-\nabla U(\mathbf{y})\rangle \\
&\quad\leqslant |\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2-a|\mathbf{x}-\mathbf{y}|^2+2\Big(\frac{a}{4}|\mathbf{x}-\mathbf{y}|^2+\frac{m_0^2\phi_+^2}{a}|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2\Big) \\
&\quad= -\frac{a}{2}|\mathbf{x}-\mathbf{y}|^2+\Big(1+\frac{2m_0^2\phi_+^2}{a}\Big)|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2,
\end{aligned} \tag{5.34}$$

where we used

$$\begin{aligned}
&\left|\iint(\mathbf{x}-\mathbf{y})\cdot\int\phi(\mathbf{x}-\mathbf{z})(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right| \\
&\leqslant \phi_+c_1|\mathbf{x}-\mathbf{y}|^2+\frac{\phi_+}{4c_1}\iint\Big(\int|(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))|\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\Big)^2\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leqslant \phi_+c_1|\mathbf{x}-\mathbf{y}|^2+\frac{\phi_+}{4c_1}\iint m_0\int|(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))|^2\rho(\mathbf{z})\,\mathrm{d}\mathbf{z}\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \\
&\leqslant \phi_+c_1|\mathbf{x}-\mathbf{y}|^2+\frac{m_0^2\phi_+}{4c_1}|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2,
\end{aligned} \tag{5.35}$$

with $c_1 = a/4\phi_+$. Combined with the energy estimate (1.5), we obtain, for any $c > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(E(t)+c\langle\mathbf{x}-\mathbf{y},\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle\big)\leqslant-\Big(\frac{\phi_-}{2}-c\big(1+\frac{2m_0^2\phi_+^2}{a}\big)\Big)|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2-\frac{ca}{2}|\mathbf{x}-\mathbf{y}|^2.$$

Then, setting

$$c := \min\left\{\frac{\phi_-/2}{1+2m_0^2\phi_+^2/a+1/2},\ \frac{\sqrt{a}}{8m_0}\right\}, \tag{5.36}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(E(t)+c\langle\mathbf{x}-\mathbf{y},\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle)\leqslant-\frac{c}{2}(|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2+a|\mathbf{x}-\mathbf{y}|^2)=-\frac{c}{2}\delta E(t).$$

Notice that, since $U(\mathbf{x}) \geqslant \frac{a}{2}|\mathbf{x}|^2$,

$$\langle \mathbf{x}-\mathbf{y}, \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle \leqslant \frac{1}{2\sqrt{a}}(a|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2)$$
$$\leqslant \frac{2m_0}{\sqrt{a}} \int (a|\mathbf{x}|^2 + |\mathbf{u}(\mathbf{x})|^2)\rho(\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \frac{4m_0}{\sqrt{a}} E(t).$$

Therefore $E(t) + c\langle \mathbf{x}-\mathbf{y}, \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle \geqslant 0$, which in turn implies that $\int_0^\infty \delta E(t)\,\mathrm{d}t =: C_0 < \infty$.

Next, for any fixed $t_1 > 0$, there exists $t_0 \leqslant t_1$ such that $\delta E(t_0) \leqslant \frac{C_0}{t_1}$ (otherwise the integral $\int_0^{t_1} \delta E(t)\,\mathrm{d}t$ would exceed C_0). Lemma 5.4 implies that there exists a partition at $t = t_0$ satisfying (5.28) and (5.29), with ϵ_1 given by $\epsilon_1 = \sqrt{\frac{2C_0}{m_0\eta t_1}}$. If t_1 is large enough, then ϵ_1 is small enough, so that we can apply Lemma 5.3 to get that (5.30) holds for all $t \geqslant t_0$. In particular, (5.30) holds for $t = t_1$. Therefore, by using (5.30) for pairs $(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}, \mathbf{y} \in S_1(t_1)$ and the uniform bound (3.3) for other pairs, we obtain ($u_{\max}$ denoting the RHS of (3.3))

$$\delta E(t_1) \leqslant m_0^2\epsilon_1 + 2m_0\eta\epsilon_1 \cdot 4(1+a)u_{\max}^2 = C\epsilon_1, \tag{5.37}$$

and the proof is finished by noticing that $\epsilon_1 = \mathcal{O}(1/\sqrt{t_1})$ for large t_1. □

6. Proof of Main Results: Existence of Global Smooth Solutions

The proof of the existence of smooth solutions proceed in two parallel tracks. Both the restricted solution and the global flow map address similar apriori bounds—the former is restricted to $\mathbf{x} \in \operatorname{supp}\{\rho(\cdot, t)\}$ and the latter applies throughout space, $\mathbf{x} \in \Omega$.

6.1. The One-Dimensional Case

The proof of the existence of global smooth solutions for 1d follows the technique of [1]: we analyze the ODE satisfied by the quantity $\partial_x u + \phi * \rho$ along characteristics.

Proof of Theorem 4.2. Write $\mathtt{d} := \partial_x u$. Differentiate the second equation of (6.1) with respect to x to get

$$\begin{aligned}
&\partial_t \rho + u\partial_x \rho = -\rho\mathtt{d}\\
&\partial_t \mathtt{d} + u\partial_x \mathtt{d} + \mathtt{d}^2 = -u\int \partial_x\phi(x-y)\rho(y)\,\mathrm{d}y - \int \phi(x-y)\partial_t\rho(y)\,\mathrm{d}y\\
&\qquad - \mathtt{d}\int \phi(x-y)\rho(y)\,\mathrm{d}y - U''(x).
\end{aligned} \tag{6.1}$$

Expressed in terms of $\mathrm{e} := \mathrm{d} + \phi * \rho$ and the time derivative along characteristics denoted by $'$, then (6.1) reads

$$\begin{aligned} \rho' &= -\rho(\mathrm{e} - \phi * \rho) \\ \mathrm{e}' &= -\mathrm{e}(\mathrm{e} - \phi * \rho) - U''. \end{aligned} \tag{6.2}$$

If $\mathrm{e} > 0$, then by (4.5),

$$\mathrm{e}' \geqslant -\mathrm{e}(\mathrm{e} - m_0\phi_-) - A = -\left(\mathrm{e} - \frac{m_0\phi_-}{2}\right)^2 + \left(\frac{(m_0\phi_-)^2}{4} - A\right).$$

Then since by (4.5) $A < (m_0\phi_-)^2/4$, one has

$$\mathrm{e}' > 0, \quad \text{for } \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - A} < \mathrm{e} < \frac{m_0\phi_-}{2} + \sqrt{\frac{(m_0\phi_-)^2}{4} - A}.$$

By (4.6), initially $\mathrm{e} > \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - A}$ for all x. Therefore the same inequality persists for all time.

Also notice that if $\mathrm{e} \geqslant 2m_0\phi_+$ then $\mathrm{e}' \leqslant -\mathrm{e}^2/2 - a$, which implies e is bounded above by $\mathrm{e} \leqslant \max\{\max_x \mathrm{e}_0, 2m_0\phi_+, \sqrt{\max\{0, -2a\}}\}$. Since $\phi * \rho$ is bounded above and below, this implies that $\partial_x u$ is uniformly bounded, and thus global smooth solution exists. □

Proof of Theorem 4.3. We start from (6.2), the dynamic of e, which is derived in the previous proof. We analyze the sign of e' in the cases of positive and negative e:

- If $\mathrm{e} \geqslant 0$, then

$$\mathrm{e}' \leqslant -\mathrm{e}(\mathrm{e} - m_0\phi_+) - a = -\left(\mathrm{e} - \frac{m_0\phi_+}{2}\right)^2 + \left(\frac{(m_0\phi_+)^2}{4} - a\right) \tag{6.3}$$

  - If (4.7) holds, then $\mathrm{e}' < 0$.
  - If (4.7) does not hold, then if

$$\mathrm{e} < \frac{m_0\phi_+}{2} - \sqrt{\frac{(m_0\phi_+)^2}{4} - a} \tag{6.4}$$

    then $\mathrm{e}' < 0$.
- If $\mathrm{e} < 0$ then

$$\mathrm{e}' \leqslant -\mathrm{e}(\mathrm{e} - m_0\phi_-) - a = -\left(\mathrm{e} - \frac{m_0\phi_-}{2}\right)^2 + \left(\frac{(m_0\phi_-)^2}{4} - a\right) \tag{6.5}$$

  - If $a > 0$, then $\mathrm{e}' < 0$.

– If $a \leqslant 0$, then if

$$\mathrm{e} < \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - a} \tag{6.6}$$

then $\mathrm{e}' < 0$.

Notice that for all the $\mathrm{e}' < 0$ cases above, we actually have $\mathrm{e}' < -\epsilon < 0$. Therefore, as long as one stays in the $\mathrm{e}' < 0$ cases, e will keep decreasing until it is negative enough so that the $-\mathrm{e}^2$ term blows it up. Therefore, we have the following situations where we can guarantee a finite time blow-up:

- If (4.7) holds, then any negative values of e will have $\mathrm{e}' < 0$ since $a > 0$, and any positive values of e will have $\mathrm{e}' < 0$.
- If (4.7) does not hold but $a > 0$ and (4.8) holds (which means (6.4) holds initially), then (6.4) will propagate since $\mathrm{e}' < 0$ for positive or negative values of e.
- If (4.7) does not hold and $a \leqslant 0$ but (4.9) holds (which means (6.6) holds initially: in particular, e starts with negative values), then (6.6) will propagate since $\mathrm{e}' < 0$ (because e stays negative). □

6.2. The Two-Dimensional Case

We follow [9], tracing the dynamics of the matrix $M_{ij} = \partial_j u_i$ associated with the solution to (1.1). Since most steps are the same as in [9, Theorem 2.1] except for the additional external potential term on the right of (1.1), we outline the derivation along the same steps as in [9] while omitting excessive details.
Step 1: M satisfies

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = -(\phi * \rho)M + R - \nabla^2 U, \tag{6.7}$$

where

$$R_{ij} = \partial_j \phi * (\rho u_i) - u_i(\partial_j \phi * \rho). \tag{6.8}$$

The divergence $\mathrm{d} = \nabla \cdot \mathbf{u}$ satisfies

$$\partial_t \mathrm{d} + \mathbf{u} \cdot \nabla \mathrm{d} + \mathrm{Tr} M^2 = -(\phi * \rho)\mathrm{d} + \mathrm{Tr} R - \Delta U. \tag{6.9}$$

The two traces in this equation are evaluated as follows: by (6.8), $\mathrm{Tr} R = -(\phi * \rho)'$; also, $\mathrm{Tr} M^2 \equiv \frac{1}{2}\big(\mathrm{d}^2 + \eta_M^2\big)$ where η_M is the spectral gap of the two eigenvalues of M. We find

$$(\mathrm{d} + \phi * \rho)' = -\frac{1}{2}\eta_M^2 - \frac{1}{2}\mathrm{d}(\mathrm{d} + 2\phi * \rho) - \Delta U. \tag{6.10}$$

Decompose M into its symmetric and anti-symmetric parts, $M = S + \Omega$, then $\eta_M^2 = \eta_S^2 - 4\omega^2$ where η_S is the spectral gap of S and $\omega = (\partial_1 u_2 - \partial_2 u_1)/2$ is the scaled vorticity. Then, by introducing $\mathrm{e} = \mathrm{d} + \phi * \rho$, we finally end up with

$$\mathrm{e}' = \frac{1}{2}(4\omega^2 + (\phi * \rho)^2 - \eta_S^2 - \mathrm{e}^2 - 2\Delta U), \qquad \mathrm{e} := \mathrm{d} + \phi * \rho. \tag{6.11}$$

Step 2: The 'e-equation' is complemented by the dynamics of the spectral gap η_S. To this end, we follow the spectral dynamics of S:

$$S' + S^2 = \omega^2 I - (\phi * \rho)S + R_{sym} - \nabla^2 U, \quad R_{sym} = \frac{1}{2}(R + R^\top);$$

where I stands for the identity matrix. The dynamics of the eigenvalues μ_i of S is given by

$$\mu_i' + \mu_i^2 = \omega^2 - (\phi * \rho)\mu_i + \langle \mathbf{s}_i, R_{sym}\mathbf{s}_i \rangle - \langle \mathbf{s}_i, \nabla^2 U \mathbf{s}_i \rangle,$$

where $\mathbf{s}_1, \mathbf{s}_2$ are the orthonormal eigenpair of S. Taking their difference,

$$\eta_S' + \mathrm{e}\eta_S = q := \langle \mathbf{s}_2, R_{sym}\mathbf{s}_2 \rangle - \langle \mathbf{s}_1, R_{sym}\mathbf{s}_1 \rangle - \langle \mathbf{s}_2, \nabla^2 U \mathbf{s}_2 \rangle + \langle \mathbf{s}_1, \nabla^2 U \mathbf{s}_1 \rangle. \tag{6.12}$$

Step 3: We need to estimate η_S based on (6.12). A good estimate of η_S will give a non-negative lower bound of e.
Step 4: Finally we need an upper bound of e. The dynamics of ω is independent of the symmetric forcing term $\nabla^2 U$,

$$\omega' + \mathrm{e}\omega = \frac{1}{2}\mathrm{Tr}(JR), \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{6.13}$$

Therefore we can bound ω in the same way as we bound η_S, and this yields an upper bound of e. This would conclude the proof of the uniform boundedness of $\mathrm{d} = \nabla \cdot \mathbf{u}$. Combined with the uniform boundedness of η_S and ω, we get the uniform boundedness of $\nabla \mathbf{u}$.

Proof of Theorem 4.4. Observe that outside the horizon range, dist $\{\mathbf{x}, \text{supp}\,\{\rho(\cdot, t)\}\} > \mathsf{R}_\infty$, the the entries of $R(\mathbf{x}, t)$ vanish since $\partial_j \widetilde{\phi}(|\mathbf{x} - \mathbf{y}|)$ does; otherwise, we have, in view of the assumed bound (4.11),

$$|R_{ij}(\mathbf{x}, t)| \leqslant 2m_0 |\phi'|_\infty u_{\max}. \tag{6.14}$$

Therefore, since $\mathbf{s}_1, \mathbf{s}_2$ are unit vectors,

$$|\langle \mathbf{s}_2, R_{sym}\mathbf{s}_2 \rangle - \langle \mathbf{s}_1, R_{sym}\mathbf{s}_1 \rangle| \leqslant 8m_0 |\phi'|_\infty u_{\max},$$

while $\Big| - \langle \mathbf{s}_2, \nabla^2 U \mathbf{s}_2 \rangle + \langle \mathbf{s}_1, \nabla^2 U \mathbf{s}_1 \rangle \Big| \leqslant A - a$, and we end up with

$$|q| \leqslant 8m_0 |\phi'|_\infty u_{\max} + A - a =: C_{max}. \tag{6.15}$$

Hence, assuming that we have the lower bound (which is true initially, by assumption (4.16a))

$$\mathrm{e} \geqslant \sqrt{C_A - \sqrt{C_A^2 - C_{max}^2}} =: c_2 > 0 \ \text{ where } \ C_A = \frac{m_0^2 \phi_-^2}{2} - 2A \tag{6.16}$$

(the quantity inside the inner square root is positive, by assumption (4.12)), η_S does not exceed

$$|\eta_S| \leqslant \max\left\{ \max_{\mathbf{x}} |(\eta_S)_0(\mathbf{x})|, \frac{C_{max}}{c_2} \right\} := \eta_{S,\max}. \tag{6.17}$$

Step 3: (6.11) implies

$$\mathrm{e}' \geqslant \frac{1}{2}(c_1^2 - \mathrm{e}^2), \qquad c_1^2 := m_0^2\phi_-^2 - \eta_{S,\max}^2 - 4A = 2C_A - \eta_{S,\max}^2, \quad (6.18)$$

with well-defined $c_1 > 0$ provided the quantity on the right is positive. In fact, assumption (4.16b) gives

$$2C_A - \max_{\mathbf{x}} |(\eta_S)_0(\mathbf{x})|^2 \geqslant C_A - \sqrt{C_A^2 - C_{max}^2} = c_2^2,$$

and by (6.16), $2C_A - \left(\frac{C_{max}}{c_2}\right)^2 = c_2^2$. Thus we have $2C_A - \eta_{S,\max}^2 = c_2^2$, and therefore c_1 is well-defined and coincides with $c_1 = c_2 > 0$. With this, (6.18) now reads $\mathrm{e}' \geqslant 1/2(c_2^2 - \mathrm{e}^2)$ and hence e is increasing whenever $\mathrm{e} \leqslant c_2$. This means the initial bound $\mathrm{e} \geqslant c_2$ can be propagated for all time, so we can identify $c_2 = \mathrm{e}_{\min}$.
Step 4: Similarly ,we obtain from (6.13) that ω is uniformly bounded:

$$|\omega| \leqslant \max\left\{\max_{\mathbf{x}} |\omega_0(\mathbf{x})|, \frac{8m_0|\phi'|_\infty \mathbf{u}\max}{c_2}\right\} =: \omega_{\max}.$$

Then (6.11) shows, since $|\Delta U| \leqslant 2A$, $\mathrm{e}' \leqslant \frac{1}{2}(4\omega_{\max}^2 + m_0^2\phi_+^2 + 4A - \mathrm{e}^2)$. Thus we get the upper bound $\mathrm{e} \leqslant \max\left\{\max_{\mathbf{x}} \mathrm{e}_0(\mathbf{x}), \sqrt{4\omega_{\max}^2 + m_0^2\phi_+^2 + 4A}\right\}$. □

Remark 6.1. (*Improved thresholds in the case of quadratic potentials*) In the special case quadratic potential, one can replace (6.17) with the uniform-in-time bound of $\max_{\mathbf{x}} |\eta_S(\mathbf{x}, t)| \leqslant \max_{\mathbf{x}} |(\eta_S)_0| + 2|\phi'|_\infty C_R$, (4.15). In particular, since the latter is independent of c_2, we can reorganize the proof with $\mathrm{e}_{\min} = c_2 = 0$, ending with global regularity in time for sub-critical data satisfying (4.16).

We close this section by noting that a key step of the existence proof, going back to [9], is the upper-bound of the residual terms

$$R_{ij}(\mathbf{x}, t) = \int_{\mathbf{y}} \partial_j\phi(|\mathbf{x} - \mathbf{y}|)\big(u_i(\mathbf{x}, t) - u_i(\mathbf{y}, t)\big)\rho(\mathbf{y}, t)\,\mathrm{d}\mathbf{y}.$$

In fact, we can be slightly more precise in bounding these terms in (6.14). By (2.15), the term $\big(u_i(\mathbf{x}, t) - u_i(\mathbf{y}, t)\big)\rho(\mathbf{y}, t)$ is exponentially small and hence can be neglected whenever $\mathbf{x} \in \text{supp}\,\{\rho(\cdot, t)\}$. Thus, for $\mathbf{x} \notin \text{supp}\,\{\rho(\cdot, t)\}$, we are left with the bound

$$|R_{ij}(\mathbf{x}, t)| \leqslant m_0|\phi'|_\infty \delta u_{\max} + \text{l.o.t.}, \qquad \delta u_{\max} := \max_{\mathbf{x},\mathbf{y} \in A_\infty} |\widetilde{\mathbf{u}}(\mathbf{x}, t) - \widetilde{\mathbf{u}}(\mathbf{y}, t)|,$$

where $A_\infty = \{\mathbf{x} : 0 < \text{dist}\{\mathbf{x}, \text{supp}\,\{\rho(\cdot, t)\}\} < \mathsf{R}_\infty\big\}$ is the vacuous horizon region surrounding supp $\{\rho(\cdot, t)\}$. However, there is no reason to expect velocity alignment inside the horizon region, where the dynamics is dictated possibly by different extension procedures. The study of such extensions is left open for future work.

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