

NEWTONIAN REPULSION AND RADIAL CONFINEMENT: CONVERGENCE TOWARDS STEADY STATE

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ABSTRACT. We investigate the large time behavior of multi-dimensional aggregation equations driven by Newtonian repulsion, and balanced by radial attraction and confinement. In case of Newton repulsion with radial confinement we quantify the algebraic convergence decay rate towards the unique steady state. To this end, we identify a one-parameter family of radial steady states, and prove dimension-dependent decay rate in energy and 2-Wasserstein distance, using a comparison with properly selected radial steady states. We also study Newtonian repulsion and radial attraction. When the attraction potential is quadratic it is known to coincide with quadratic confinement. Here we study the case of perturbed radial quadratic attraction, proving that it still leads to one-parameter family of unique steady states. It is expected that this family to serve for a corresponding comparison argument which yields algebraic convergence towards steady repulsive-attractive solutions.

CONTENTS

1. Introduction	1
2. Main results	3
2.1. Newtonian repulsion with external confining potential	3
2.2. Newtonian repulsion with attraction	6
3. Equilibration of Newtonian repulsion with confining potential	6
4. Uniqueness of steady state for Newtonian repulsion with near-quadratic attraction	11
5. Appendix	15
5.1. 1D steady state are not unique	15
5.2. Steady states must have compact support	16
References	19

1. INTRODUCTION

In this paper we study the large time behavior of the first-order aggregation equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \mathbf{u}(t, \mathbf{x}) = -\nabla \Phi(t, \mathbf{x}), \quad (1.1)$$

subject to prescribed initial distribution, $\rho(0, \mathbf{x}) = \rho_0(\mathbf{x})$, with mass

$$m_0 = \int \rho_0(\mathbf{x}) \, d\mathbf{x} = \int \rho(t, \mathbf{x}) \, d\mathbf{x} > 0, \quad \forall t > 0. \quad (1.2)$$

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The dynamics we have in mind for (1.1) governs the interaction of infinitesimal mass elements, $\rho(t, \mathbf{x}) d\mathbf{x}$, which are dominated by repulsion near in the immediate neighborhood of $\mathbf{x} \in \mathbb{R}^d$ and balanced by attraction and confinement which dominate away from \mathbf{x} . This reflects “social” interactions encountered in applications — describing collective dynamics in ecology, human interactions or sensor-based crowds, [CMV03, CMV06, FHK11, KSUB11, BCLR13, BCY14, CFT14, CFP17], In this paper, we consider the case of Newtonian repulsion $\nabla(-\Delta)^{-1}\rho(t, \mathbf{x})$ coupled with attraction $\nabla\mathcal{W} * \rho(t, \mathbf{x})$ and confinement $\mathcal{V}(\mathbf{x})$,

$$\mathbf{u}(t, \mathbf{x}) = -\nabla\Phi(t, \mathbf{x}), \quad \Phi(t, \mathbf{x}) := \int \mathcal{N}(\mathbf{x}-\mathbf{y})\rho(t, \mathbf{y}) d\mathbf{y} + \int \mathcal{W}(\mathbf{x}-\mathbf{y})\rho(t, \mathbf{y}) d\mathbf{y} + \mathcal{V}(\mathbf{x}). \quad (1.3)$$

Here, $\rho(t, \mathbf{x}) \geq 0$ is the large crowd density distribution of “agents”, varying in time-space $(t, \mathbf{x}) \in (\mathbb{R}_+ \times \mathbb{R}^d)$, \mathcal{N} is the Newtonian potential satisfying $\Delta\mathcal{N} = -\delta$,

$$\mathcal{N}(\mathbf{x}) = \begin{cases} -\frac{1}{2}|\mathbf{x}|, & d = 1 \\ -\frac{1}{2\pi} \log |\mathbf{x}|, & d = 2 \\ \frac{c_d}{|\mathbf{x}|^{d-2}}, & c_d > 0, d \geq 3 \end{cases} \quad (1.4)$$

and $\mathcal{V}(\mathbf{x}) = V(r)$ and $\mathcal{W}(\mathbf{x}) = W(r)$, $r = |\mathbf{x}|$ are confining external potential and, respectively, a pairwise attraction potential, both are assumed radial, smooth and with Pareto tail at infinity

$$\lim_{r \rightarrow \infty} V'(r)r^{d-1} = \infty, \quad (1.5)$$

so that the external potential (— and likewise, the pairwise interaction potential) dominates the Newtonian Repulsion at infinity, $\lim_{R \rightarrow \infty} V(R)/N(R) = \infty$.

This paper is concerned with the large time behavior of the aggregation equation (1.1), when Newtonian repulsion is balanced by the presence of either \mathcal{V} or \mathcal{W} . Observe that a steady state of (1.1), ρ_∞ , is characterized¹ by a velocity field which vanishes on the support of ρ , i.e.,

$$-\int \nabla\mathcal{N}(\mathbf{x}-\mathbf{y})\rho_\infty(\mathbf{y}) d\mathbf{y} - \int \nabla\mathcal{W}(\mathbf{x}-\mathbf{y})\rho_\infty(\mathbf{y}) d\mathbf{y} - \nabla\mathcal{V}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \text{supp } \rho_\infty. \quad (1.6)$$

Taking divergence, then (1.6) implies

$$\rho_\infty(\mathbf{x}) = \int \Delta\mathcal{W}(\mathbf{x}-\mathbf{y})\rho_\infty(\mathbf{y}) d\mathbf{y} + \Delta\mathcal{V}(\mathbf{x}), \quad \forall \mathbf{x} \in \text{supp } \rho_\infty, \quad (1.7)$$

which appears to be a key property of steady states. The set of steady states is not empty: indeed, (1.1) is the 2-Wasserstein gradient flow of the total energy

$$E[\rho] = \frac{1}{2} \iint \mathcal{N}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\rho(\mathbf{x}) d\mathbf{y} d\mathbf{x} + \frac{1}{2} \iint \mathcal{W}(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\rho(\mathbf{x}) d\mathbf{y} d\mathbf{x} + \int \mathcal{V}(\mathbf{x})\rho(\mathbf{x}) d\mathbf{x},$$

i.e., its solution $\rho(t, \mathbf{x})$ satisfies the energy dissipation law

$$\frac{d}{dt}E(t) = - \int |\mathbf{u}(t, \mathbf{x})|^2 \rho(t, \mathbf{x}) d\mathbf{x} := -\mathcal{D}[\rho(t, \cdot)], \quad E(t) := E[\rho(t, \cdot)]. \quad (1.8)$$

¹A steady solution of (1.1), $\nabla \cdot (\rho_\infty \nabla \Phi_\infty) = 0$, implies $\int \rho_\infty |\nabla \Phi_\infty|^2 d\mathbf{x} = 0$, i.e., \mathbf{u}_∞ vanishes on $\text{supp } \rho_\infty$ in agreement with (1.8) below.

By compactness arguments $E[\rho]$ admits a global energy minimizer, $\{\rho_\infty : \mathcal{D}[\rho_\infty] = 0\}$, which is a steady state of (2.1). The main question, therefore, is whether the steady state ρ_∞ is unique, and whether the solution $\rho(t, \cdot)$ converges to ρ_∞ as $t \rightarrow \infty$.

2. MAIN RESULTS

We will use C and c to denote positive constants, being large and small respectively, which may depend on \mathcal{V} , \mathcal{W} , and ρ_0 , but otherwise, are independent of the other parameters; their specific values may change from one equation to the next. For notation simplicity, we will assume $d \geq 2$ in the rest of this paper. The counterparts of all results for $d = 1$ are rather straightforward, and outlined in the Appendix. \mathbb{B}_R denotes the d -dimensional ball $\mathbb{B}_R = \{\mathbf{x} : |\mathbf{x}| \leq R\}$.

2.1. Newtonian repulsion with external confining potential. We first present the results for (1.1) with $\mathcal{W} = 0$, i.e., the model with Newtonian repulsion and external confining potential

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \mathbf{u}(t, \mathbf{x}) = - \int \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho(t, \mathbf{y}) \, d\mathbf{y} - \nabla \mathcal{V}(\mathbf{x}). \quad (2.1)$$

We first state the result on the uniqueness of steady state.

Theorem 2.1. *Consider the aggregation equation (2.1) with radially-symmetric confinement $\mathcal{V}(\mathbf{x}) = V(r)$, satisfying $\Delta \mathcal{V}(\mathbf{x}) > 0, \forall \mathbf{x}$. Then for each $m_0 > 0$, (2.1) admits a unique compactly supported steady state with total mass m_0 , and it is radially-symmetric.*

Remark 2.1. *In the Appendix, consult proposition 5.2, it is shown under a restrictive tail condition, $V'(r) \gtrsim r^{-\frac{d-1}{d+1}}$ for $r \geq R_0$, that a steady solution of (2.1) must be compactly supported. The gap between (1.5) and this tale condition remains open.*

The repulsion-confinement (2.1) is the gradient flow of the corresponding energy dissipation law

$$E[\rho] = \frac{1}{2} \iint \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \rho(\mathbf{x}) \, d\mathbf{y} \, d\mathbf{x} + \int \mathcal{V}(\mathbf{x}) \rho(\mathbf{x}) \, d\mathbf{x}. \quad (2.2)$$

It is straightforward to show that the *global energy minimizer* of (2.2) is unique for *any* external potential $\mathcal{V}(\mathbf{x})$. In fact, given any two minimizers ρ_0 and ρ_1 with the same total mass, then considering the homotopy

$$\rho_s(\mathbf{x}) := (1 - s)\rho_0(\mathbf{x}) + s\rho_1(\mathbf{x}), \quad 0 \leq s \leq 1, \quad (2.3)$$

one can verify the the convexity $\frac{d^2}{ds^2} E[\rho_s] > 0$, which implies uniqueness of the global energy minimizer.

However, the uniqueness of global energy minimizer does *not* imply the uniqueness of steady state. In fact, Appendix shows that in 1D, if \mathcal{V} is not convex, then generally speaking steady states may not be unique, despite the uniqueness of global energy minimizer. This suggests that the conclusion of Theorem 2.1 is far from trivial.

Proof. As a first step we record the following family of radially symmetric steady states parameterized by a cut-off radius $R > 0$

$$\rho_R(\mathbf{x}) := \Delta \mathcal{V}(\mathbf{x}) \chi_{|\mathbf{x}| \leq R}(\mathbf{x}).$$

Indeed, the total potential field generated by $\rho_R(\mathbf{x})$

$$\Phi_R(\mathbf{x}) := \int \mathcal{N}(\mathbf{x} - \mathbf{y})\rho_R(\mathbf{y}) \, d\mathbf{y} + \mathcal{V}(\mathbf{x}) = \int \mathcal{N}(\mathbf{x} - \mathbf{y})\Delta\mathcal{V}(\mathbf{y})\chi_{|\mathbf{y}| \leq R}(\mathbf{y}) \, d\mathbf{y} + \mathcal{V}(\mathbf{x}),$$

is radially symmetric and harmonic in \mathbb{B}_R

$$-\Delta\Phi_R(\mathbf{x}) = \Delta\mathcal{V}(\mathbf{x})\chi_{|\mathbf{x}| \leq R}(\mathbf{x}) - \Delta\mathcal{V}(\mathbf{x}) = 0, \quad \forall |\mathbf{x}| \leq R.$$

Therefore $\Phi_R(\mathbf{x})$ is constant in $|\mathbf{x}| \leq R$ and $\mathbf{u}_R = -\nabla\Phi_R$ vanishes there,

$$\int \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\Delta\mathcal{V}\chi_{\mathbb{B}_R}(\mathbf{y}) \, d\mathbf{y} + \nabla\mathcal{V}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathbb{B}_R, \quad (2.4)$$

which means that $\rho_R = \Delta\mathcal{V}\chi_{\mathbb{B}_R}$, satisfying (1.6), is a steady state. Observe that this family of steady-states can be equally parametrized by their total mass: for any $m_0 > 0$, there exists a uniquely determined $R_0 = R_0(m_0) > 0$ such that² (\mathbb{S}^{d-1} denoting the d -dimensional unit sphere)

$$\frac{1}{|\mathbb{S}^{d-1}|} \int \Delta\mathcal{V}\chi_{\mathbb{B}_{R_0}} \, d\mathbf{y} = \int_0^{R_0} \frac{\partial}{\partial r} (r^{d-1}V'(r)) \, dr = R_0^{d-1}V'(R_0) = m_0.$$

In the second step we consider a compactly supported steady state ρ_∞ : we will show that it must coincide with ρ_R for properly chosen R . To this end recall that according to (1.7) (with $\mathcal{W} = 0$), a steady state of (2.1) satisfies

$$\rho_\infty(\mathbf{x}) = \Delta\mathcal{V}(\mathbf{x})\chi_{\text{supp } \rho_\infty}(\mathbf{x}), \quad (2.5)$$

and by (1.6) with $\mathcal{W} = 0$, it is characterized by

$$-\int \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\Delta\mathcal{V}(\mathbf{y})\chi_{\text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} - \nabla\mathcal{V}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \text{supp } \rho_\infty. \quad (2.6)$$

Let R_∞ denote its finite diameter $R_\infty = \max_{\mathbf{x} \in \text{supp } \rho_\infty} |\mathbf{x}|$. We turn to compare ρ_∞ with the steady solution $\rho_{R_\infty} = \Delta\mathcal{V}\chi_{\mathbb{B}_{R_\infty}}$. By our first step, the latter is a steady state, hence it also satisfies (1.6) (with $\mathcal{W} = 0$), namely

$$-\int \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\Delta\mathcal{V}(\mathbf{y})\chi_{\mathbb{B}_{R_\infty}}(\mathbf{y}) \, d\mathbf{y} - \nabla\mathcal{V}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \mathbb{B}_{R_\infty}. \quad (2.7)$$

By definition, $\mathbb{B}_{R_\infty} \supset \text{supp } \rho_\infty$ and there exists $\mathbf{x} \in \text{supp } \rho_\infty$ such that $|\mathbf{x}| = R_\infty$. Taking the difference between (2.6) and (2.7) and multiply by that \mathbf{x} gives

$$-\int \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\Delta\mathcal{V}(\mathbf{y})\chi_{\mathbb{B}_{R_\infty} \setminus \text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} = 0. \quad (2.8)$$

Now, with $\nabla\mathcal{N}(\mathbf{x}) = -c_d|\mathbf{x}|^{-d}\mathbf{x}$ we compute that for any $|\mathbf{y}| < R_\infty$, consult figure 1 below,

$$\mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y}) = -\frac{c_d}{|\mathbf{x} - \mathbf{y}|^d} \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) = -\frac{c_d}{|\mathbf{x} - \mathbf{y}|^d} (R_\infty^2 - \mathbf{x} \cdot \mathbf{y}) < 0, \quad |\mathbf{y}| < R_\infty. \quad (2.9)$$

Thus, the first integrand in (2.8) does not vanish; by assumption, the second integrand is strictly positive, and consequently the third inregrand must vanish,

$$\text{supp } \rho_\infty = \{\mathbf{y} : |\mathbf{y}| \leq R_\infty\}. \quad (2.10)$$

Therefore, the steady state ρ_∞ is uniquely determined as the radially symmetric $\rho_\infty = \Delta\mathcal{V}(\mathbf{x})\chi_{|\mathbf{x}| \leq R_\infty}(\mathbf{x})$. \square

²We make a minimal growth assumption $r^{d-1}V'(r) \xrightarrow{r \rightarrow \infty} \infty$

A similar comparison argument has been used in [BLL12, §3.1] in the case of quadratic potential $\mathcal{V}(\mathbf{x}) = |\mathbf{x}|^2$. Here we extend this argument to general radially-symmetric potentials. Moreover, we pursue a considerably more intricate comparison argument to study the rate of equilibration of (2.1). This is the content of our next result.

Theorem 2.2. *Consider the aggregation equation (2.1) with a C^3 radially-symmetric confining potential $\mathcal{V}(\mathbf{x}) = V(r)$, satisfying*

$$0 < a \leq \Delta \mathcal{V}(\mathbf{x}) \leq A < \infty, \quad \forall \mathbf{x}, \quad (2.11)$$

and subject to compactly supported initial data ρ_0 with uniform lower-bound³

$$\rho_0(\mathbf{x}) \geq \rho_{\min} > 0, \quad \forall \mathbf{x} \in \text{supp } \rho_0.$$

Then its energy $E(t) = E[\rho(t, \cdot)]$ decays towards the limiting energy E_∞ ,

$$E(t) - E_\infty \leq C_\gamma (1+t)^{-\gamma}, \quad \gamma < \frac{d+2}{(d-2)(d+1)}, \quad t \geq 0, \quad E_\infty = E[\rho_\infty]. \quad (2.12)$$

The proof, provided in section 3, proceeds by comparing between the family of steady solutions, $\rho_{R(t)}$ with $R(t) := \max_{\mathbf{x} \in \text{supp } \rho(t, \cdot)} |\mathbf{x}|$ associated with the given solution $\rho(t, \cdot)$, and the steady state ρ_∞ . Compared with the argument outlined in Theorem 2.1, here we lack the steady state characterization (2.5): in fact, even if (2.5) is assumed to hold for the initial data, $\rho_0 = \Delta \mathcal{V} \chi_{\text{supp } \rho_0}$, it does not necessarily propagate in time. We resolve this difficulty by introducing the functional

$$F(t) := \frac{1}{2} \int \left(\rho(t, \mathbf{x}) - \Delta \mathcal{V}(\mathbf{x}) \right)^2 \rho(t, \mathbf{x}) \, d\mathbf{x}, \quad (2.13)$$

which measures the discrepancy of $\rho(t, \mathbf{x})$ from satisfying (2.5). Then, we design a Lyapunov-type modified energy functional, \tilde{E} by combining $E(t) - E_\infty$, $F(t)$ and the discrepancy of radius $R(t) - R_\infty$ where

$$R(t) = \max_{\mathbf{x} \in \text{supp } \rho(t, \cdot)} |\mathbf{x}|, \quad R_\infty = \max_{\mathbf{x} \in \text{supp } \rho_\infty} |\mathbf{x}|. \quad (2.14)$$

Verifying the algebraic decay rate of \tilde{E} implies the result (2.12), as well as quantifies the algebraic rate of $R(t) - R_\infty$,

$$(R(t) - R_\infty)_+ \lesssim C_\gamma (1+t)^{-\frac{d+2}{d(d-2)(d+1)}}.$$

The proof of Theorem 2.2 tells us that the aggregation solution $\rho(t, \cdot)$ approaches the unique steady state ρ_∞ in the sense of 2-Wasserstein distance with algebraic convergence rate. Note that in the case $d = 2$ this algebraic rate γ can be arbitrarily large, while for higher spatial dimensions, γ is restricted by a d -dependent constant.

Remark 2.2. *The same methodology may also apply to $\mathcal{V}(\mathbf{x})$ which is not radially-symmetric, as long as the first step in our proof of Theorem 2.1 goes through. To be precise, assume the existence of a parameterized family of steady states, $\{\rho_\infty(\mathbf{x}; p)\}$, such that (i) $\text{supp } \rho_\infty(\cdot; p)$ is convex, and (ii) the following monotonicity condition holds, $\text{supp } \rho_\infty(\cdot; p_1) \subset \text{supp } \rho_\infty(\cdot; p_2)$ whenever $p_1 < p_2$ (and as before, there is one-to-one correspondence with the initial mass $p = p(m_0)$). Then one can obtain the uniqueness of steady states for fixed p_0 , and derive the equilibration rate via a similar approach. It remains open to explore more general class of external potentials which give rise to the existence of such a family of steady states.*

³Note that ρ_0 is therefore discontinuous on $\partial \text{supp } \rho_0$ while assumed bounded away from vacuum on $\text{supp } \rho_0$.

2.2. Newtonian repulsion with attraction. We apply the ideas in the previous subsection to study the aggregation equation (1.1),(1.3) with pairwise interaction potential Φ given by sum of Newtonian repulsion and smooth attraction potential \mathcal{W} ,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \mathbf{u}(t, \mathbf{x}) = -\nabla \Phi, \quad \Phi = \mathcal{N} * \rho + \mathcal{W} * \rho. \quad (2.15)$$

Observe that being a solution of the dynamics with pairwise attraction equation (2.15), ρ can be also viewed as a solution of the external potential equation (2.1) with a ρ -dependent potential $\mathcal{V}_\rho = \mathcal{W} * \rho(t, \cdot)$. The distinction is that \mathcal{V}_ρ is time-dependent, except in the case of quadratic pairwise attraction, $\mathcal{W}_2 := \frac{1}{2}|\mathbf{x}|^2$. Indeed, since (2.15) preserves the center of mass $\mathbf{c}_0 := \int \mathbf{x} \rho_0(\mathbf{x}) d\mathbf{x} = \int \mathbf{x} \rho(t, \mathbf{x}) d\mathbf{x}$, one may assume $\mathbf{c}_0 = 0$ without loss of generality, hence

$$\nabla(\mathcal{W}_2 * \rho)(t, \mathbf{x}) = \int (\mathbf{x} - \mathbf{y}) \rho(t, \mathbf{y}) d\mathbf{y} = m_0 \mathbf{x} = -\nabla \mathcal{V}_2(\mathbf{x}), \quad \mathcal{V}_2 := \frac{1}{2} m_0 |\mathbf{x}|^2.$$

Thus, the forcing induced by pairwise quadratic attraction is equivalent to aggregation with quadratic confinement, $-\nabla \Phi = -\nabla \mathcal{N} * \rho - \nabla \mathcal{W}_2 * \rho = -\nabla \mathcal{N} * \rho - \nabla \mathcal{V}_2$. The following theorem states the uniqueness of steady states of pairwise attraction (2.15) for potentials, \mathcal{W} , close to quadratic.

Theorem 2.3. *Consider the aggregation equation (2.15) with an attraction potential*

$$\mathcal{W}(\mathbf{x}) = \frac{|\mathbf{x}|^2}{2d} + w(\mathbf{x}), \quad |\Delta w(\mathbf{x})| \leq \epsilon, \quad (2.16)$$

where $w(\mathbf{x}) = w(|\mathbf{x}|)$ is a radially-symmetric perturbation of “order” $\epsilon > 0$, depending on d . Then for each $m_0 > 0$, (2.15) admits a unique steady state (up to translation) with total mass m_0 , and it is radially-symmetric.

The case $w \equiv 0$ corresponds to the Theorem 2.1 with $\Phi = \mathcal{N} * \rho + \mathcal{V}_2$, Theorem 2.3 can be viewed as a perturbation of Theorem 2.1, $\Phi = \mathcal{N} * \rho + \mathcal{V}$, with a perturbed potential $\mathcal{V} = \mathcal{V}_2 + w * \rho$, satisfying $\Delta \mathcal{V} = d + \Delta w * \rho > 1 - \epsilon m_0 > 0$. Alternatively, this can be viewed as aggregation driven by quadratic external forcing, $\Phi = \mathcal{N}_\epsilon * \rho + \mathcal{V}_2$, with perturbed Newtonian repulsion $\mathcal{N}_\epsilon := \mathcal{N} + w$.

We expect that an explicit algebraic equilibration rate can be obtained by the same method as the previous subsection, and this is left as future work.

3. EQUILIBRATION OF NEWTONIAN REPULSION WITH CONFINING POTENTIAL

In this section we prove Theorem 2.2. We first prepare a quantitative version of (2.9).

Lemma 3.1. *For any \mathbf{x} with $|\mathbf{x}| = R > 0$, there holds*

$$\mathbf{x} \cdot \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \leq -\frac{c}{R^{d-2}}, \quad \forall \mathbf{y} \neq \mathbf{x}, |\mathbf{y}| \leq R, \quad d \geq 2. \quad (3.1)$$

Indeed, since $(\mathbf{x} - \mathbf{y}) \cdot \mathbf{x} \equiv \frac{1}{2}(|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{x}|^2 - |\mathbf{y}|^2) \geq \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2$, (3.1) follows in view of

$$\mathbf{x} \cdot \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) = -\frac{(d-2)c_d}{|\mathbf{x} - \mathbf{y}|^d} \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) \leq -\frac{(d-2)c_d}{2|\mathbf{x} - \mathbf{y}|^{d-2}} \leq -\frac{c}{R^{d-2}}, \quad c = (d-2)c_d 2^{1-d},$$

with the proper adjustment of $c > 0$ in the 2D case. Below, we use $L^{p,q}$ denote the usual notation of Lorentz space, e.g., [BS88].

We will also need the following interpolation bound.

Lemma 3.2. *For compactly supported $g \in L_c^\infty(\mathbb{R}^d)$ there holds,*

$$\|g\|_{L^{d,1}} \lesssim \begin{cases} C_d \|g\|_{L^2}^{\frac{2}{d}} \times \|g\|_{L^\infty}^{1-\frac{2}{d}}, & d > 2, \\ C_p \|g\|_{L^2}^{\frac{p}{2}} \times \|g\|_{L^\infty}^{1-\frac{p}{2}}, & d = 2, \forall p < 2. \end{cases} \quad (3.2)$$

Indeed, if $\lambda_g(s) = |\{\mathbf{x} : |g(\mathbf{x})| > s\}|$ is the distribution function associated with g , then for any $1 < p < r < \infty$,

$$\begin{aligned} \|g\|_{L^{r,1}} &= r \int_0^{\|g\|_{L^\infty}} \lambda_g^{1/r}(s) \, ds \\ &\lesssim \left(\int_0^\infty s^p \lambda_g(s) \frac{ds}{s} \right)^{1/r} \times \left(\int_0^{\|g\|_{L^\infty}} s^{-\left(\frac{p-1}{r}\right)r'} \, ds \right)^{1/r'} = C_{p,r} \|g\|_{L^p}^{\frac{p}{r}} \times \|g\|_{L^\infty}^{1-\frac{p}{r}}, \end{aligned}$$

and (3.2) follows with $(r, p) = (d, 2)$. When $d = 2$ we use it with $r = 2$ and any $p < 2$, so that for compactly supported g 's,

$$\|g\|_{L^{2,1}} \lesssim C_p \|g\|_{L^p}^{\frac{p}{2}} \times \|g\|_{L^\infty}^{1-\frac{p}{2}} \lesssim C_p \|g\|_{L^2}^{\frac{p}{2}} \times \|g\|_{L^\infty}^{1-\frac{p}{2}}, \quad \forall p < 2.$$

Proof of Theorem 2.2. The assumptions of Theorem 2.1 are satisfied, and hence a unique radial steady state ρ_∞ with prescribed mass m_0 exists, satisfying $\rho_\infty = \Delta \mathcal{V} \chi_{|\mathbf{x}| \leq R_\infty}$.

STEP 1 — Upper and lower bounds of ρ . Tracing (2.1) along characteristics,

$$\rho' := \partial_t \rho + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} = \rho(\Delta \mathcal{V} - \rho), \quad 0 < a \leq \Delta \mathcal{V} \leq A,$$

implies that after a certain time t_0 (which may depend on a, A but otherwise is independent⁴ of $\max \rho_0$), there holds

$$\frac{a}{2} \leq \rho(t, \mathbf{x}) \leq 2A, \quad \forall t \geq t_0, \quad \forall \mathbf{x} \in \text{supp } \rho(t, \cdot),$$

Therefore, by shifting the initial time if necessary, we may assume that without loss of generality, that we have the uniform bounds

$$0 < \rho_{\min} \leq \rho(t, \mathbf{x}) \leq \rho_{\max}, \quad \forall t \geq 0, \quad \forall \mathbf{x} \in \text{supp } \rho(t, \cdot). \quad (3.3)$$

STEP 2 — Estimate the discrepancy functional $F(t)$ in (2.13). A straightforward computation yields

$$\begin{aligned} \frac{d}{dt} F(t) &= \int (\rho - \Delta \mathcal{V}) \partial_t \rho \cdot \rho \, d\mathbf{x} + \frac{1}{2} \int (\rho - \Delta \mathcal{V})^2 \partial_t \rho \, d\mathbf{x} \\ &= - \int (\rho - \Delta \mathcal{V}) \nabla \cdot (\rho \mathbf{u}) \rho \, d\mathbf{x} + \int (\rho - \Delta \mathcal{V}) \nabla(\rho - \Delta \mathcal{V}) \cdot \mathbf{u} \rho \, d\mathbf{x} \\ &= \int (-\nabla \rho \cdot \mathbf{u} - \rho \nabla \cdot \mathbf{u} + \nabla \rho \cdot \mathbf{u} - \nabla \Delta \mathcal{V} \cdot \mathbf{u}) (\rho - \Delta \mathcal{V}) \rho \, d\mathbf{x} \\ &= \int (-\rho(\rho - \Delta \mathcal{V}) - \nabla \Delta \mathcal{V} \cdot \mathbf{u}) (\rho - \Delta \mathcal{V}) \rho \, d\mathbf{x} \\ &\leq -\rho_{\min} F(t) - \int \nabla \Delta \mathcal{V} \cdot \mathbf{u} (\rho - \Delta \mathcal{V}) \rho \, d\mathbf{x}. \end{aligned}$$

⁴for example, take $t_0 \gtrsim \max\{|\log \frac{\min \rho_0}{2a}|, \frac{1}{A}\}$.

The second term on the right can be bounded in terms of the energy dissipation rate \mathcal{D} in (1.8),

$$\left| \int (-\nabla \Delta \mathcal{V} \cdot \mathbf{u})(\rho - \Delta \mathcal{V}) \rho \, d\mathbf{x} \right| \leq \|\mathcal{V}\|_{C^3} \int |\mathbf{u}| \cdot |\rho - \Delta \mathcal{V}| \rho \, d\mathbf{x} \leq \|\mathcal{V}\|_{C^3} \left(\frac{\rho_{\min}}{2\|\mathcal{V}\|_{C^3}} F + \frac{2\|\mathcal{V}\|_{C^3}}{\rho_{\min}} \mathcal{D} \right),$$

and we end up with $\frac{d}{dt} F(t) \leq -\frac{\rho_{\min}}{2} F + C\mathcal{D}$. This implies that F is bounded: in fact, since $\mathcal{D} = -\frac{d}{dt} E$ it follows that $F + C(E - E_\infty) \leq F_0 + C(E_0 - E_\infty)$. Hence we seek the large time behavior for quantities $F, (E - E_\infty)$ (and likewise $R - R_\infty$ in the next step) which depending on their vanishing order $\ll 1$. Observe with small enough $\epsilon_1 > 0$ there follows

$$\frac{d}{dt} \left((E(t) - E_\infty) + \epsilon_1 F(t) \right) \leq -\mathcal{D} + \epsilon_1 \left(-\frac{\rho_{\min}}{2} F + C\mathcal{D} \right) \leq -c(\mathcal{D} + F). \quad (3.4)$$

To close this inequality, we will need to take into account the further discrepancy between $\text{supp } \rho(t, \cdot)$ and $\text{supp } \rho_\infty$.

STEP 3 — Estimate of $R'(t)$. Recall that $R(t)$ is the radius of $\text{supp } \rho(t, \cdot)$, (2.14) and assume for a moment that $R(t) \geq R_\infty$, see figure 1 for a typical configuration⁵.

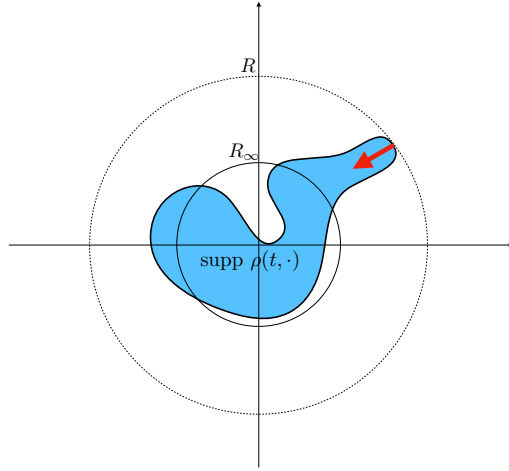


FIGURE 1. The support of $\rho(t, \cdot)$ inscribed in \mathbb{B}_R vs. the limiting ball \mathbb{B}_{R_∞} .

Fix \mathbf{x} on the edge of $\text{supp } \rho(t, \cdot)$, $|\mathbf{x}| = R$. Then by (2.4) the velocity \mathbf{u} in (1.3) amounts to

$$\begin{aligned} \mathbf{u}(t, \mathbf{x}) &= - \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho(t, \mathbf{y}) \, d\mathbf{y} - \nabla \mathcal{V}(\mathbf{x}) \\ &= - \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) (\rho(t, \mathbf{y}) - \Delta \mathcal{V}(\mathbf{y})) \, d\mathbf{y} - \left(\int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \Delta \mathcal{V}(\mathbf{y}) \, d\mathbf{y} + \nabla \mathcal{V}(\mathbf{x}) \right) \\ &= - \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) (\rho(t, \mathbf{y}) - \Delta \mathcal{V}(\mathbf{y})) \, d\mathbf{y}. \end{aligned}$$

⁵Note that $\text{supp } \rho_0$ and hence $\text{supp } \rho$ need *not* be simply connected.

We estimate the last term by examining separately⁶, $\mathbf{u}_\pm := -\nabla\mathcal{N} * ((\rho - \Delta\mathcal{V})_\pm \chi_{\mathbb{B}_R})$. We begin by estimating the discrepancy from below, $(\rho - \Delta\mathcal{V})_- \chi_{\mathbb{B}_R}$. By Lemma 3.1,

$$\begin{aligned}
 \mathbf{x} \cdot \mathbf{u}_-(t, \mathbf{x}) &= - \int_{|\mathbf{y}| \leq R} \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y}) (\rho(t, \mathbf{y}) - \Delta\mathcal{V}(\mathbf{y}))_- \, d\mathbf{y} \\
 &\leq \frac{c}{R^{d-2}} \int_{|\mathbf{y}| \leq R} (\rho(t, \mathbf{y}) - \Delta\mathcal{V}(\mathbf{y})) \, d\mathbf{y} \\
 &= \frac{c}{R^{d-2}} \left(\int_{|\mathbf{y}| \leq R_\infty} \Delta\mathcal{V} \, d\mathbf{y} - \int_{|\mathbf{y}| \leq R} \Delta\mathcal{V} \, d\mathbf{y} \right) \\
 &= -\frac{c}{R^{d-2}} \int_{R_\infty \leq |\mathbf{y}| \leq R} \Delta\mathcal{V} \, d\mathbf{y} \\
 &\leq -\frac{c}{R^{d-2}} \frac{a}{d} (R^d - R_\infty^d) \\
 &\lesssim -R(R - R_\infty),
 \end{aligned} \tag{3.5}$$

where the second equality uses the fact that $\int_{|\mathbf{y}| \leq R} \rho(\mathbf{y}) \, d\mathbf{y} = m_0 = \int_{|\mathbf{y}| \leq R_\infty} \Delta\mathcal{V} \, d\mathbf{y}$ and the second inequality uses the lower bound $\Delta\mathcal{V} \geq a$.

Next, we estimate the discrepancy from above, $g = (\rho - \Delta\mathcal{V})_+$. Since $\nabla\mathcal{N} \in L^{d', \infty}$ then $\|\nabla\mathcal{N} * g\|_{L^\infty} \lesssim \|g\|_{L^{d,1}}$. Recall that g is uniformly bounded, supported in \mathbb{B}_R and satisfies the L^2 bound $\|g\|_{L^2}^2 \leq 1/\rho_{\min} F(t)$, so Lemma 3.2 implies the existence of finite C_d, C_p such that

$$\begin{aligned}
 \frac{\mathbf{x}}{R} \cdot \mathbf{u}_+(t, \mathbf{x}) &= - \int \frac{\mathbf{x}}{R} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y}) (\rho(t, \mathbf{y}) - \Delta\mathcal{V}(\mathbf{y}))_+ \, d\mathbf{y} \\
 &\leq \|\rho(t, \cdot) - \Delta\mathcal{V}\|_{L^{d,1}} \leq \begin{cases} C_d (F(t))^{1/d}, & d > 2 \\ C_p (F(t))^{p/4}, & \forall p < d = 2. \end{cases}
 \end{aligned} \tag{3.6}$$

Using the bounds (3.5), (3.6) we find

$$\frac{d}{dt} (R(t) - R_\infty)_+ = \sup_{|\mathbf{x}|=R, \mathbf{x} \in \text{supp } \rho} \mathbf{u}(t, \mathbf{x}) \cdot \frac{\mathbf{x}}{R} \leq -c(R(t) - R_\infty)_+ + C \begin{cases} (F(t))^{1/d}, & d > 2 \\ (F(t))^{p/4}, & \forall p < d = 2. \end{cases}$$

Now fix an arbitrary $m > d$. By Young's inequality we have

$$\begin{aligned}
 \frac{d}{dt} (R(t) - R_\infty)_+^m &\lesssim -(R(t) - R_\infty)_+^m + (R(t) - R_\infty)_+^{(m-1)s'} + F(t), \\
 s &= \begin{cases} d, & d > 2 \\ \frac{4}{p}, & \forall p < d = 2. \end{cases}
 \end{aligned} \tag{3.7}$$

Since $m > d$ then $(m-1)s' > m$: indeed, when $d > 2$ then $s = d$ and $(m-1)d' > m$, and when $d = 2$ then we can always choose p so that $4/m < p < 2$ and with $s = 4/p$ we then

⁶Here and below we let z_-, z_+ denote the negative and respectively positive parts of a real z .

have $(m-1)(4/p)' > m$. In either case, the first term on the right of (3.7) dominates the second and we end up with

$$\frac{d}{dt}(R(t) - R_\infty)_+^m \lesssim -(R(t) - R_\infty)_+^m + F(t), \quad \forall m > d, d \geq 2. \quad (3.8)$$

STEP 4 — We form the Lyapunov functional, $\tilde{E}(t)$, as a suitable linear combination of

$$\tilde{E}(t) := (E(t) - E_\infty) + \epsilon_1 F(t) + \epsilon_2 (R(t) - R_\infty)_+^m,$$

with fixed $\epsilon_1 \gg \epsilon_2 > 0$ which are yet to be chosen. Choosing , the corresponding combination of (1.8), (3.4) and (3.8) then yield, with small enough ϵ_2 ,

$$\frac{d}{dt} \tilde{E} \leq -c(\mathcal{D} + F) - c\epsilon_2 (R - R_\infty)_+^m + C\epsilon_2 F \leq -\frac{1}{2}cF - c\epsilon_2 (R - R_\infty)_+^m. \quad (3.9)$$

where the constants $c \ll 1 \ll C$ are independent of ϵ_2 .

STEP 5 — Close the estimate. We aim to show that

$$E[\rho(t)] - E_\infty \leq C_q ((R(t) - R_\infty)_+^{2/q} + F(t)), \quad \begin{cases} q = \frac{2d}{d+2}, & d > 2 \\ \text{any } q > 1, & d = 2. \end{cases} \quad (3.10)$$

Combined with (3.9), we obtain, noticing that $\alpha := \frac{m}{2/q} > 1$ and adjusting $\epsilon_2 \ll 1$ if necessary,

$$\frac{d}{dt} \tilde{E} \leq -c\tilde{E}^\alpha, \quad \alpha = \frac{mq}{2} > \begin{cases} d \frac{d}{d+2}, & d > 2 \\ 1, & d = 2, \end{cases}$$

which recovers (2.12), $E(t) - E_\infty \leq \tilde{E} \lesssim (1+t)^{-\gamma}$ with $\gamma = 1/(\alpha - 1)$.

It remains to prove (3.10). Let ρ_1 denote the discrepancy of ρ from the steady state $\rho_\infty = \Delta \mathcal{V} \chi_{\mathbb{B}_{R_\infty}}$,

$$\rho_1 := \rho - \Delta \mathcal{V} \chi_{\mathbb{B}_{R_\infty}}, \quad \int \rho_1 \, d\mathbf{x} = 0. \quad (3.11)$$

Observe that ρ_1 is uniformly bounded since $\Delta \mathcal{V}$ and ρ are, and that is supported in \mathbb{B}_R ; more precisely $\rho_1 = \rho \chi_{\mathbb{B}_R \setminus \mathbb{B}_{R_\infty}} - (\Delta \mathcal{V} - \rho) \chi_{\mathbb{B}_{R_\infty}}$ hence

$$\begin{aligned} \|\rho_1(t, \cdot)\|_{L^1} &= \int_{\mathbb{B}_R \setminus \mathbb{B}_{R_\infty}} \rho \, d\mathbf{x} + \int_{\mathbb{B}_{R_\infty}} |\Delta \mathcal{V} - \rho| \, d\mathbf{x} \\ &\lesssim C \rho_{\max} (R - R_\infty) + \left(\frac{R_\infty^d}{\rho_{\min}} \right)^{1/2} \left(\int |\Delta \mathcal{V} - \rho|^2 \rho(t, \mathbf{x}) \, d\mathbf{x} \right)^{1/2} \\ &\lesssim (R(t) - R_\infty)_+ + F^{1/2}(t). \end{aligned} \quad (3.12)$$

Expressed in terms of ρ_1 , the discrepancy of the energy is given by

$$E[\rho] - E_\infty = \int \Phi_\infty(\mathbf{x}) \rho_1(\mathbf{x}) \, d\mathbf{x} + \frac{1}{2} \iint \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho_1(\mathbf{x}) \rho_1(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y}. \quad (3.13)$$

Let us first bound the first linear term on the right of (3.13). Here $\Phi_\infty(\mathbf{x}) := \int \mathcal{N}(\mathbf{x} - \mathbf{y}) \Delta \mathcal{V}(\mathbf{y}) \chi_{\mathbb{B}_{R_\infty}}(\mathbf{y}) \, d\mathbf{y} + \mathcal{V}(\mathbf{x})$ is the total potential generated by the steady state and as before, being radial and harmonic it remains constant in \mathbb{B}_{R_∞} . Let $\Phi_\infty(R_\infty \frac{\mathbf{x}}{|\mathbf{x}|})$ be the radial extension

of this constant throughout \mathbb{B}_R : since ρ_1 has zero mean on \mathbb{B}_R then $\int_{\mathbb{B}_R} \Phi_\infty\left(R_\infty \frac{\mathbf{x}}{|\mathbf{x}|}\right) \rho_1(\mathbf{x}) \, d\mathbf{x} = 0$, and since $\Phi_\infty(\mathbf{x})$ is Lipschitz outside \mathbb{B}_{R_∞} (because we assume that $\Delta\mathcal{V}$ is), then (3.12) implies

$$\begin{aligned} \left| \int \Phi_\infty(\mathbf{x}) \rho_1(\mathbf{x}) \, d\mathbf{x} \right| &= \left| \int_{\mathbb{B}_R} \left(\Phi_\infty(\mathbf{x}) - \Phi_\infty\left(R_\infty \frac{\mathbf{x}}{|\mathbf{x}|}\right) \right) \rho_1(\mathbf{x}) \, d\mathbf{x} \right| \\ &= \left| \int_{\mathbb{B}_R \setminus \mathbb{B}_{R_\infty}} \left(\Phi_\infty(\mathbf{x}) - \Phi_\infty\left(R_\infty \frac{\mathbf{x}}{|\mathbf{x}|}\right) \right) \rho_1(\mathbf{x}) \, d\mathbf{x} \right| \\ &\lesssim (R - R_\infty)_+ \|\rho_1\|_{L^1} \\ &\lesssim (R - R_\infty)_+^2 + F(t). \end{aligned} \quad (3.14)$$

To estimate the quadratic term in (3.13), we separate between the cases $d > 2$ and $d = 2$. For the former, set $q = \frac{2d}{d+2} \in (1, 2)$ and use Hardy-Littlewood-Sobolev with $\mathcal{N} \in L^{\frac{d}{d-2}, \infty}$ to conclude

$$\begin{aligned} \left| \int \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho_1(\mathbf{x}) \rho_1(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \right| &\lesssim \|\rho_1\|_{L^q}^2 \lesssim \left(\int_{\mathbb{B}_R \setminus \mathbb{B}_{R_\infty}} \rho^q \, d\mathbf{x} \right)^{\frac{2}{q}} + \left(\int_{\mathbb{B}_{R_\infty}} |\Delta\mathcal{V} - \rho|^q \, d\mathbf{x} \right)^{\frac{2}{q}} \\ &\leq C \rho_{max}^2 (R - R_\infty)_+^{\frac{2}{q}} + \left(\frac{R_\infty^d}{\rho_{min}} \right)^{\frac{2/q}{(2/q)'}} \int_{\mathbb{B}_{R_\infty}} |\Delta\mathcal{V} - \rho|^2 \rho \, d\mathbf{x} \\ &\lesssim (R - R_\infty)_+^{\frac{2}{q}} + F, \quad q = \frac{2d}{d+2} \in (1, 2). \end{aligned} \quad (3.15)$$

For the remaining case $d = 2$ we recall that ρ_1 has zero mean, hence the 2D embedding $\|\rho_1\|_{\dot{H}^{-1}} \leq C_q \|\rho_1\|_{L_{loc}^q}$ recovers (3.15) for any $q > 1$

$$\left| \int \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho_1(\mathbf{x}) \rho_1(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \right| = \|\rho_1\|_{\dot{H}^{-1}}^2 \lesssim C_q \|\rho_1\|_{L_{loc}^q}^2 \lesssim (R - R_\infty)_+^{\frac{2}{q}} + F, \quad \forall q > 1.$$

Now (3.10) follows from (3.13), (3.14) and (3.15). \square

4. UNIQUENESS OF STEADY STATE FOR NEWTONIAN REPULSION WITH NEAR-QUADRATIC ATTRACTION

First notice that (2.16) implies that for any $r > 0$,

$$\left| \mathbf{w}'(r) \int_{|\mathbf{x}|=r} dS \right| = \left| \int_{|\mathbf{x}|=r} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla w(\mathbf{x}) \, dS \right| = \left| \int_{|\mathbf{x}| \leq r} \Delta w(\mathbf{x}) \, d\mathbf{x} \right| \leq \epsilon |\mathbb{B}_r|, \quad (4.1)$$

Therefore

$$|\mathbf{w}'(r)| \leq \epsilon \cdot \frac{r}{d}. \quad (4.2)$$

Proof of Theorem 2.3. Let ρ_∞ be the global energy minimizer of $E[\rho]$ among all radially-symmetric density distributions with total mass m_0 . Since the gradient flow (2.15) preserves the radial symmetry, ρ_∞ is clearly a steady state of (2.15).

Assume $\rho(\mathbf{x})$ is a steady state of (2.15) with total mass m_0 (and assume its center of mass $\int \mathbf{x}\rho(\mathbf{x}) \, d\mathbf{x} = 0$ without loss of generality), and we aim to show $\rho = \rho_\infty$.

Denote $R = \max_{\mathbf{x} \in \text{supp } \rho} |\mathbf{x}|$ and let

$$\tilde{\mathcal{V}}(\mathbf{x}) = \int \mathcal{W}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y}, \quad \tilde{\mathcal{V}}_\infty(\mathbf{x}) = \int \mathcal{W}(\mathbf{x} - \mathbf{y})\rho_\infty(\mathbf{y}) \, d\mathbf{y}, \quad (4.3)$$

be the attractive potential fields generated by ρ and ρ_∞ . Here $\tilde{\mathcal{V}}_\infty$ is radially-symmetric because ρ_∞ is. Then $\rho(\mathbf{x})$ is a steady state of (2.1) with \mathcal{V} replaced by $\tilde{\mathcal{V}}$, which implies

$$\rho = \Delta \tilde{\mathcal{V}} \chi_{\text{supp } \rho}, \quad - \int \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \Delta \tilde{\mathcal{V}}(\mathbf{y}) \chi_{\text{supp } \rho}(\mathbf{y}) \, d\mathbf{y} - \nabla \tilde{\mathcal{V}}(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \text{supp } \rho. \quad (4.4)$$

Similarly

$$\begin{aligned} \rho_\infty &= \Delta \tilde{\mathcal{V}}_\infty \chi_{\text{supp } \rho_\infty} : \\ &- \int \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \Delta \tilde{\mathcal{V}}_\infty(\mathbf{y}) \chi_{\text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} - \nabla \tilde{\mathcal{V}}_\infty(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in \text{supp } \rho_\infty. \end{aligned} \quad (4.5)$$

The assumptions on \mathcal{W} imply that

$$1 - \epsilon \leq \Delta \mathcal{W}(\mathbf{x}) \leq 1 + \epsilon, \quad \forall \mathbf{x}, \quad (4.6)$$

and therefore

$$m_0(1 - \epsilon) \leq \Delta \tilde{\mathcal{V}}(\mathbf{x}) \leq m_0(1 + \epsilon), \quad m_0(1 - \epsilon) \leq \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x}) \leq m_0(1 + \epsilon). \quad (4.7)$$

Next we compute

$$\begin{aligned} \tilde{\mathcal{V}}(\mathbf{x}) - \tilde{\mathcal{V}}_\infty(\mathbf{x}) &= \int w(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} \\ &- \int w(\mathbf{x} - \mathbf{y})\rho_\infty(\mathbf{y}) \, d\mathbf{y} \\ &= \int w(\mathbf{x} - \mathbf{y})\Delta \tilde{\mathcal{V}}(\mathbf{y})\chi_{\text{supp } \rho}(\mathbf{y}) \, d\mathbf{y} - \int w(\mathbf{x} - \mathbf{y})\Delta \tilde{\mathcal{V}}_\infty(\mathbf{y})\chi_{\text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} \\ &= \int w(\mathbf{x} - \mathbf{y})(\Delta \tilde{\mathcal{V}}(\mathbf{y}) - \Delta \tilde{\mathcal{V}}_\infty(\mathbf{y}))\chi_{\text{supp } \rho \cap \text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} \\ &+ \int w(\mathbf{x} - \mathbf{y})\Delta \tilde{\mathcal{V}}(\mathbf{y})\chi_{\text{supp } \rho \setminus \text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} \\ &- \int w(\mathbf{x} - \mathbf{y})\Delta \tilde{\mathcal{V}}_\infty(\mathbf{y})\chi_{\text{supp } \rho_\infty \setminus \text{supp } \rho}(\mathbf{y}) \, d\mathbf{y} \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (4.8)$$

STEP 1 — estimate $\|\Delta \mathcal{V} - \Delta \mathcal{V}_\infty\|_{L^\infty}$.

We take the Laplacian of (4.8):

$$\Delta \tilde{\mathcal{V}}(\mathbf{x}) - \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x}) = \Delta I_1 + \Delta I_2 + \Delta I_3, \quad (4.9)$$

and estimate the three terms on the RHS.

$$\begin{aligned} |\Delta I_1| &= \left| \int \Delta w(\mathbf{x} - \mathbf{y})(\Delta \tilde{\mathcal{V}}(\mathbf{y}) - \Delta \tilde{\mathcal{V}}_\infty(\mathbf{y}))\chi_{\text{supp } \rho \cap \text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} \right| \\ &\leq \epsilon \cdot |\text{supp } \rho_\infty| \cdot \|\Delta \mathcal{V} - \Delta \mathcal{V}_\infty\|_{L^\infty}, \end{aligned} \quad (4.10)$$

by (2.16).

$$\begin{aligned}
 |\Delta I_2| &= \left| \int \Delta w(\mathbf{x} - \mathbf{y}) \Delta \tilde{\mathcal{V}}(\mathbf{y}) \chi_{\text{supp } \rho \setminus \text{supp } \rho_\infty}(\mathbf{y}) \, d\mathbf{y} \right| \\
 &\leq \epsilon \cdot m_0(1 + \epsilon) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right|,
 \end{aligned} \tag{4.11}$$

by (2.16) and (4.7).

To estimate I_3 , we first use the fact that ρ and ρ_∞ have the same total mass, and obtain

$$\begin{aligned}
 0 &= \int \rho(\mathbf{x}) \, d\mathbf{x} - \int \rho_\infty(\mathbf{x}) \, d\mathbf{x} \\
 &= \int \Delta \tilde{\mathcal{V}}(\mathbf{x}) \chi_{\text{supp } \rho}(\mathbf{x}) \, d\mathbf{x} - \int \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x}) \chi_{\text{supp } \rho_\infty}(\mathbf{x}) \, d\mathbf{x} \\
 &= \int (\Delta \tilde{\mathcal{V}}(\mathbf{x}) - \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x})) \chi_{\text{supp } \rho \cap \text{supp } \rho_\infty}(\mathbf{x}) \, d\mathbf{x} \\
 &\quad + \int \Delta \tilde{\mathcal{V}}(\mathbf{x}) \chi_{\text{supp } \rho \setminus \text{supp } \rho_\infty}(\mathbf{x}) \, d\mathbf{x} - \int \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x}) \chi_{\text{supp } \rho_\infty \setminus \text{supp } \rho}(\mathbf{x}) \, d\mathbf{x}.
 \end{aligned} \tag{4.12}$$

Therefore

$$\begin{aligned}
 &\left| \int \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x}) \chi_{\text{supp } \rho_\infty \setminus \text{supp } \rho}(\mathbf{x}) \, d\mathbf{x} \right| \\
 &= \left| \int (\Delta \tilde{\mathcal{V}}(\mathbf{x}) - \Delta \tilde{\mathcal{V}}_\infty(\mathbf{x})) \chi_{\text{supp } \rho \cap \text{supp } \rho_\infty}(\mathbf{x}) \, d\mathbf{x} + \int \Delta \tilde{\mathcal{V}}(\mathbf{x}) \chi_{\text{supp } \rho \setminus \text{supp } \rho_\infty}(\mathbf{x}) \, d\mathbf{x} \right| \\
 &\leq |\text{supp } \rho_\infty| \cdot \|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} + m_0(1 + \epsilon) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right|.
 \end{aligned} \tag{4.13}$$

This implies

$$\begin{aligned}
 |\Delta I_3| &= \left| \int \Delta w(\mathbf{x} - \mathbf{y}) \Delta \tilde{\mathcal{V}}_\infty(\mathbf{y}) \chi_{\text{supp } \rho_\infty \setminus \text{supp } \rho}(\mathbf{y}) \, d\mathbf{y} \right| \\
 &\leq \epsilon \cdot |\text{supp } \rho_\infty| \cdot \|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} + \epsilon \cdot m_0(1 + \epsilon) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right|.
 \end{aligned} \tag{4.14}$$

Finally, use these in (4.9) we conclude that

$$\|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} \leq 2\epsilon \cdot |\text{supp } \rho_\infty| \cdot \|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} + 2\epsilon \cdot m_0(1 + \epsilon) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right|. \tag{4.15}$$

If ϵ is small enough so that $|\text{supp } \rho_\infty| \cdot 2\epsilon < 1$, then

$$\|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} \leq \frac{2\epsilon \cdot m_0(1 + \epsilon)}{1 - |\text{supp } \rho_\infty| \cdot 2\epsilon} \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right|. \tag{4.16}$$

As a byproduct, this shows that unless $\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty = 0$ which implies the conclusion, we always have $\text{supp } \rho \not\subset \text{supp } \rho_\infty = \{\mathbf{x} : |\mathbf{x}| \leq R_\infty\}$ and therefore $R > R_\infty$. Now we will show that the option $R > R_\infty$ is impossible.

STEP 2 — use comparison principle. Assume on the contrary that $R > R_\infty$. Taking ∇ on (4.8) and conducting similar estimates gives

$$|\nabla\tilde{\mathcal{V}}(\mathbf{x}) - \nabla\tilde{\mathcal{V}}_\infty(\mathbf{x})| \leq \epsilon \cdot \frac{2R}{d} \cdot 2 \left(|\text{supp } \rho_\infty| \cdot \|\Delta\tilde{\mathcal{V}} - \Delta\tilde{\mathcal{V}}_\infty\|_{L^\infty} + m_0(1 + \epsilon) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right| \right), \quad \forall |\mathbf{x}| \leq R, \quad (4.17)$$

using $|\nabla w(\mathbf{x} - \mathbf{y})| \leq \epsilon \frac{|\mathbf{x} - \mathbf{y}|}{d} \leq \epsilon \cdot \frac{2R}{d}$ by (4.2).

The fact that $\Delta\tilde{\mathcal{V}}_\infty \chi_{|\mathbf{x}| \leq R}$ is a steady state of (2.1) with $\tilde{\mathcal{V}}_\infty$ implies

$$- \int \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) \chi_{|\mathbf{y}| \leq R}(\mathbf{y}) \, d\mathbf{y} - \nabla\tilde{\mathcal{V}}_\infty(\mathbf{x}) = 0, \quad \forall |\mathbf{x}| \leq R. \quad (4.18)$$

Taking difference with (4.4) and evaluating at $\mathbf{x} \in \text{supp } \rho$ with $|\mathbf{x}| = R$ (such an \mathbf{x} exists due to the definition of R) gives

$$\begin{aligned} & - \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) (\Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) - \rho(\mathbf{y}))_+ \, d\mathbf{y} \\ & - \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) (\Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) - \rho(\mathbf{y}))_- \, d\mathbf{y} - (\nabla\tilde{\mathcal{V}}_\infty(\mathbf{x}) - \nabla\tilde{\mathcal{V}}(\mathbf{x})) = 0. \end{aligned} \quad (4.19)$$

Since $\text{supp } \rho \subset \mathbb{B}_R$ and $\rho = \Delta\tilde{\mathcal{V}} \chi_{\text{supp } \rho}$,

$$|(\Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) - \rho(\mathbf{y}))_-| \leq \|\Delta\tilde{\mathcal{V}}_\infty - \Delta\tilde{\mathcal{V}}\|_{L^\infty}, \quad \forall |\mathbf{y}| \leq R. \quad (4.20)$$

Also notice that since $R \geq R_\infty$, we have $\int_{|\mathbf{y}| \leq R} \rho(\mathbf{y}) \, d\mathbf{y} = m_0 = \int_{|\mathbf{y}| \leq R_\infty} \Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) \, d\mathbf{y}$, which implies

$$\begin{aligned} & \int_{|\mathbf{y}| \leq R} \Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) \, d\mathbf{y} - \int_{|\mathbf{y}| \leq R} \rho(\mathbf{y}) \, d\mathbf{y} \\ & = \int_{|\mathbf{y}| \leq R} \Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) \, d\mathbf{y} - \int_{|\mathbf{y}| \leq R_\infty} \Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) \, d\mathbf{y} \geq m_0(1 - \epsilon) |\{R_\infty \leq |\mathbf{y}| \leq R\}|. \end{aligned}$$

Therefore

$$\int_{|\mathbf{y}| \leq R} (\Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) - \rho(\mathbf{y}))_+ \, d\mathbf{y} \geq m_0(1 - \epsilon) |\{R_\infty \leq |\mathbf{y}| \leq R\}|. \quad (4.21)$$

Take inner product of (4.19) with \mathbf{x} . Lemma 3.1 with (4.21) shows that

$$-\mathbf{x} \cdot \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) (\Delta\tilde{\mathcal{V}}_\infty(\mathbf{y}) - \rho(\mathbf{y}))_+ \, d\mathbf{y} \geq \frac{c_d}{R^{d-2}} \cdot m_0(1 - \epsilon) |\{R_\infty \leq |\mathbf{y}| \leq R\}|. \quad (4.22)$$

Then we estimate the other two terms in (4.19), after taking inner product with \mathbf{x} :

$$\begin{aligned}
 & \left| \mathbf{x} \cdot \int_{|\mathbf{y}| \leq R} \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) (\Delta \tilde{\mathcal{V}}_\infty(\mathbf{y}) - \rho(\mathbf{y}))_- \, d\mathbf{y} - \mathbf{x} \cdot (\nabla \tilde{\mathcal{V}}_\infty(\mathbf{x}) - \nabla \tilde{\mathcal{V}}(\mathbf{x})) \right| \\
 & \leq \|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} \cdot \int (-\mathbf{x}) \cdot \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \chi_{|\mathbf{y}| \leq R}(\mathbf{y}) \, d\mathbf{y} + R |\nabla \tilde{\mathcal{V}}(\mathbf{x}) - \nabla \tilde{\mathcal{V}}_\infty(\mathbf{x})| \\
 & \leq \|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} \cdot \frac{R^2}{d} + R \cdot \epsilon \cdot \frac{2R}{d} \cdot 2 \left(|\text{supp } \rho_\infty| \cdot \|\Delta \tilde{\mathcal{V}} - \Delta \tilde{\mathcal{V}}_\infty\|_{L^\infty} \right. \\
 & \quad \left. + m_0(1 + \epsilon) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right| \right) \\
 & \leq 2\epsilon \cdot m_0(1 + \epsilon) \frac{R^2}{d} \cdot \left(\frac{1 + 4\epsilon \cdot |\text{supp } \rho_\infty|}{1 - |\text{supp } \rho_\infty| \cdot 2\epsilon} + 2d \right) \cdot \left| \text{supp } \rho \setminus \text{supp } \rho_\infty \right| \\
 & \leq 2\epsilon \cdot m_0(1 + \epsilon) \frac{R^2}{d} \cdot \left(\frac{1 + 4\epsilon \cdot |\text{supp } \rho_\infty|}{1 - |\text{supp } \rho_\infty| \cdot 2\epsilon} + 2d \right) \cdot \min\{|\{R_\infty \leq |\mathbf{y}| \leq R\}|, |\text{supp } \rho|\},
 \end{aligned} \tag{4.23}$$

where the first inequality uses the fact that $(-\mathbf{x}) \cdot \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \geq 0$ by Lemma 3.1, the second inequality uses (4.17) and the fact that $\chi_{|\mathbf{y}| \leq R}$ is a steady state of (2.1) with $\mathcal{V}(\mathbf{x}) = |\mathbf{x}|^2/(2d)$, and the third inequality uses (4.16).

If $R \leq 2R_\infty$, then (4.22) and (4.23) contradict (4.19). In fact, if $R > R_\infty$, and ϵ is small enough such that

$$2\epsilon \cdot \frac{1 + \epsilon}{1 - \epsilon} \cdot \frac{1}{d} \left(\frac{1 + 4\epsilon \cdot |\text{supp } \rho_\infty|}{1 - |\text{supp } \rho_\infty| \cdot 2\epsilon} + 2d \right) < \frac{c_d}{(2R_\infty)^d}, \tag{4.24}$$

then the RHS of (4.22) is greater than that of (4.23), which gives the contradiction.

If $R > 2R_\infty$, then by the estimates

$$|\text{supp } \rho| \leq \frac{1}{1 - \epsilon}, \quad |\{R_\infty \leq |\mathbf{y}| \leq R\}| \geq \frac{2^d - 1}{2^d} |\mathbb{B}_1| \cdot R^d, \quad \forall R > 2R_\infty. \tag{4.25}$$

(4.22) and (4.23) contradict (4.19), if ϵ is small enough such that

$$2\epsilon \cdot \frac{1 + \epsilon}{(1 - \epsilon)^2} \cdot \frac{1}{d} \left(\frac{1 + 4\epsilon \cdot |\text{supp } \rho_\infty|}{1 - |\text{supp } \rho_\infty| \cdot 2\epsilon} + 2d \right) < c_d \frac{2^d - 1}{2^d}. \tag{4.26}$$

Notice the estimate

$$|\text{supp } \rho_\infty| \leq \frac{1}{1 - \epsilon}, \quad R_\infty \leq \frac{c_d}{(1 - \epsilon)^{1/d}}, \tag{4.27}$$

which implies the smallness conditions (4.24) and (4.26) on ϵ only depend on d . \square

Remark 4.1. *Compared to the proof of Theorem 2.1, the main new ingredient in the above proof is a contraction argument, which can be seen in the derivation from (4.15) to (4.16).*

5. APPENDIX

5.1. 1D steady state are not unique. In the Appendix we give a description of the steady states (2.1) when $d = 1$. In this case, one can write (2.1) as

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad u(t, x) = - \int \mathcal{N}'(x - y) \rho(t, y) \, dy - \mathcal{V}'(x). \tag{5.1}$$

Define $m(t, x)$ as the primitive of $\rho(t, x)$:

$$m(t, x) := \int_{-\infty}^x \rho(t, y) dy - \frac{m_0}{2}. \quad (5.2)$$

We have (omitting t -dependence)

$$\int_{-\infty}^x \partial_y(\rho u) dy = \rho(x)u(x) = \rho(x) \left(- \int \mathcal{N}'(x-y)\rho(y) dy - \mathcal{V}'(x) \right), \quad (5.3)$$

and

$$\begin{aligned} - \int \mathcal{N}'(x-y)\rho(y) dy &= - \int_{-\infty}^{\infty} \mathcal{N}'(x-y)\partial_y m(y) dy \\ &= - \lim_{y \rightarrow \infty} \mathcal{N}'(x-y)m(y) + \lim_{y \rightarrow -\infty} \mathcal{N}'(x-y)m(y) - \int_{-\infty}^{\infty} \mathcal{N}''(x-y)m(y) dy \\ &= - \frac{1}{2} \cdot \frac{m_0}{2} + \left(-\frac{1}{2}\right) \cdot \left(-\frac{m_0}{2}\right) + m(x) = m(x). \end{aligned} \quad (5.4)$$

Therefore, by integrating (5.1) in x , we see that $m(t, x)$ satisfies

$$\partial_t m + (m(x) - \mathcal{V}'(x))\partial_x m = 0. \quad (5.5)$$

For fixed t , since $m(t, x)$ is an increasing function in x , one can define $X(t, m)$ as its inverse function, except a countable set of values of m . Then $X(t, m)$, for almost all $m \in (-m_0/2, m_0/2)$, satisfies an ODE

$$\frac{d}{dt} X(t, m) = m - \mathcal{V}'(X). \quad (5.6)$$

Therefore, as long as \mathcal{V} is super-linear:

$$\lim_{x \rightarrow \infty} \mathcal{V}'(x) = \infty, \quad \lim_{x \rightarrow -\infty} \mathcal{V}'(x) = -\infty. \quad (5.7)$$

(5.6) drives $X(t, m)$ to the equilibrium point x with $\mathcal{V}'(x) = m$, which lies in the same basin of attraction as the initial data $X_{in}(m)$. If \mathcal{V} is strictly convex, then there is a unique x with $\mathcal{V}'(x) = m$; otherwise there may be more than one x . Therefore we conclude:

Proposition 5.1. *If \mathcal{V} is super-linear, then the solution to (5.1) with compactly supported initial data converges to a steady state as $t \rightarrow \infty$, in the sense that $\lim_{t \rightarrow \infty} X(t, m) = X_\infty(m)$ for almost all $m \in (-m_0/2, m_0/2)$, for some $X_\infty(m)$ with $\mathcal{V}'(X_\infty(m)) = m$.*

If in addition, \mathcal{V} is strictly convex, then the steady state is unique for each fixed m_0 ; if $\mathcal{V}''(x) \geq a > 0, \forall x$, then the convergence rate of the limit $\lim_{t \rightarrow \infty} X(t, m) = X_\infty(m)$ is exponential, being uniform in m .

If \mathcal{V} is not convex, then the steady state may fail to be unique.

5.2. Steady states must have compact support.

Proposition 5.2. *Let $d \geq 2$, and \mathcal{V} be a radial potential satisfying $\|\Delta \mathcal{V}\|_{L^\infty} < \infty$ and the condition:*

$$V'(r) \geq c_V r^{-\frac{d-1}{d+1}}, \quad \forall r \geq R_0, \quad (5.8)$$

for some $R_0 > 0$, where $c_V > 0$. Then any steady state of (2.1) has compact support.

Proof. Let $\rho = \Delta\mathcal{V}\chi_{\text{supp}\rho}$ be a steady state, and take $R > 0$. We aim to prove that when R is large enough, then $\text{supp}\rho \cap \{|\mathbf{x}| = R\} = \emptyset$. In the rest of the proof, we denote

$$\epsilon_R = \int_{|\mathbf{x}| > R} \rho(\mathbf{x}) \, d\mathbf{x}, \quad \text{satisfying} \quad \lim_{R \rightarrow \infty} \epsilon_R = 0. \quad (5.9)$$

Suppose the contrary, then we take $\mathbf{x} \in \text{supp}\rho \cap \{|\mathbf{x}| = R\}$, and we may assume $\mathbf{x} = (R, 0, \dots, 0)^T$ without loss of generality. The steady state equation (1.6) implies

$$- \int \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} - \nabla\mathcal{V}(\mathbf{x}) = 0. \quad (5.10)$$

Taking inner product with \mathbf{x} gives

$$- \int \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} - V'(R)R = 0. \quad (5.11)$$

We aim to show that the LHS is negative which leads to a contradiction. We first write

$$\begin{aligned} - \int \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} &= c \int \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) \, d\mathbf{y} \leq c \int_{y_1 \leq R} \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) \, d\mathbf{y} \\ &\leq - \int_{|\mathbf{y}| \leq R} \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} \\ &\quad + c \int_{R - \delta \leq y_1 \leq R} \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) \, d\mathbf{y} \\ &\quad + c \int_S \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) \, d\mathbf{y}, \end{aligned} \quad (5.12)$$

where y_1 denotes the first component of \mathbf{y} , $\delta > 0$ is small, to be determined, and

$$S := \{\mathbf{y} : y_1 \leq R\} \setminus \left(\mathbb{B}_R \cup \{\mathbf{y} : R - \delta \leq y_1 \leq R\} \right). \quad (5.13)$$

Now we estimate the three terms on the RHS of (5.12) separately:

The first term (combined with the term $V'(R)R$ in (5.11)). Similar to STEP 3 of the proof Theorem 2.2, we write

$$\begin{aligned} - \int_{|\mathbf{y}| \leq R} \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} - V'(R)R &= \int_{\mathbb{B}_R \setminus \text{supp}\rho} \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\Delta\mathcal{V}(\mathbf{y}) \, d\mathbf{y} \\ &\leq - \frac{c}{R^{d-2}} \int_{\mathbb{B}_R \setminus \text{supp}\rho} \Delta\mathcal{V}(\mathbf{y}) \, d\mathbf{y}. \end{aligned} \quad (5.14)$$

Notice that by the assumption (5.8),

$$\int_{|\mathbf{y}| \leq R} \Delta\mathcal{V}(\mathbf{y}) \, d\mathbf{y} = \int_{|\mathbf{y}|=R} \nabla\mathcal{V}(\mathbf{y}) \cdot \vec{n} \, dS(\mathbf{y}) = cR^{d-1}V'(R) \geq cR^{d-1-\frac{d-1}{d+1}}, \quad (5.15)$$

for R sufficiently large, and

$$\int_{\text{supp}\rho} \Delta\mathcal{V}(\mathbf{y}) \, d\mathbf{y} = m_0. \quad (5.16)$$

Therefore, since $d - 1 - \frac{d-1}{d+1} > 0$, we get

$$- \int_{|\mathbf{y}| \leq R} \mathbf{x} \cdot \nabla\mathcal{N}(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \, d\mathbf{y} - V'(R)R \leq - \frac{c}{R^{d-2}} \cdot R^{d-1-\frac{d-1}{d+1}} = -cR^{\frac{2}{d+1}}. \quad (5.17)$$

The second term. One can show that for fixed $y_1 < R$, writing $\mathbf{y} = (y_1, \mathbf{y}')$, $\mathbf{y}' \in \mathbb{R}^{d-1}$,

$$\frac{\mathbf{x}}{|\mathbf{x}|} \cdot \int_{\mathbb{R}^{d-1}} \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} d\mathbf{y}' = C, \quad (5.18)$$

is independent of y_1 . In fact,

$$\begin{aligned} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \int_{\mathbb{R}^{d-1}} \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} d\mathbf{y}' &= \int_{\mathbb{R}^{d-1}} \frac{R - y_1}{((R - y_1)^2 + (\mathbf{y}')^2)^{d/2}} d\mathbf{y}' \\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + (\mathbf{y}')^2)^{d/2}} d\mathbf{y}' = C. \end{aligned} \quad (5.19)$$

Therefore, using the assumption $\|\Delta\mathcal{V}\|_{L^\infty} < \infty$, we get

$$\int_{R-\delta \leq y_1 \leq R} \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) d\mathbf{y} \leq CR \int_{R-\delta \leq y_1 \leq R} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \int_{\mathbb{R}^{d-1}} \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} d\mathbf{y}' dy_1 \leq C\delta R. \quad (5.20)$$

The third term. We claim that

$$|\mathbf{x} - \mathbf{y}| \geq \sqrt{\delta R}, \quad \forall \mathbf{y} \in S. \quad (5.21)$$

For those \mathbf{y} with $y_1 < 0$, this is clear because $|\mathbf{x} - \mathbf{y}| \geq R$ in this case. For those $\mathbf{y} = (y_1, \mathbf{y}')$ with $y_1 \geq 0$, notice that

$$|\mathbf{x} - \mathbf{y}|^2 = (R - y_1)^2 + |\mathbf{y}'|^2 \geq |\mathbf{y}'|^2 = |\mathbf{y}|^2 - y_1^2. \quad (5.22)$$

By the definition of S , we have $|\mathbf{y}|^2 \geq R^2$ and $y_1^2 \leq (R - \delta)^2$. Therefore

$$|\mathbf{x} - \mathbf{y}|^2 \geq R^2 - (R - \delta)^2 = 2\delta R - \delta^2 \geq \delta R, \quad (5.23)$$

using the smallness of δ . This proves the claim.

Using (5.21), we get

$$\frac{|\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^d} \leq R \cdot \frac{1}{|\mathbf{x} - \mathbf{y}|^{d-1}} \leq R \cdot (\delta R)^{-(d-1)/2} = \delta^{-(d-1)/2} R^{-(d-3)/2}, \quad (5.24)$$

which together with the assumption $\|\Delta\mathcal{V}\|_{L^\infty} < \infty$, gives the estimate

$$\int_S \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) d\mathbf{y} \leq C\epsilon_R \delta^{-(d-1)/2} R^{-(d-3)/2}, \quad (5.25)$$

using the fact that $S \cap \mathbb{B}_R = \emptyset$.

Now we take

$$\delta = \epsilon_R^{2/(d+1)} R^{-(d-1)/(d+1)}, \quad (5.26)$$

to equate the second and third terms, and finally obtain the estimate

$$\begin{aligned} 0 &\leq - \int_{|\mathbf{y}| \leq R} \mathbf{x} \cdot \nabla \mathcal{N}(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y} - V'(R)R + c \int_{R-\delta \leq y_1 \leq R} \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) d\mathbf{y} \\ &\quad + c \int_S \frac{\mathbf{x} \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^d} \rho(\mathbf{y}) d\mathbf{y} \leq -cR^{2/(d+1)} + C\epsilon_R^{2/(d+1)} R^{2/(d+1)}. \end{aligned} \quad (5.27)$$

This gives the desired contradiction for large enough R , in view of (5.9). □

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