Communications in Contemporary Mathematics
(2025) 2540006 (23 pages)
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DOI: 10.1142/S0219199725400061

Decrease of entropy and emergence of order in collective dynamics

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Received 21 June 2025
Revised 22 October 2025
Accepted 29 October 2025
Published

In memory of Haïm Brezis

We study the hydrodynamic description of collective dynamics driven by velocity *alignment*. It is known that such Euler alignment systems must flock toward a limiting "flocking" velocity, provided their solutions remain globally smooth. To address this question of global existence we proceed in two steps. (i) Entropy and closure. The system lacks a closure, reflecting lack of detailed energy balance in collective dynamics. We discuss the decrease of entropy and the asymptotic behavior toward a mono-kinetic closure; and (ii) Mono-kinetic closure. We prove that global regularity persists for all time for a large class of initial conditions satisfying a critical threshold condition, which is intimately linked to the decrease of entropy. The result applies in any number of spatial dimensions, thus addressing the open question of existence beyond two dimensions.

Keywords: Alignment; hydrodynamics; entropy; mono-kinetic closure; existence.

Mathematics Subject Classification 2020: 35L65, 35Q31, 76N10, 92D25

Dedication

Haïm Brezis was one of the leading French analysts in 20th century mathematics. In addition to his profound contributions to mathematics, he had two characteristics that left a lasting impact on both his contemporaries and future generations. Brezis was a *scholar*: when one encounters an inequality or a theorem in his papers, Brezis ensures that his reader does not merely understand *why* is it true, but also *where* and *how* it arises. His scholarly approach provided context, insight and clarity into the origins and applications of mathematical concepts. Further, Brezis was a gifted *master of presentation*, with exceptional ability to present mathematical ideas in a coherent and engaging manner: when one reads his papers and books, his writings create a unique feeling of being part of a seamless narrative. Two examples are in order. His 2003 JAMS paper with Bourgain[a] is such a joy to read, that it inspired me to work on this topic. Everything becomes interconnected. When reading his 2024 CCM paper with Mawhin and Mironescu,[b] one has the feeling of being journeying through mathematical history with Jacobi, Sylvester, Poincaré, Hadamard, …everything flows so naturally in the hands of Brezis.

No wonder that his books on Functional Analysis and PDEs have influenced and, to a large extent, shaped a whole generation of French mathematicians. The scientific legacy of Haïm Brezis will continue to inspire mathematicians around the world.

1. Introduction and Statement of Main Results

We are concerned with the existence and large-time behavior of global smooth solutions for the multi-dimensional Euler alignment system

$$\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u} + \mathbb{P}) = \tau \displaystyle\int \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(t,\mathbf{y}) - \mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{x})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}. \end{cases} \tag{1}$$

A solution pair of density–velocity, $(\rho, \mathbf{u}) : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \times \mathbb{R}^n$, is sought subject to compactly supported initial data $(\rho(0,\mathbf{x}), \mathbf{u}(0,\mathbf{x})) = (\rho_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x}))$, either in the whole space, $\Omega = \mathbb{R}^n$, or in the n-dimensional torus $\Omega = \mathbb{T}^n$. System (1) is the large-crowd hydrodynamic description of "social agents" identified by positions and velocities, $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1}^N$, $N \gg 1$, governed by the Cucker–Smale alignment model [8]

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{\tau}{N} \displaystyle\sum_{j=1}^N \phi(\mathbf{x}_i, \mathbf{x}_j)(\mathbf{v}_j - \mathbf{v}_i). \end{cases} \tag{2}$$

Alignment dynamics is a canonical model governing emergence phenomena in collective dynamics of flocking, swarming, etc. The dynamics is driven by a non-negative symmetric *communication kernel*, $\phi(\mathbf{x},\mathbf{y}) = \phi(\mathbf{y},\mathbf{x}) \geq 0$, with amplitude $\tau > 0$. We have two main examples of symmetric kernels in mind — the canonical Cucker–Smale class of *metric kernels*, see [8]

$$\phi(\mathbf{x},\mathbf{y}) = \phi(|\mathbf{x} - \mathbf{y}|), \tag{3}$$

[a]J. Bourgain and H. Brezis, On the equation div $Y = f$ and application to control of phases, *J. AMS* **16**(20) (2003) 393–426.
[b]H. Brezis, J. Mawhin and P. Mironescu, A brief history of the Jacobian, *Commun. Contemp. Math.* **26**(2) (2024) 2330001.

and the class of topologically-based kernels[38]

$$\phi(\mathbf{x},\mathbf{y}) = \phi_1(|\mathbf{x}-\mathbf{y}|)\phi_2(d_\rho(\mathbf{x},\mathbf{y})), \quad d_\rho(\mathbf{x},\mathbf{y}) := \int_{\mathcal{C}(\mathbf{x},\mathbf{y})} \rho(t,\mathbf{z})\mathrm{d}\mathbf{z},$$

reflecting the dependence on the mass in an intermediate "domain of communication", $\mathcal{C}(\mathbf{x},\mathbf{y})$, enclosed between $\mathbf{x}$ and $\mathbf{y}$.

The Euler alignment system (1) involves the pressure $\mathbb{P}$ — a symmetric positive definite tensor which should encode the thermodynamics of large-crowd collective dynamics. But (1) is not a closed system: it lacks closure of $\mathbb{P}$ in terms of the macroscopic variables ρ and $\mathbf{u}$. This reflects the fact that unlike physical particles, the social agents engaged in collective dynamics form a thermo-dynamically open system which is far from equilibrium and does not admit a universal closure. Indeed, most of the relevant literature *assumes* a mono-kinetic closure, $\mathbb{P} \equiv 0$. Accordingly, our study of solutions of (1) proceeds in two main stages:

(i) We investigate a rather general class of so-called *entropic pressures* introduced in [41] and conclude that their strong solutions must approach mono-kinetic closure. In particular, it is shown that strong Euler alignment solutions with isentropic closure experience a uniform entropy *decrease* to $-\infty$, and if we reject such a scenario then we must impose *mono-kinetic closure*, $\mathbb{P} \equiv 0$.

(ii) We consider the pressure-less or mono-kinetic Euler alignment, proving existence of global smooth solutions under certain *threshold conditions*. This addresses the open question of existence for $n \geq 3$, extending the known results for dimensions $n = 1, 2$ [4, 17, 42].

1.1. *The road to mono-kinetic closure*

To investigate this mono-kinetic assumption we let ρE denote the (total) energy associated with the pressure in (1) (here and below, $|\mathbf{w}|$ denotes the ℓ_2-norm of $\mathbf{w}$ and $|\mathbf{w}|_\infty$ denote its L^∞-norm),

$$\rho E := \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e, \quad \rho e := \frac{1}{2}\mathrm{trace}(\mathbb{P}). \tag{4}$$

Since the social agents engaged in collective dynamics are often driven by energy received from the "outside", the detailed energy balance associated with (1) may be less relevant [43, §1.1]. Instead, lack of thermal equilibrium in the form of closure *equalities*, can be relaxed to certain *inequalities*, which are compatible with the decay of *energy fluctuations*. We impose the notion of an *entropic pressure* in which $\mathbb{P}$, augmented with arbitrary "heat-flux" vector function $\mathbf{q}$, are required to satisfy the inequality

$$\begin{aligned}(\rho E)_t &+ \nabla_{\mathbf{x}} \cdot (\rho E\mathbf{u} + \mathbb{P}\mathbf{u} + \mathbf{q}) \\ &\leq -\tau \int \phi(\mathbf{x},\mathbf{y})(2\rho E(t,\mathbf{x}) - \rho\mathbf{u}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{y}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}.\end{aligned} \tag{5}$$

This can be expressed in an equivalent form[c] in terms of an entropic inequality for the internal energy, ρe, stating [41, Definition 1.1]

$$(\rho e)_t + \nabla_{\mathbf{x}} \cdot (\rho e\mathbf{u} + \mathbf{q}) + \mathrm{trace}(\mathbb{P}\nabla\mathbf{u}) \leq -2\tau \int \phi(\mathbf{x},\mathbf{y})\rho e(t,\mathbf{x})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}. \tag{6}$$

The entropic inequalities (5) or (6) are flexible enough to cover pressure tensors derived from an underlying kinetic formulation discussed in Sec. 2, as well as the mono-kinetic closure $\mathbb{P} \equiv 0$. They imply *depletion of fluctuations* and in particular, as we shall see later on, that the pressure in entropic alignment dynamics approaches the mono-kinetic closure. In Sec. 3, we discuss several results which quantify this statement of "approach toward mono-kinetic closure". To state one of these results, we need the following notations.

Notations. We let $\nabla_S \mathbf{u}$ denote the symmetric gradient, $(\nabla_S \mathbf{u})_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. We order the real eigenvalues of $\nabla_S \mathbf{u}$: $\lambda_- := \lambda_1 \leq \lambda_2 < \cdots \leq \lambda_n =: \lambda_+$, and in general, given a closed set $X \subset \mathbb{R}$ we let X_+ and X_- denote its largest, respectively smallest elements. We use η_c to denote different positive constants. Finally, we use $\phi * \rho$ to denote the average density, or *thickness*,

$$\phi * \rho(t,\mathbf{x}) := \int \phi(\mathbf{x},\mathbf{y})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y},$$

of course, in case of metric kernel, this coincides with the usual notion of convolution. In Sec. 3, we prove the following.

Theorem 1 (Decay toward mono-kinetic closure). *Let $(\rho,\mathbf{u},\mathbb{P})$ be a strong entropic solution of (1) and assume the following threshold condition holds: there exists a constant, $\eta_c > 0$, such that*

$$\lambda_-(\nabla_S \mathbf{u})(t,\cdot) + \tau\phi * \rho(t,\cdot) \geq \eta_c > 0. \tag{7}$$

Then $\int \|\mathbb{P}(t,\mathbf{x})\|\mathrm{d}\mathbf{x} \leq e^{-\eta_c t}\int \|\mathbb{P}_0(\mathbf{x})\|\mathrm{d}\mathbf{x}$.

1.2. *Communications kernels and thickness*

Different classes of communication kernels $\phi(\mathbf{x},\mathbf{y})$ are treated in the literature, classified according to their short- and long-range behavior where $\mathbf{x} \approx \mathbf{y}$, and, respectively, $|\mathbf{x}-\mathbf{y}| \gg 1$. The class of singular kernels $\phi(\mathbf{x},\mathbf{y}) \sim |\mathbf{x}-\mathbf{y}|^{-\alpha}$ was treated in [12] where it was shown that the enstrophy associated with strongly singular kernels, $\alpha \in [n, n+2)$, enforced mono-kinetic closure. Existence and

[c]The formal manipulation $(1)_2 \cdot \mathbf{u} - (1)_1 \times \frac{|\mathbf{u}|^2}{2}$ yields

$$\left(\frac{1}{2}\rho|\mathbf{u}|^2\right)_t + \nabla_{\mathbf{x}} \cdot \left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u} + \mathbb{P}\mathbf{u}\right) - \mathrm{trace}(\mathbb{P}\nabla\mathbf{u})$$
$$= -\tau \int \phi(\mathbf{x},\mathbf{y})\rho\mathbf{u}(t,\mathbf{x}) \cdot (\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}.$$

Combining this equality with (6) is equivalent with (5).

flocking behavior of the 1D mono-kinetic case with strongly singular kernels was studied in [10, 35–37].

In this paper, we restrict attention to the case of bounded kernels, $\phi(\cdot,\cdot) \leq \phi_+$. The results can be extended to the case of integrable kernel with weak singularity, $\phi(\mathbf{x},\mathbf{y}) \sim |\mathbf{x}-\mathbf{y}|^{-\alpha}$ with $\alpha < n$. Alignment with such weakly singular kernels were studied in [25].

Next, we make the distinction between long-range and short-range kernels. Assume that $\phi(\cdot,\cdot)$ admits a metric lower envelope, $\phi(\mathbf{x},\mathbf{y}) \gtrsim \varphi(|\mathbf{x}-\mathbf{y}|)$, with decreasing radial φ. Heavy-tailed kernels is the subclass of long-range kernels that naturally arise in connection with the unconditional flocking behavior of (1), [5, 6, 15, 16]

$$\int^{\infty} \varphi(r)\mathrm{d}r = \infty.$$

Heavy-tails rule out short-range kernels with finite support which are important in applications. In this latter case, flocking is secured if the Euler alignment dynamics remains uniformly non-vacuous, $\rho \geq \rho_- > 0$ [40, Theorem 3; 33, §7.3].

We proceed by making the following thickness condition. This corresponds to the notion of ball-thickness introduced in [33, §3.7.2]. The scenario of (uniform) thickness covers both cases of long-range (— heavy-tailed) kernels and short-range (non-vacuous) kernels; this will be discussed in Sec. 4 below.

Assumption 1 (Thickness). The following thickness condition holds:

$$\int^{\infty} \min_{\mathbf{x}} \phi * \rho(t,\mathbf{x})\mathrm{d}t = \infty.$$

In particular, a *uniform thickness* condition holds if there exists a constant, $c_* > 0$, such that

$$\phi * \rho(t,\cdot) \geq c_* > 0. \tag{8}$$

1.3. *Decrease of entropy and emergence of order*

We restrict attention to scalar pressure, $\mathbb{P} = p\mathbb{I}_{n\times n}$. In this case, we have the isentropic closure, $p = \frac{1}{n}\mathrm{trace}(\mathbb{P}) = \frac{2}{n}\rho e$, and the entropic inequality (6) reads

$$p_t + \mathbf{u}\cdot\nabla_{\mathbf{x}} p + \gamma p \nabla_{\mathbf{x}}\cdot\mathbf{u} \leq -2\tau p\phi * \rho, \quad \gamma := 1 + \frac{2}{n}.$$

Next we manipulate — multiplying the last inequality by $\rho^{-\gamma}$ and adding a multiple of the mass equation, $-\gamma\rho^{\gamma-1}p \times (1)_1$, to find

$$(p\rho^{-\gamma})_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}(p\rho^{-\gamma}) \leq -2\tau p\rho^{-\gamma}\phi * \rho.$$

We conclude that the entropy inequality we imposed in (6) amounts to an inequality on the specific entropy, $\ln(p\rho^{-\gamma})$,

$$S_t + \mathbf{u}\cdot\nabla_{\mathbf{x}} S \leq -2\tau\phi * \rho < 0, \quad S := \ln(p\rho^{-\gamma}). \tag{9}$$

The conclusion that entropy decreases in time is quite striking in the sense that it counters the usual tendency of entropy to quantify disorganization as expressed by the second law of entropy increase in physical systems. Ever since [30] stipulated that life maintains its own organization by extracting order from its environment, stating "*life ... feeds on negative entropy*", there have been many attempts to link living systems to a reversed second law which drives organisms toward higher ordered content [1, 7]. We have shown here that the self-organization of alignment dynamics decreases the entropy by communicating "information" in the whole flowfield through decrease of fluctuations. In Sec. 3, we further elaborate on implications of the entropy decrease in alignment dynamics. Here is one manifestation of (9) proved in Sec. 3.1.

Theorem 2. *Let* $(\rho, \mathbf{u})$ *be a strong entropic solution of* (1) *with isentropic closure* $\mathbb{P} = \frac{2}{n}\rho e \mathbb{I}_{n\times n}$, *and assume the thickness condition holds. Then there is an entropy decay* $S(t,\cdot) \stackrel{t\to\infty}{\longrightarrow} -\infty$, *unless* $\mathbb{P} \equiv 0$.

Theorem 2 expresses the following dichotomy between two possible scenarios for a non-vacuous strong isentropic solutions: either, for $\rho, p > 0$, there is large time decay toward a mono-kinetic closure — in fact the uniform thickness (8) implies exponential decay *uniformly for all* $\mathbf{x}$

$$\max_{\mathbf{x}} p\rho^{-\gamma}(t,\mathbf{x}) \lesssim e^{-2\tau c_* t} \stackrel{t\to\infty}{\longrightarrow} 0 \tag{10}$$

or else a mono-kinetic closure $p \equiv 0$. If we reject the first scenario, then we conclude Euler alignment system can admit global strong solutions only in the setting of mono-kinetic closure.

1.4. *Strong solutions with mono-kinetic closure*

We now turn to investigate the existence of strong solutions of the alignment dynamics system (1) with mono-kinetic closure $\mathbb{P} \equiv 0$,

$$\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \tau \displaystyle\int \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(t,\mathbf{y}) - \mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})d\mathbf{y}. \end{cases} \tag{11}$$

The 1D system (11) with metric kernel (3) admits global smooth solution if and only if the initial data satisfies the critical threshold condition $u_0' + \tau\phi * \rho_0 > 0$ [4]. This was extended to the class of uni-directional flows, $\mathbf{u}(t,\mathbf{x}) := (u(t,\mathbf{x}), 0, \ldots, 0)$ with $u : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$, in [21], and to 2D metric case in [17]. In Sec. 4, we address the open question of existence of solution of Euler alignment system (11) with general symmetric kernels in $n > 2$ spatial dimensions. The case of metric kernels is summarized in the following.

Theorem 3 (Global strong solutions with sub-critical data). *Consider the mono-kinetic Euler alignment* (11) *with metric communication kernel*

$\phi(\mathbf{x},\mathbf{y}) = \phi(|\mathbf{x}-\mathbf{y}|) > 0$ *satisfying a uniform thickness* (8), $\phi * \rho(t,\cdot) \geq c_* > 0$. *The system is subject to initial conditions* $(\rho_0, \mathbf{u}_0) \in L^1_+ \times W^{1,\infty}$ *with velocity fluctuations which do not exceed,*

$$8|\phi'|_\infty \cdot \delta u_0 < \tau c_*^2, \quad \delta u(t) := \sup_{\mathbf{x},\mathbf{y}\in \mathrm{supp}\{\rho(t,\cdot)\}} |\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})|. \tag{12}$$

Assume that the initial data $(\rho_0, \mathbf{u}_0)$ *satisfy a sub-critical threshold condition*

$$\lambda_-(\nabla_S \mathbf{u}_0)(\mathbf{x}) + \tau\phi * \rho_0(\mathbf{x}) \geq \eta_c > 0, \quad \eta_c := \frac{1}{2}\tau c_*. \tag{13}$$

Then the Euler alignment system (11)–(13) *admits a global smooth solution,* $(\rho(\cdot,t), \mathbf{u}(\cdot,t)) \in L^1_+ \times W^{1,\infty}(\mathbb{R}^n)$, *with uniformly bounded velocity gradient,* $|\nabla\mathbf{u}(t,\cdot)|_{L^\infty} \leq \max\{|\nabla\mathbf{u}_0|_{L^\infty}, c_*, C_0\} < \infty$.

Remark 1 (What does the threshold condition mean?). Considering the limiting case $\tau = 0$ then the threshold condition (13) requires $\lambda_-(\nabla_S \mathbf{u}_0)(\mathbf{x}) > 0$. In the one-dimensional case, this reflects the fact that the inviscid Burgers' equation admits global smooth solution for increasing profile $u_0' > 0$; however, this is a rather restricted set of initial profiles. Similarly, in the n-dimensional case, the threshold (13) with $\tau = 0$ reflects global smooth solutions of the pressure-less Euler $\mathbf{u}_t + \mathbf{u}\cdot\nabla_\mathbf{x}\mathbf{u} = 0$ for the restricted class of initial configurations $\lambda_-(\nabla_S \mathbf{u}_0)(\mathbf{x}) > 0$ (for which $\nabla_\mathbf{x}\cdot\mathbf{u}_0 > 0$ excludes, for example, $|\mathbf{u}(t,\cdot)| \overset{|\mathbf{x}|\to\infty}{\longrightarrow} 0$). Thus, the essence of the threshold condition (13) is securing global existence for a large set of initial configurations, by allowing $\lambda_-(\nabla_S \mathbf{u}_0)$ (or $\nabla_\mathbf{x}\cdot\mathbf{u}_0$) to admit negative values dictated by the local thickness, $\tau\phi*\rho$. Observe that according to (13), the admissible values of $\lambda_-(\nabla_S \mathbf{u}_0)$ include the negative range

$$\lambda_-(\nabla_S \mathbf{u}_0) \geq \eta_c - \tau\phi*\rho, \quad \eta_c - \tau\phi*\rho \leq -\frac{1}{2}\tau c_*, \ \ \eta_c = \frac{1}{2}\tau c_*.$$

This type of critical threshold which secures global regularity with negative "initial slopes" as long as they are "not too negative" was introduced in the context of Euler–Poisson equations in [11, 24] and is found useful for Euler alignment models [4, 42]; see [33] and the references therein.

Remark 2 (Comparing the threshold conditions). It is instructive that the threshold condition for global existence sought in (13), $\lambda_-(\nabla_S \mathbf{u}_0) + \tau\phi*\rho_0 \geq \eta_c > 0$, is the same condition we met earlier, (7), in the context of decay toward mono-kinetic closure.

We compare this threshold condition with the results available in current literature on the global regularity of mono-kinetic Euler alignment system (11) in one- and two dimensions. Global smooth solutions in the 1D case, and in the more general setup of *uni-directional* flows, exist if and only if the initial configuration satisfies the initial threshold $u_0'(x) + \tau\phi*\rho_0(x) \geq 0$ [4, 21, 22]. This corresponds to the limiting case $\eta_c = 0$. For the role of the zero set $\{x \,|\, u_0'(x) + \tau\phi*\rho_0(x) = 0\}$ we refer to [23].

A sufficient threshold for 2D regularity, [17, Theorem 2.1], requires the initial threshold $\nabla\cdot\mathbf{u}_0+\tau\phi*\rho_0>0$ *and* $(\lambda_+-\lambda_-)(\nabla_S\mathbf{u}_0)\leq\tau\delta_0$ with $\delta_0=\frac{1}{2}m_0\phi(D_\infty)$. Noting that $\nabla\cdot\mathbf{u}=\lambda_++\lambda_-$ then

$$\begin{aligned}\lambda_-(\nabla_S\mathbf{u}_0)+\tau\phi*\rho_0&=\frac{\lambda_++\lambda_-}{2}+\tau\phi*\rho_0-\frac{\lambda_+-\lambda_-}{2}\\&\geq\frac{\tau\phi*\rho}{2}-\frac{\tau\delta_0}{2},\quad \phi*\rho\geq m_0\phi(D_\infty).\end{aligned}$$

Thus, this 2D result is covered by Theorem 3 subject to threshold $\lambda_-(\nabla_S\mathbf{u}_0)+\tau\phi*\rho_0\geq\eta_c>0$ with $\eta_c=\frac{1}{2}\delta_0$.

2. Kinetic Formulation

The passage from the agent-based description (2) to the hydrodynamic description (1) goes through a kinetic formulation,

$$\partial_t f+\mathbf{v}\cdot\nabla_{\mathbf{x}}f=-\tau\nabla_{\mathbf{v}}\cdot Q_\phi(f,f),\quad (t,\mathbf{x},\mathbf{v})\in\mathbb{R}_+\times\Omega\times\mathbb{R}^n, \tag{14}$$

which governs the empirical distribution

$$f=f_N(t,\mathbf{x},\mathbf{v}):=\frac{1}{N}\sum_{j=1}^N\delta_{\mathbf{x}-\mathbf{x}_j(t)}\otimes\delta_{\mathbf{v}-\mathbf{v}_j(t)},$$

and is driven by pairwise communication protocol on the right of $(2)_2$

$$Q_\phi(f,f):=\iint_{\mathbb{R}^n\times\Omega}\phi(\mathbf{x},\mathbf{y})(\mathbf{v}'-\mathbf{v})f(t,\mathbf{x},\mathbf{v})f(t,\mathbf{y},\mathbf{v}')\mathrm{d}\mathbf{v}'\mathrm{d}\mathbf{y}. \tag{15}$$

The formal derivation of (14) was introduced in [16] and was justified in increasing order of rigor in [2, 5, 13, 15, 27–29]. For large crowds of N agents, $N\gg 1$, the dynamics (14) is captured by its first three limiting moments which are assumed to exist in a proper sense: the density $\rho(t,\mathbf{x}):=\lim_{N\to\infty}\int f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$, momentum $\rho\mathbf{u}(t,\mathbf{x}):=\lim_{N\to\infty}\int\mathbf{v}f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$ and pressure $\mathbb{P}(t,\mathbf{x}):=\lim_{N\to\infty}\int(\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u})f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$, governed by the Euler alignment system (1).

To address the lack of closure, we consider the energy balance associated with the limiting quadratic moment of (14) (which is assumed to exist) $\rho E(t,\mathbf{x}):=\lim_{N\to\infty}\int\frac{1}{2}|\mathbf{v}|^2f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$,

$$(\rho E)_t+\nabla_{\mathbf{x}}\cdot(\rho E\mathbf{u}+\mathbb{P}\mathbf{u}+\mathbf{q})=-\tau\int\phi(\mathbf{x},\mathbf{y})(2\rho E(t,\mathbf{x})-\rho\mathbf{u}(t,\mathbf{x})\cdot\rho\mathbf{u}(t,\mathbf{y}))\mathrm{d}\mathbf{y}.$$

The total energy ρE admits a decomposition into its kinetic and internal parts, $\rho E=\frac{1}{2}\rho|\mathbf{u}|^2+\rho e$, corresponding to the two-term decomposition[d] $|\mathbf{v}|^2$ "=" $|\mathbf{u}|^2+|\mathbf{v}-\mathbf{u}|^2$, namely,

$$\rho E=\frac{1}{2}\rho|\mathbf{u}|^2+\rho e,\quad (\rho e)(t,\mathbf{x}):=\lim_{N\to\infty}\frac{1}{2}\int|\mathbf{v}-\mathbf{u}|^2f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}=\frac{1}{2}\mathrm{trace}(\mathbb{P}).$$

[d]Here and below, "=" is interpreted as "equality modulo linear moments", noting that linear moments vanish $\int(\mathbf{v}-\mathbf{u})f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}=0$.

We note in passing that the internal energy coincides with the modulated kinetic energy, $\int |\mathbf{v}-\mathbf{u}|^2 f(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$, which enabled [3] a direct passage from agent-based description to a mean-field limit identified with pressure-less Euler alignment.

The energy flux on the left, $\rho E\mathbf{u} + \mathbb{P}\mathbf{u} + \mathbf{q}$, is recovered as the quadratic moment of (14) corresponding to the three-term decomposition

$$|\mathbf{v}|^2\mathbf{v} \text{``}=\text{''}\ |\mathbf{v}|^2\mathbf{u} + 2(\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u})\mathbf{u} + |\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u}).$$

This results in an energy flux expressed in terms of the pressure $\mathbb{P}$ and "heat flux" $\mathbf{q}(t,\mathbf{x}) := \lim_{N\to\infty} \frac{1}{2}\int (\mathbf{v}-\mathbf{u})|\mathbf{v}-\mathbf{u}|^2 f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$. To address the lack of closure problem, we impose the notion of an entropic pressure which relaxes the energy equality, requiring $(\mathbb{P},\mathbf{q})$ to satisfy the corresponding inequality (5)

$$\begin{aligned}&(\rho E)_t + \nabla_{\mathbf{x}}\cdot(\rho E\mathbf{u} + \mathbb{P}\mathbf{u} + \mathbf{q})\\ &\qquad\le -\tau\int \phi(\mathbf{x},\mathbf{y})(2\rho E(t,\mathbf{x}) - \rho\mathbf{u}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{y}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}.\end{aligned}$$

3. Decay of Fluctuations toward Mono-kinetic Closure

We discuss various scenarios in which an entropic pressure decays toward mono-kinetic closure.

Mono-kinetic closure with heavy-tailed kernels. The formulation of the entropy inequality in terms of the total energy is equivalent to imposing an instantaneous decay of energy fluctuations. Indeed, integration of (5) implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}\rho|\mathbf{u}|^2(t,\mathbf{x}) + \rho e(t,\mathbf{x})\right)\mathrm{d}\mathbf{x}\\ &\qquad\le -\tau\iint \phi(\mathbf{x},\mathbf{y})(\rho|\mathbf{u}|^2(t,\mathbf{x}) + 2\rho e(t,\mathbf{x}) - \rho\mathbf{u}(t,\mathbf{x})\cdot\mathbf{u}(t,\mathbf{y}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}.\end{aligned}$$

Symmetrization of the integrand on the left using the fact that $\int \rho\mathbf{u}(t,\mathbf{x})\mathrm{d}\mathbf{x} \equiv m_0$, and symmetrization of the integrand of the right using the assumed symmetry of ϕ, finally yields the decay of fluctuations

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2m_0}\iint\left(\frac{1}{2}|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2 + e(t,\mathbf{x}) + e(t,\mathbf{y})\right)\rho(\mathbf{x})\rho(\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\ &\qquad\le -\tau\iint \phi(\mathbf{x},\mathbf{y})\left(\frac{1}{2}|\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{y})|^2 + e(t,\mathbf{x}) + e(t,\mathbf{y})\right)\rho(\mathbf{x})\rho(\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.\end{aligned} \tag{16}$$

We conclude the following result of [41, Theorem 4.1].

Theorem 4. *Consider the Euler alignment system with heavy-tailed kernel*

$$\phi(\mathbf{x},\mathbf{y}) \gtrsim \langle|\mathbf{x}-\mathbf{y}|\rangle^{-\theta}, \quad \theta\le 1, \quad \langle r\rangle := (1+r^2)^{1/2}. \tag{17}$$

Let $(\rho, \mathbf{u})$ *be a non-vacuous strong solution of* (1) *with entropic pressure* $\mathbb{P}$. *Let* $D(t)$ *denote the diameter of* $\operatorname{supp}\rho(t,\cdot)$, *and assume the dispersion bound*

$$\int \langle D(t)\rangle^{-\theta}\mathrm{d}t = \infty, \quad D(t) := \max_{\mathbf{x},\mathbf{y}\in\operatorname{supp}\rho(t,\cdot)} |\mathbf{x}-\mathbf{y}|. \tag{18}$$

Then unconditional flocking holds

$$\iint \left(\frac{1}{2}|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})|^2 + e(t,\mathbf{x}) + e(t,\mathbf{y})\right)\rho(t,\mathbf{x})\rho(t,\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \stackrel{t\to\infty}{\longrightarrow} 0,$$

and in particular, asymptotic mono-kinetic closure holds, $\int \|\mathbb{P}(t,\mathbf{x})\|\mathrm{d}\mathbf{x} \stackrel{t\to\infty}{\longrightarrow} 0$.

This shows that under the dispersion bound (18), the same mechanism that is responsible for unconditional flocking, $\|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})\|_{L^2(\mathrm{d}\rho(\mathbf{x})\mathrm{d}\rho(\mathbf{y})} \stackrel{t\to\infty}{\longrightarrow} 0$, drives the alignment dynamics with *any* entropic pressure toward mono-kinetic closure[e] $\|\mathbb{P}(t,\cdot)\|_{L^1} \leq 2\int \rho e(,\mathbf{x})\mathrm{d}\mathbf{x} \stackrel{t\to\infty}{\longrightarrow} 0$.

Mono-kinetic closure with a threshold condition. The dispersion bound sought in (18) can be replaced by a threshold condition. To this we revisit the entropy inequality (6) which is re-written in terms of the re-scaled pressure $\overline{\mathbb{P}} = \frac{\mathbb{P}}{2\rho e}$

$$\partial_t(\rho e) + \nabla_{\mathbf{x}}\cdot(\rho e\mathbf{u} + \mathbf{q}) \leq -(\operatorname{trace}(\overline{\mathbb{P}}\nabla\mathbf{u}) + \tau\phi * \rho)2\rho e(t,\mathbf{x}), \quad \overline{\mathbb{P}} := \frac{\mathbb{P}}{2\rho e}. \tag{19}$$

Observe that $\overline{\mathbb{P}}$ is a symmetric positive definite matrix with unit trace. For such unit-trace matrices, we have

$$\operatorname{trace}(\overline{\mathbb{P}}M) \geq \lambda_-(M_S), \quad M_S := \frac{1}{2}(M + M^\top).$$

Indeed, if we let $\{\lambda_i > 0, \mathbf{w}_i\}$ be the complete eigen-system of $\overline{\mathbb{P}}$ then

$$\begin{aligned}\operatorname{trace}(\overline{\mathbb{P}}M) &= \sum_i \langle\overline{\mathbb{P}}M\mathbf{w}_i, \mathbf{w}_i\rangle = \sum_i \lambda_i(\overline{\mathbb{P}})\langle M\mathbf{w}_i, \mathbf{w}_i\rangle \\ &\geq \sum_i \lambda_i(\overline{\mathbb{P}})\lambda_-(M_S) = \lambda_-(M_S).\end{aligned}$$

In particular, therefore, (19) yields

$$\partial_t(\rho e) + \nabla_{\mathbf{x}}\cdot(\rho e\mathbf{u} + \mathbf{q}) \leq -2(\lambda_-(\nabla_S\mathbf{u}) + \tau\phi * \rho)\rho e(t,\mathbf{x}).$$

We conclude that if the threshold condition (7) holds

$$\eta(t,\mathbf{x}) \equiv \eta(\rho(t,\mathbf{x}),\mathbf{u}(t,\mathbf{x})) := \lambda_-(\nabla_S\mathbf{u})(t,\mathbf{x}) + \tau\phi * \rho(t,\mathbf{x}) \geq \eta_c > 0, \tag{20}$$

then the decay of internal energy follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho e(t,\mathbf{x})\mathrm{d}\mathbf{x} \leq -2\eta_c\int \rho e(t,\mathbf{x})\mathrm{d}\mathbf{x},$$

[e]Since $\mathbb{P}$ is symmetric positive-definite $\|\mathbb{P}(t,\mathbf{x})\| = \max_\lambda \lambda(\mathbb{P}) \leq \operatorname{trace}(\mathbb{P}) = 2\rho e(t,\mathbf{x})$.

which in turn proves Theorem 1. The key question is whether such threshold inequality $\eta(t, \mathbf{x}) \geq \eta_c > 0$ persists in time. Observe that (20) is independent of the thermo-dynamical state $\{e, \mathbb{P}\}$, and in particular, therefore, applies to the mono-kinetic closure $\mathbb{P} = 0$. This motivates our search in Sec. 4 for a *critical threshold* in the space of initial configurations, $\eta(\rho_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x})) \geq \eta_c > 0$.

3.1. *Entropy decrease and mono-kinetic closure*

We restrict attention to the case of isentropic closure $p = (\gamma - 1)\rho e$ which led to the reverse entropy inequality (9).

$$S_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} S \leq -2\tau\phi * \rho < 0, \quad S = p\rho^{-\gamma}. \tag{21}$$

The Euler alignment system (1), (5) can be viewed as hyperbolic system of conservation laws for $\mathbf{w} = (\rho, \rho\mathbf{u}, \rho E)$, which we abbreviate as

$$\mathbf{w}_t + \operatorname{div} \mathbf{f}(\mathbf{w}) = \tau\mathbb{A}(\mathbf{w}), \quad \tau > 0,$$

where $\mathbf{f}(\mathbf{w})$ is the flux and $\mathbb{A}(\mathbf{w})$,

$$\mathbb{A}(\mathbf{w}) := \begin{bmatrix} 0 \\ \displaystyle\int \phi(\mathbf{x}, \mathbf{y})(\mathbf{u}(t, y) - \mathbf{u}(t, \mathbf{x}))\rho(t, \mathbf{x})\rho(t, \mathbf{y})\mathrm{d}\mathbf{y} \\ \displaystyle\leq \int \phi(\mathbf{x}, \mathbf{y})(2\rho E(t, \mathbf{x})\rho(t, \mathbf{y}) - \rho\mathbf{u}(t, \mathbf{x}) \cdot \rho\mathbf{u}(t, \mathbf{y}))\mathrm{d}\mathbf{y} \end{bmatrix},$$

encodes the alignment terms on the right of (1), (5). In this context, $(-\rho S, -\rho\mathbf{u}S)$ forms an *entropy pair*: combining (21) and the mass equation, (21)$\times\rho$ + $(1)_1 \times S$, implies that the convex entropy $-\rho S$ is *increasing*,

$$U(\mathbf{w})_t + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{w}) > 0, \quad U(\mathbf{w}) = -\rho S, \ \ \mathbf{F}(\mathbf{w}) = -\rho\mathbf{u}S,$$

in contrast to the celebrated statement of convex entropy decrease in presence of diffusion, $\mathbb{D}(\mathbf{w})$, [19; 14; 18, §7; 20; 9],

$$\mathbf{w}_t + \operatorname{div} \mathbf{f}(\mathbf{w}) = \sigma\mathbb{D}(\mathbf{w}) \rightsquigarrow U(\mathbf{w})_t + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{w}) < 0, \quad \sigma > 0.$$

It follows that alignment and diffusion compete in driving the dynamics in different directions of increasing order and, respectively, disorder. This is realized in the reversed entropy inequality (21) which implies the maximum principle, $S(t, \cdot) \leq \max S_0$, in contrast to the minimum principle in the vanishing difussive Euler equations [39]

$$S_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} S \geq 0 \rightsquigarrow S(t, \cdot) \geq \min S_0.$$

In fact, uniform thickness in (21) implies

$$S^\eta_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} S^\eta < 0, \quad S^\eta(t, \mathbf{x}) := S(t, \mathbf{x}) + 2\tau c_* t,$$

which in turn enforces the decay of $S(t,\cdot)$ asserted in Theorem 2. We verify this by setting[f] $\mathcal{H}(S) := (S - S_*)^+$ with S_* to be determined. Since $\mathcal{H}(\cdot)$ is non-decreasing then $(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\mathcal{H}(S^\eta) \leq 0$, and the entropy inequality follows:

$$(\rho\mathcal{H}(S^\eta))_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\mathcal{H}(S^\eta)) \leq 0.$$

Now let $S_* := \max_{\mathbf{x}}(S^\eta)_0 = \max_{\mathbf{x}} S_0$, then integration of the entropy inequality yields

$$\int_{\mathbf{x}} \rho(S^\eta(t,\mathbf{x}) - S_*)^+ \mathrm{d}\mathbf{x} \leq \int_{\mathbf{x}} \rho(S_0(\mathbf{x}) - S_*)^+ \mathrm{d}\mathbf{x} = 0,$$

implying that the non-negative integrand on the left must vanish. Hence $S^\eta(t,\mathbf{x}) \leq S_* \leq \max_{\mathbf{x}} S_0$ and (10) follows:

$$p\rho^{-\gamma}(t,\mathbf{x}) \leq Ke^{-2\tau c_* t}, \quad K = e^{\max_{\mathbf{x}} S_0}.$$

Similar arguments apply to the uniform decay $S(t,\cdot) \stackrel{t\to\infty}{\longrightarrow} -\infty$ asserted in Theorem 2 under a general thickness condition.

The "competition" between entropy production and entropy dissipation is demonstrated in the one-dimensional Navier–Stokes alignment (NSA) equations which we abbreviate

$$\begin{aligned} \mathbf{w}_t + \mathbf{f}(\mathbf{w})_x &= \tau\mathbb{A}(\mathbf{w}) + \sigma\mathbb{D}(\mathbf{w}), \quad \sigma\mathbb{D}(\mathbf{w}) \\ &:= \sigma_1 \begin{bmatrix} 0 \\ u \\ \dfrac{1}{2}u^2 \end{bmatrix}_{xx} + \sigma_2 \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix}_x \quad C_v T = e, \end{aligned} \tag{22}$$

with the corresponding entropy balance

$$(-\rho S)_t + (-\rho u S + \sigma_2(\ln T)_x)_x = 2\tau(\phi * \rho)\rho - \sigma_1\frac{u_x^2}{T} - \sigma_2\left(\frac{T_x}{T}\right)^2. \tag{23}$$

The entropy increase rate on the first term on the right of (23) follows from (21); the terms involved the temperature T encode the usual entropy decrease associated with the diffusion term in Navier–Stokes, $\sigma\mathbb{D}(\mathbf{w})$. Observe that whenever the NSA dynamics (22) with vanishing amplitudes $\tau, \sigma \ll 1$ develops sharp gradients, then the diffusive entropy dissipation terms, $\sigma_1 u_x^2, \sigma_2 T_x^2 \gg 1$, dominate the bounded entropy increase due to alignment, $\phi * \rho \leq \text{Const}$.

Finally, we note that the "competition" between entropy production and entropy dissipation is realized already at the kinetic level. We consider the kinetic formulation (14) driven by both alignment and diffusion [33, §6; 34],

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \tau\nabla_{\mathbf{v}}\cdot Q_\phi(f,f) = \sigma\Delta_{\mathbf{v}} f.$$

[f] $X^+ := \begin{cases} X & X > 0 \\ 0 & X \leq 0 \end{cases}$ denotes the "positive part of X".

It follows that $H(f) := f \log f - f$ satisfies

$$\partial_t \int H(f)\mathrm{d}\mathbf{v} + \nabla_{\mathbf{x}} \cdot \int \mathbf{v}H(f)\mathrm{d}\mathbf{v} = \tau\phi * \rho \int f\mathrm{d}\mathbf{v} - 4\sigma \int |\nabla_{\mathbf{v}}\sqrt{f}|^2\mathrm{d}\mathbf{v}. \tag{24}$$

The first term on the right shows that alignment increases the kinetic entropy functional $\int H(f)\mathrm{d}\mathbf{v}$ due to communication satisfying the uniform thickness assumption (8), $\tau\phi * \rho f > \tau c_* f > 0$. This reversed H theorem was already observed by us in [16, §6]. In contrast, the decrease of the kinetic entropy functional due to diffusion is dictated by Fisher information $-4\sigma|\nabla_{\mathbf{v}}\sqrt{f}|^2 < 0$. They balance each other to zero entropy production with a Maxwellian profile

$$f(t,\mathbf{x},\mathbf{v}) = \frac{\rho}{(2\pi\theta)^{n/2}}e^{-\frac{|\mathbf{v}-\mathbf{u}|^2}{2\theta}}, \quad \theta(t,\mathbf{x}) \sim \left(\frac{\sigma}{\tau}\frac{\rho}{\phi * \rho}\right)^{\frac{(\gamma-1)}{\gamma}}.$$

4. Alignment with Mono-kinetic Closure

Our main result settles the open question of existence of strong solutions of the multi-dimensional mono-kinetic Euler alignment system (1) in $n \geq 3$ dimensions subject to general C^1 symmetric communication kernels.

4.1. *Existence of strong solutions*

The mono-kinetic closure reduces the momentum equation $(1)_2$ to $(11)_2$

$$\begin{aligned}&\mathbf{u}_t(t,\mathbf{x}) + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u}(t,\mathbf{x})\\ &\quad = \int \phi(\mathbf{x},\mathbf{y})(\mathbf{u}(t,\mathbf{y}) - \mathbf{u}(t,\mathbf{x}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}, \quad \mathbf{x} \in \operatorname{supp}(\rho(t,\cdot)).\end{aligned} \tag{25}$$

The existence result involves three quantities which do not increase in time: the mass $m(t) := \int \rho(t,\mathbf{x})\mathrm{d}\mathbf{x} = m_0$, the velocity fluctuation, $\delta u(t)$ (see (37) below)

$$\delta u(t) \leq \delta u_0, \quad \delta u(t) := \sup_{\mathbf{x},\mathbf{y}}\{|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})| : \mathbf{x},\mathbf{y} \in \operatorname{supp}\{\rho(t,\cdot)\}\}, \tag{26}$$

and the total kinetic energy, $\mathcal{E}(t) := \int \frac{1}{2}\rho|\mathbf{u}|^2(t,\mathbf{x})\mathrm{d}\mathbf{x} \leq \mathcal{E}(0)$. We assume that their initial amplitudes are not too large relative to the uniform thickness, $c_* > 0$,

$$(8\alpha_0 + 4\beta_0)m_0 < \tau c_*^2, \quad \begin{cases}\alpha_0 := |\nabla_{\mathbf{x}}\phi|_\infty|\delta u_0|_\infty,\\ \beta_0 := |(\nabla_{\mathbf{x}} + \nabla_{\mathbf{y}})\phi|_\infty\left(\frac{2\mathcal{E}_0}{m_0}\right)^{1/2}.\end{cases} \tag{27}$$

Theorem 5 (Global strong solutions with sub-critical data). *Consider the multiD Euler alignment system* (11) *with C^1 symmetric communication kernel, $\phi(\cdot,\cdot) > 0$, satisfying the uniform thickness* (8), $\phi * \rho(t,\cdot) \geq c_* > 0$, *and subject to initial conditions $(\rho_0, \mathbf{u}_0) \in L^1_+ \times W^{1,\infty}$ with bounded amplitudes* (27).

Assume that the threshold condition holds:

$$\eta(\rho_0, \mathbf{u}_0) = \lambda_-(\nabla_S \mathbf{u}_0)(\mathbf{x}) + \tau\phi * \rho_0(\mathbf{x}) \geq \eta_c > 0, \quad \eta_c = \frac{1}{2}\tau c_*. \tag{28}$$

Then, Euler alignment (11), (27) *and* (28) *admits a global smooth solution*, $(\rho(\cdot,t), \mathbf{u}(\cdot,t)) \in L^1_+ \times W^{1,\infty}(\mathbb{R}^n)$, *and the following uniform bound holds*:

$$|\nabla \mathbf{u}(t,\cdot)|_{L^\infty} \leq \max\{|\nabla \mathbf{u}_0|_{L^\infty}, c_*, C_0\}.$$

The constant $C_0 = C(\max_{\mathbf{x}} \|\nabla \mathbf{u}_0\|, m_0, \phi_+)$ *is specified in* (35) *below.*

In the canonical case of metric kernels, Theorem 5 holds with $\beta_0 = 0$. This is summarized in Theorem 3.

Remark 3 (Large class of sub-critical initial data). Observe that the threshold condition (28) is the same threshold condition which secured decay toward mono-kinetic closure for general class of entropic pressures in Theorem 1. It allows a "large" set of sub-critical initial configurations $(\rho_0, \mathbf{u}_0)$ in (28) such that $\lambda_-(\nabla_S \mathbf{u}_0)$ admits negative values, $\lambda_-(\nabla_S \mathbf{u}_0) > \eta_c - \tau\phi * \rho$ with $\eta_c - \tau\phi * \rho \leq -\frac{1}{2}\tau c_*$; consult Remark 1. One can state the existence result with a smaller threshold η_c (thus allowing even larger range of "negative slopes" $\lambda_-(\nabla_S \mathbf{u}_0)$) at the expense of more restricted range of initial amplitudes (α_0, β_0).

Proof of Theorem 5. Our purpose is to show that the derivatives $\{\partial_j u_i\}$ are uniformly bounded. We proceed in four steps.

Step 1. We begin by identifying an invariant region associated with the threshold $\eta(t,\mathbf{x}) = \lambda_-(\nabla_S \mathbf{u}) + \tau\phi * \rho$. The goal is to show that the threshold condition (28) secures a sub-critical region in configuration space which persists in time, $\eta(\rho_0, \mathbf{u}_0) \geq \eta_c > 0 \rightsquigarrow \eta(\rho(t,\cdot), \mathbf{u}(t,\cdot)) \geq \eta_c > 0$.

Let $'$ abbreviate differentiation along particle path,

$$\square' := (\partial_t + \mathbf{u} \cdot \nabla)\square.$$

Differentiation of (25) implies that the $n \times n$ velocity gradient matrix, $M = \nabla \mathbf{u}$, satisfies

$$M' + M^2 + \tau\phi * \rho M = \tau R, \tag{29}$$

where R is the $n \times n$ matrix

$$R_{ij} := \int \frac{\partial \phi}{\partial x_j}(\mathbf{x}, \mathbf{y})(u_i(t,\mathbf{y}) - u_i(t,\mathbf{x}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}.$$

The following observation of the residual R is at the heart of matter; in the special case of *metric* kernels it goes back to [4] in the 1D case, and to [17] in the 2D case,

$$\operatorname{trace} R = -(\phi * \rho)' + \psi * \rho, \quad \psi(t,\mathbf{x},\mathbf{y}) := \sum_i \left(\frac{\partial \phi}{\partial x_i} + \frac{\partial \phi}{\partial y_i}\right) u_i(t,\mathbf{y}). \tag{30}$$

Verification of (30): Integration by parts followed by the mass equation $(1)_1$ yield

$$
\begin{aligned}
\operatorname{trace} R &= \int \sum_i \frac{\partial\phi}{\partial x_i}(\mathbf{x},\mathbf{y})(u_i(t,\mathbf{y})-u_i(t,\mathbf{x}))\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}\\
&= -\int \sum_i \frac{\partial\phi}{\partial y_i}(\mathbf{x},\mathbf{y})u_i(t,\mathbf{y})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}\\
&\quad -\int \sum_i \frac{\partial\phi}{\partial x_i}(\mathbf{x},\mathbf{y})u_i(t,\mathbf{x})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}\\
&\quad +\int \sum_i \left(\frac{\partial\phi}{\partial x_i}+\frac{\partial\phi}{\partial y_i}\right)(\mathbf{x},\mathbf{y})u_i(t,\mathbf{y})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}\\
&= \int \phi(\mathbf{x},\mathbf{y})\nabla_{\mathbf{y}}\cdot(\rho\mathbf{u})(t,\mathbf{y})\mathrm{d}\mathbf{y} - \mathbf{u}\cdot\nabla_{\mathbf{x}}\int \phi(\mathbf{x},\mathbf{y})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}\\
&\quad +\int \psi(\mathbf{x},\mathbf{y})\rho(t,\mathbf{y})\mathrm{d}\mathbf{y}\\
&= \int \phi(\mathbf{x},\mathbf{y})\times -\rho_t(t,\mathbf{y})\mathrm{d}\mathbf{y} - \mathbf{u}\cdot\nabla_{\mathbf{x}}\phi*\rho+\psi*\rho\\
&= -(\partial_t+\mathbf{u}\cdot\nabla_{\mathbf{x}})\phi*\rho+\psi*\rho.
\end{aligned}
$$

We decompose M into its symmetric and skew-symmetric arts, $M = S + \Omega$ and trace the symmetric part of (29), $S = \nabla_S\mathbf{u}$, to find

$$S' + S^2 + \Omega^2 + \tau\phi*\rho S = \tau R_S, \quad R_S := \frac{1}{2}(R+R^\top). \tag{31}$$

Since Ω is skew-symmetric then $S' + S(S+\tau\phi*\rho\mathbb{I}) \geq \tau R_S$. Hence, the minimal eigen-pair of S, $\lambda_- := \min_\lambda \lambda(S)$ with the corresponding unit eigenvector $\mathbf{w}_-$, satisfies

$$\lambda'_-(t,\mathbf{x}) + \lambda_-(t,\mathbf{x})(\lambda_-(t,\mathbf{x})+\tau\phi*\rho(t,\mathbf{x})) \geq \tau\langle R_S\mathbf{w}_-,\mathbf{w}_-\rangle. \tag{32}$$

Here comes the main motivation for (30): we use it to express (32) in terms of $\eta := \lambda_- + \tau\phi*\rho$ as follows:

$$\eta' + (\eta-\tau\phi*\rho)\eta \geq \tau\chi, \quad \chi := \langle (R_S - \operatorname{trace} R_S)\mathbf{w}_-,\mathbf{w}_-\rangle + \psi*\rho,$$

and since by uniform thickness $\phi*\rho \geq c_*$ hence,

$$\eta'(t,\mathbf{x}) \geq \eta(\tau c_* - \eta) - \tau|\chi|. \tag{33}$$

Our purpose is to secure that the quantity on the right of (33) is positive for the threshold $\eta \downarrow \eta_c$, so that $\eta(t,\cdot)$ always increases whenever it approaches $\eta_c = \frac{1}{2}\tau c_*$ from above, and hence $\eta(\rho(t,\cdot),\mathbf{u}(t,\cdot)) \geq \eta_c$ remains an invariant region in time. To this end, we need to upper-bound $|\chi|$.

The rank-one integrand in R implies $|\langle R_S\mathbf{w}_-,\mathbf{w}_-\rangle| \le |\nabla_{\mathbf{x}}\phi|_\infty \delta u(t) m_0$, and the velocity fluctuations bound $(26)_2$ yields[g]

$$|\langle (R_S - \operatorname{trace} R)\mathbf{w}_-,\mathbf{w}_-\rangle| \le 2\alpha_0 m_0, \quad \alpha_0 = |\nabla_{\mathbf{x}}\phi|_\infty \delta u_0, \tag{34}$$

the kinetic energy bound $(26)_3$ yields

$$|\psi * \rho| \le \beta_0 m_0, \quad \beta_0 = |(\nabla_{\mathbf{x}} + \nabla_{\mathbf{y}})\phi|_\infty \sqrt{\frac{2\mathcal{E}_0}{m_0}}.$$

Therefore, since the initial amplitudes assumed in (27) are not too large,

$$\tau|\chi| \le \tau|\langle (R_S - \operatorname{trace} R_S)\mathbf{w}_-,\mathbf{w}_-\rangle| + \tau|\psi * \rho| \le \frac{1}{4}(\tau c_*)^2,$$

and (33) yields

$$\eta'(t,\mathbf{x}) > \eta(\tau c_* - \eta) - \frac{1}{4}(\tau c_*)^2_{|\eta=\eta_c} = 0, \quad \eta_c = \frac{1}{2}\tau c_*.$$

Thus, the initial threshold bound $\eta(\rho_0,\mathbf{u}_0) \ge \eta_c$ with $\eta_c = \frac{1}{2}\tau c_*$, will persist for all time, $\eta(\rho,\mathbf{u}) \ge \eta_c$.

Step 2. Next we upper-bound the skew-symmetric part of (29)

$$\Omega' + \frac{1}{2}(S\Omega + \Omega S) + \tau\phi * \rho\Omega = \tau R_\Omega, \quad R_\Omega := \frac{1}{2}(R - R^\top).$$

If $(\mu_+,\mathbf{z}_+)$ is the maximal eigen-pair of Ω with purely imaginary eigenvalue μ_+ such that $|\mu_+| = \max_\mu |\mu(\Omega)|$ and normalized eigenvector $\mathbf{z}_+$, then

$$\mu_+'(t,\mathbf{x}) + \mu_+(t,\mathbf{x})(\langle S\mathbf{z}_+,\mathbf{z}_+\rangle + \tau\phi * \rho) = \tau\langle R_\Omega \mathbf{z}_+,\mathbf{z}_+\rangle.$$

The threshold established in Step 1 tells us $\langle S\mathbf{z}_+,\mathbf{z}_+\rangle + \tau\phi * \rho \ge \eta_c > 0$ and hence

$$\begin{aligned}|\mu_+|'(t,\mathbf{x}) + \eta_c|\mu_+(t,\mathbf{x})| &\le |\mu_+|'(t,\mathbf{x}) + |\mu_+(t,\mathbf{x})|(\langle S\mathbf{z}_+,\mathbf{z}_+\rangle + \tau\phi * \rho)\\ &\le \tau|\langle R_\Omega \mathbf{z}_+,\mathbf{z}_+\rangle|.\end{aligned}$$

As before, $|\langle R_\Omega \mathbf{z}_+,\mathbf{z}_+\rangle| \le \alpha_0 m_0 < \frac{1}{8}\tau c_*^2$ and we end up with a uniform bound of the vorticity which involves the constant $\frac{1}{8}\tau^2 c_*^2/\eta_c = \frac{1}{4}\tau c_*$,

$$\|\Omega(t,\mathbf{x})\| \le \gamma_0, \quad \gamma_0 := \max\left\{\max_{\mathbf{x}} \|\Omega_0(\mathbf{x})\|, \frac{1}{4}\tau c_*\right\}.$$

Step 3. Now we can bound $\|S(t,\mathbf{x})\|$, that is, the maximal eigenvalue of $\lambda_+ = \max_\lambda \lambda(S)$. We revisit (31), this time with the upper bound of $-\Omega^2$, to derive the

[g]The entries integrated in $R_{ij} \mapsto r_i s_j$ with $r_i = \phi_{x_i}$ and $s_j = u_i(t,\mathbf{y}) - u_i(t,\mathbf{x})$ yields $\langle \mathbf{r},\mathbf{w}_-\rangle\langle \mathbf{s},\mathbf{w}_-\rangle - \sum r_i s_i \le 2|\nabla_{\mathbf{x}}\phi|\delta u(t)$.

reverse inequality (33) for $\zeta(t,\mathbf{x}) := \lambda_+(t,\mathbf{x}) + \tau\phi * \rho(t,\mathbf{x})$,

$$\zeta'(t,\mathbf{x}) + \tau c_* \zeta(t,\mathbf{x}) \leq \zeta' + \tau\phi * \rho\zeta \leq \tau|\langle R_S \mathbf{w}_+, \mathbf{w}_+\rangle| + \|\Omega\|^2.$$

Again, since the initial amplitudes assumed in (27) are not too large, $\tau|\langle R_S \mathbf{w}_+, \mathbf{w}_+\rangle| \leq \frac{1}{8}(\tau c_*)^2$ and together with $\|\Omega\| \leq \gamma_0$, this yields

$$\max_{\mathbf{x}} \lambda_+(t,\mathbf{x}) \leq \max\left\{\max_{\mathbf{x}} \lambda_+(S_0)(\mathbf{x}), \delta_0\right\}, \quad \delta_0 = \frac{1}{8}\tau c_* + \frac{\gamma_0^2}{\tau c_*}.$$

And finally, combined with the lower threshold $\lambda_-(S) \geq -\tau\phi * \rho \geq -\tau\phi_+ m_0$ we conclude

$$\|S(t,\mathbf{x})\| \leq \max\left\{\max_{\mathbf{x}} \lambda_+(S_0)(\mathbf{x}), \delta_0, \tau\phi_+ m_0\right\}.$$

Step 4. The bounds of S and Ω imply that $\nabla\mathbf{u}$ is uniformly bounded in time

$$\left|\frac{\partial u_i}{\partial x_j}(t,\mathbf{x})\right| \leq \max\left\{\max_{\mathbf{x}} \|\nabla\mathbf{u}_0\|, \frac{1}{4}\tau c_*, \delta_0, \tau\phi_+ m_0\right\}. \tag{35}$$

□

Remark 4. Observe that the velocity fluctuations $\delta u(t)$ decay exponentially in time, see (41) below. If we use this improved bound in (34) then one can deduce an improved threshold condition with larger set of restricted initial fluctuations, e.g., [31, Remark 6.1].

4.2. *Uniform thickness and flocking*

We now turn to the question of uniform thickness assumed in Theorem 5 which is addressed in the next sections for the two main classes of heavy-tailed and short-range kernels.

4.2.1. *Uniform thickness with heavy-tailed kernels*

Consider communication kernels, quantified in terms of the Pareto-type tail

$$\phi(\mathbf{x},\mathbf{y}) \geq C\langle|\mathbf{x}-\mathbf{y}|\rangle^{-\theta}, \quad \theta \in (0,1). \tag{36}$$

A key feature of such heavy-tailed kernels is that their alignment dynamics maintains global communication: each part of the crowd with mass distribution $\rho(\mathbf{x})\mathrm{d}\mathbf{x}$ communicates *directly* with every other part with mass distribution $\rho(\mathbf{y})\mathrm{d}\mathbf{y}$. Indeed, in this case,

$$\phi_-(t) := \min_{\mathbf{x},\mathbf{y}\in\mathrm{supp}\{\rho(t,\cdot)\}} \phi(\mathbf{x},\mathbf{y}) \gtrsim \langle D(t)\rangle^{-\theta} > 0, \quad D(t) := \mathrm{diam}(\mathrm{supp}\{\rho(t,\cdot)\})$$

and for $\theta < 1$ this implies that $\phi_-(t)$ remains uniformly bounded in time away from zero. The uniform-in-time bound follows by combining two standard arguments:

(1) Decay of velocity fluctuations. In the case of mono-kinetic closure, $\mathbb{P} \equiv 0$, the momentum equation $(1)_2$ decouples into n scalar equations, each of which satis-

fies a maximum principle; in fact there is a decay quantified in [42, Theorem 2.1; 17, Theorem 1.1]

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\delta u(t) \leq -(\tau m_0 \phi_-(t))\delta u(t), \\ &\delta u(t) = \max_{\mathbf{x},\mathbf{y}\in \operatorname{supp}\{\rho(t,\cdot)\}} |\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})|. \end{aligned} \tag{37}$$

(2) The decay rate of $\delta u(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta u(t) \leq -C\tau m_0 \langle D(t)\rangle^{-\theta}\delta u(t) \tag{38a}$$

is dictated by the dispersion of $\operatorname{supp}\{\rho(t,\cdot)\}$, which in turn does not exceed

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leq \delta u(t). \tag{38b}$$

It follows [15] that $H(t) := C\tau m_0\langle D(t)\rangle^{1-\theta} + (1-\theta)\delta u(t)$ is non-increasing, $\dot{H}(t) \leq 0$, and therefore $\operatorname{supp}\{\rho(t,\cdot)\}$ is kept uniformly bounded in time,

$$D(t) \leq D_\infty := \left(\frac{H_0}{\tau m_0}\right)^{\frac{1}{1-\theta}}, \quad H_0 = C\tau m_0\langle D_0\rangle^{1-\theta} + (1-\theta)\delta u_0. \tag{39}$$

In particular, $\phi_-(t) \geq C\langle D(t)\rangle^{-\theta} \geq C\langle D_\infty\rangle^{-\theta} > 0$ and uniform thickness (8) follows:

$$\phi * \rho(t) \geq \phi_-(t)\int \rho(t,\mathbf{x})\mathrm{d}\mathbf{x} \geq c_* := C\langle D_\infty\rangle^{-\theta}m_0. \tag{40}$$

An alternative derivation of the uniform-in-time bounds is outlined in Sec. 4.3 below.

Remark 5 (Flocking I). The dispersion bound (39) implies an exponential decay of velocity fluctuations

$$\delta u(t) \lesssim e^{-\tau m_0 D_\infty^{-\theta} t}\delta u_0. \tag{41}$$

Since the mean velocity is time-invariant, $\overline{\mathbf{u}}(t) := \frac{1}{m_0}\int \rho\mathbf{u}(t,\mathbf{x})\mathrm{d}\mathbf{x}$, there follows the flocking behavior

$$\max_{\mathbf{x}\in \operatorname{supp}\{\rho(t,\cdot)\}} |\mathbf{u}(t,\mathbf{x}) - \overline{\mathbf{u}}_0|_\infty \lesssim e^{-\tau m_0 D_\infty^{-\theta} t}|\delta\mathbf{u}_0|_\infty. \tag{42}$$

Flocking behavior for heavy-tailed metric-based kernels $\phi(\mathbf{x},\mathbf{y}) \mapsto \phi(|\mathbf{x}-\mathbf{y}|)$ goes back to Cucker–Smale [8, 15, 16]. Here we observe that it extends to general heavy-tailed symmetric kernels. It corresponds to the flocking behavior at the level of agent-based description e.g., [26, Definition 1.1], in which a cohesive flock of a finite diameter $\max_{i,j}|x_i(t) - x_j(t)| \leq D_\infty < \infty$, is approaching a limiting velocity, $\max_{i,j}|\mathbf{v}_i(t) - \mathbf{v}_j(t)| \to 0$ as $t \to \infty$.

Remark 6 (Flocking II). The enstrophy in (16) drives the decay of the energy fluctuations

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathcal{E}(t) \leq -\frac{C\tau m_0}{\langle D(t)\rangle^\theta}\delta\mathcal{E}(t),$$

$$\delta\mathcal{E}(t) := \left(\iint\left(\frac{1}{2}|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|^2 + e(\mathbf{x}) + e(\mathbf{y})\right)\mathrm{d}\rho(\mathbf{x})\mathrm{d}\rho(\mathbf{y})\right)^{1/2}.$$

Observe that the last inequality applies to general Euler alignment systems (1), *independent* of the specific closure for the pressure tensor. In particular, if we can control the dispersion

$$D(t) \lesssim \langle t\rangle^\gamma, \quad \gamma \geq 0, \tag{43}$$

then we would conclude that both — the L^2-velocity fluctuations and internal energy fluctuations decay to zero

$$\delta\mathcal{E}(t) \lesssim e^{-\eta t^{1-\gamma\theta}}\delta\mathcal{E}(0), \quad \eta = \frac{C\tau m_0}{2(1-\gamma\theta)u_\infty}, \quad \gamma\theta < 1. \tag{44}$$

We mention three examples.

(1) (A uniformly bounded velocity). $|\mathbf{u}(t,\mathbf{x})| \leq u_\infty$ implies — in view of (38b), that (43) holds with $\gamma = 1$, $D(t) \leq D_0 + 2u_\infty t$, and L^2-flocking follows for $\theta < 1$. This covers the mono-kinetic scenario discussed in the previous remark.
(2) (Matrix communication kernels). When (1) is driven by symmetric positive definite *matrix* kernel, $\phi \mapsto \Phi(\mathbf{x},\mathbf{y})$, then a dispersion bound is secured with $\gamma = 2/3$, and (44) follows for $\theta < 2/3$ [32, Proposition 3.1; 41, Appendix C].
(3) (Singular kernels). When $\phi(\mathbf{x},\mathbf{y}) \sim |\mathbf{x}-\mathbf{y}|^{-\alpha}$, $\alpha \in (n, n+2)$, their enstrophy enforces that (43) holds with $\gamma = \gamma_{\alpha,n} > 1$ [41, Appendix E], and flocking follows for a restricted set of $\theta < 1/\gamma$. This proves flocking independent of the specific closure for the pressure and independent whether the kernel $\phi(\mathbf{x},\mathbf{y})$ is long- or short-range.

4.2.2. *Uniform thickness with short-range kernels*

We restrict attention to compactly supported metric kernels (3). In this case, alignment takes place in local neighborhoods of size $\leq D_0$, which is assumed much smaller than the diameter of the ambient space — the 2π-periodic torus $\Omega = \mathbb{T}^n$. With lack of global, direct communication, existence and large-time behavior of strong solution of (1) depends on having a bounded, non-vacuous density,

$$0 < \rho_- \leq \rho(t,\mathbf{x}) \leq \rho_+ < \infty. \tag{45}$$

In particular, uniform thickness (8) follows:

$$\phi * \rho(t) \geq c_* := \int \phi(|\mathbf{x}|)\mathrm{d}\mathbf{x}\,\rho_- > 0. \tag{46}$$

Remark 7 (Flocking III). We mention the flocking of alignment dynamics with short-range kernels in finite tours, as long as the non-vacuous condition (45) holds [40, Theorem 3; 33, §4.4]. Again, flocking of non-vacuous dynamics holds independent of the closure of pressure: the key question is securing the uniform-in-time bound (46) or, as noted in [40, Remark, p. 499] at least $\int^{\infty} \rho_-(t)\mathrm{d}t = \infty$ which in turn would lead to (non-uniform) thickness, $\int^{\infty} \min_{\mathbf{x}} \phi * \rho(t,\mathbf{x})\mathrm{d}t = \infty$. This question was addressed in the case of 1D singular kernel in [10, 35], but is open for bounded short-kernels.

4.3. *Uniform bounds revisited*

A necessary main ingredient in the analysis of (1) is the uniform-in-time bounds of $\mathrm{diam}(\mathrm{supp}\{\rho(t,\cdot)\}), \phi_-(t)$, and the amplitude of velocity $\max_{\mathbf{x}\in\mathrm{supp}\{\rho(t,\cdot)\}} |\mathbf{u}(\mathbf{x},t)|$. An alternative approach to the standard arguments in Sec. 4.2.1 is advocated in our work [31, Lemma 3.2]. Here, we extend our argument to general symmetric kernels. The next lemma shows that whenever one has a uniform bound of $|\mathbf{u}(\mathbf{x},t)| + |\mathbf{x}|$ for the *restricted* class of lower-bounded ϕ's which scales like $\mathcal{O}(1/\min\phi)$, then it implies a uniform bound of $|\mathbf{u}(\mathbf{x},t)|+|\mathbf{x}|$ for the general class of admissible ϕ's (36).

Lemma 1 (The reduction to lower-bounded ϕ's). *Consider* (1) *with a with the restricted class of uniformly lower-bounded ϕ's*:

$$0 < \phi_- \le \phi(\mathbf{x},\mathbf{y}) \le \phi_+ < \infty. \tag{47}$$

Assume that the solutions $(\widetilde{\rho},\widetilde{\mathbf{u}})$ *associated with the restricted* (1), (47), *satisfy the uniform bound* (*with constants $C_\pm$ depending on ϕ_+, m_0 and $\mathcal{E}_0$*)

$$\max_{t\ge 0,\, \mathbf{x}\in\mathrm{supp}\widetilde{\rho}(\cdot,t)} (|\widetilde{\mathbf{u}}(\mathbf{x},t)| + |\mathbf{x}|) \le \max\left\{C_+ \cdot \max_{\mathbf{x}\in\mathrm{supp}\ \widetilde{\rho}_0} (|\widetilde{\mathbf{u}}_0(\mathbf{x})| + |\mathbf{x}|), \frac{C_-}{\phi_-}\right\}. \tag{48}$$

Then the following holds for solutions associated with a general class of admissible kernels ϕ satisfying (36): *if* $(\rho,\mathbf{u})$ *is a smooth solution of* (1), *then there exists* $\beta > 0$ *depending on the initial data* $(\rho_0,\mathbf{u}_0)$, *such that* $(\rho,\mathbf{u})$ *coincides with the solution*, $(\widetilde{\rho}_\beta,\widetilde{\mathbf{u}}_\beta)$, *associated with the lower-bounded*

$$\phi_\beta(\mathbf{x},\mathbf{y}) := \max\{\phi(\mathbf{x},\mathbf{y}),\beta\}.$$

This means that if ϕ belongs to the general class of admissible kernels (36), then we can assume, without loss of generality, that ϕ coincides with the lower bounded ϕ_β and hence the uniform bound (48) holds with $\phi_- = \beta$. The justification of this reduction step is outlined below.

Proof of Lemma 1. By (36), one could take large enough r such that $r \cdot \min_{|\mathbf{x}-\mathbf{y}|\le r} \phi(\mathbf{x},\mathbf{y}) \ge 2C_-$ and

$$r \ge 2C_+ \cdot \max_{\mathbf{x}\in\mathrm{supp}\ \rho_0} (|\mathbf{u}_0(\mathbf{x})| + |\mathbf{x}|). \tag{49}$$

Let $\beta := \min_{|\mathbf{x}-\mathbf{y}|\leq r}\phi(\mathbf{x},\mathbf{y})$. By assumption, (48) holds for the lower-bounded ϕ_β, so that

$$\max_{t\geq 0,\,\mathbf{x}\in\text{supp }\rho_\beta(\cdot,t)}(|\mathbf{u}_\beta(\mathbf{x},t)|+|\mathbf{x}|)\leq\max\left\{C_+\cdot\max_{\mathbf{x}\in\text{supp }\rho_0}(|\mathbf{u}_0(\mathbf{x})|+|\mathbf{x}|),\frac{C_-}{\beta}\right\},\quad(50)$$

where $(\rho_\beta,\mathbf{u}_\beta)$ is the smooth solution of (1) with interaction kernel ϕ_β, which we assume to exist. By definition,

$$\frac{C_-}{\beta}=\frac{C_-}{\min_{|\mathbf{x}-\mathbf{y}|\leq r}\phi(\mathbf{x},\mathbf{y})}\leq\frac{r}{2}.\quad(51)$$

Fix $t\geq 0$, then (49)–(51) imply that the distance for any $\mathbf{x},\mathbf{y}\in\text{supp }\rho_\beta(\cdot,t)$ does not exceed

$$|\mathbf{x}-\mathbf{y}|\leq|\mathbf{x}|+|\mathbf{y}|\leq 2\max\left\{C_+\cdot\max_{\mathbf{x}\in\text{supp }\rho_0}(|\mathbf{u}_0(\mathbf{x})|+|\mathbf{x}|),\frac{C_-}{\beta}\right\}\leq r.\quad(52)$$

Thus, for any $\mathbf{x},\mathbf{y}\in\text{supp }\rho_\beta(\cdot,t)$ there holds $\phi(\mathbf{x},\mathbf{y})\geq\beta$ and hence ϕ_β coincides with ϕ for $(\mathbf{x},\mathbf{y})\in\text{supp}\{\rho(t,\cdot)\}$, so that the dynamics of $(\rho_\beta,\mathbf{u}_\beta)$ coincides with $(\rho,\mathbf{u})$. □

Example. If the uniform lower bound $\phi(\mathbf{x},\mathbf{y})\geq\phi_-$ holds then according to (37)

$$\delta u(t)\leq\delta u_0\cdot e^{-(\tau m_0\phi_-)t},$$

and hence $D(t)\leq D_0+\frac{1}{\tau m_0\phi_-}$. Therefore, existence of strong solutions and their flocking behavior follows long range ϕ's with Pareto's tail (36).

Acknowledgments

I thank Thomas Chen for helpful discussions. Research was supported by ONR grant N00014-2412659, NSF grant DMS-2508407 and by the Fondation Sciences Mathématiques de Paris (FSMP) while being hosted by LJLL at Sorbonne University.

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