On the Mathematics of Swarming

Emergent Behavior in Alignment Dynamics

Eitan Tadmor

Introduction

The starting point of our discussion is the celebrated Cucker-Smale (CS) model, [CS07a, CS07b], which describes the dynamics of N entities, referred to as agents, with time-dependent positions and velocities $(\mathbf{x}_i(t), \mathbf{v}_i(t)) : \mathbb{R}_+ \to (\Omega, \mathbb{R}^d)$ governed by

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i(t), \\ \dot{\mathbf{v}}_i(t) = \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N} \phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)), \end{cases} \tag{1}$$

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and subject to prescribed initial conditions, $(\mathbf{x}_i(0), \mathbf{v}_i(0)) = (\mathbf{x}_{i0}, \mathbf{v}_{i0}) \in (\Omega, \mathbb{R}^d)$. The ambient space of positions $\Omega \subset \mathbb{R}^d$ will refer to one of two main scenarios—either $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. System (1) is a canonical model for *alignment dynamics* in which pairwise interactions steer towards average heading. Alignment originated in pioneering works [Rey87, VCBCS95, CS07a, CS07b], as a key ingredient in *self-organization*—a unity from within which leads to the emergence of higher-order, large-scale patterns. It is found in ecology—from flocking of birds and schooling of fish to swarming bacteria and insects; in social dynamics of human interactions—from alignment of pedestrians and emerging consensus of opinions to markets and marketing; and in sensor-based networks—from swarming of mobile robots and control of UAVs to macromolecules and metallic rods.

Figure 1. Flocking of birds.

Figure 2. Alignment of pedestrians.

The dynamics (1) governs pairwise interactions, $\phi_{ij}(t) := \phi(\mathbf{x}_i(t), \mathbf{x}_j(t))$, dictated by a scalar *communication kernel*, $\phi(\cdot,\cdot)$ with amplitude $\kappa > 0$. We assume that $\phi(\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$ is a nonnegative symmetric kernel, properly normalized,

$$\int_\Omega \phi(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}' \equiv 1, \quad \phi(\mathbf{x},\mathbf{x}') = \phi(\mathbf{x}',\mathbf{x}) \geq 0. \qquad (2)$$

The role for the kernel ϕ is context-dependent: its approximate shape is either derived empirically, deduced from higher-order principles, learned from the data, or postulated based on phenomenological arguments, e.g., [Bal08, CFTV10, CDMBC07, CS07a, KTIHC11, ST21, VZ12] and the references therein. The specific structure of ϕ, however, is not necessarily known. Instead, we ask how different *classes* of communication kernels affect the emergent behavior of (1). Here are a few examples for different communication protocols.

A major part of current literature is devoted to the generic class of *metric-based* kernels, $\phi(\mathbf{x},\mathbf{x}') = \varphi(|\mathbf{x}-\mathbf{x}'|)$. Another example is the class of *topologically-based* kernels,

Figure 3. Swarming of drones.

[Bal08, ST20], where $\phi(\mathbf{x},\mathbf{x}') = \varphi(\mu(\mathbf{x},\mathbf{x}'))$ is dictated by the size of the crowd in an intermediate domain of communication $\mathcal{C}(\mathbf{x},\mathbf{x}')$ enclosed between $\mathbf{x}$ and $\mathbf{x}'$,

$$\mu(\mathbf{x},\mathbf{x}') := \frac{1}{N}\#\{k : \mathbf{x}_k \in \mathcal{C}(\mathbf{x},\mathbf{x}')\}. \qquad (3)$$

In particular, if the domain of communication $\mathcal{C}$ is shifted to an R-ball centered at $\mathbf{x}$, one ends up with the *nonsymmetric* topological kernel [MT11] $\phi(\mathbf{x},\mathbf{x}') = \varphi(|\mathbf{x}-\mathbf{x}'|)/\mu(B_R(\mathbf{x}))$. A still larger class of pairwise interactions consists of symmetric *matrix* communication kernels, e.g., [ST21]. Other important protocols of communication which will not be analyzed here include the class of *singular* kernels,

$$\phi(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|^\beta}, \qquad 0 < \beta < d+2, \qquad (4)$$

in which communication heavily emphasizes near-by neighbors over those farther away, e.g., [MMPZ19] and the references therein, and communication based on various random-based protocols found in chemo- and phototactic dynamics, the Elo rating system, voter and related opinion-based models, a random-batch method, and consensus-based optimization, to name but a few.[1]

In addition to alignment, pairwise interactions may include *repulsions* and *attractions*,

$$\dot{\mathbf{v}}_i = \frac{\kappa}{N}\sum_{j=1}^N \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i) - \frac{1}{N}\sum_{j=1}^N \nabla U(|\mathbf{x}_j - \mathbf{x}_i|), \qquad (5)$$

which are encoded here by a radial potential U. A general protocol for rules of engagement, with pairwise interactions driven by alignment, repulsion, and attraction which are dominant in three different zones of proximity, was proposed in [Rey87]. We shall focus here on

[1] *In view of the limited bibliographic scope available for this article, we refer the reader to* [Tad21] *for a complementary bibliography source.*

the emergent behavior of alignment dynamics, and refer to [CFTV10, CDMBC07, ST21, Tad21] and the references therein for results related to more general protocols. To date, we still lack a mathematical theory which analyzes the emergent behavior of the general class of 3Zone models for collective dynamics.

Connectivity. The large-time behavior of (1) depends on the time-dependent weighted graph with agents at the N vertices $\mathsf{V}(t) = \{i \,|\, \mathbf{x}_i(t)\}$ and time-dependent edges $\mathsf{E}(t) = \{(i,j) \,|\, i \neq j \,:\, \phi_{ij}(t) > 0\}$. The energy fluctuations associated with (1),

$$\delta\mathscr{E}(t) := \frac{1}{N^2}\sum_{i,j=1}^{N} |\mathbf{v}_i(t) - \mathbf{v}_j(t)|^2,$$

satisfy

$$\begin{aligned}\frac{\mathsf{d}}{\mathsf{d}t}\delta\mathscr{E}(t) &= -\frac{2\kappa}{N^2}\sum_{(i,j)\in\mathsf{E}(t)} \phi_{ij}(t)|\mathbf{v}_i(t) - \mathbf{v}_j(t)|^2 \\ &\leq -\lambda_2(t)\frac{2\kappa}{N^2}\sum_{i,j=1}^{N}|\mathbf{v}_i(t) - \mathbf{v}_j(t)|^2.\end{aligned} \tag{6}$$

The first equality follows directly from (1) and the assumed symmetry of the adjacency matrix $\Phi(t) = \{\phi_{ij}(t)\}$. The second inequality is a sharp bound in terms of the *spectral gap*, $\lambda_2(t) := \lambda_2(\Delta_{\Phi(t)})$, of the graph Laplacian associated with $\Phi(t)$. Here the graph Laplacian, $(\Delta_\Phi)_{\alpha\beta} := (\sum_{\gamma\neq\alpha}\phi_{\alpha\gamma})\delta_{\alpha\beta} - \phi_{\alpha\beta}(1-\delta_{\alpha\beta})$, and its spectral gap coincide with the Fiedler number which encodes the connectivity properties of the weighted graph of agents $(\mathsf{V}(t), \mathsf{E}(t))$, e.g., [CS07a]. Indeed, (6) tells us that energy fluctuations are depleted as long as the graph remains (algebraically) *connected*,

$$\delta\mathscr{E}(t) \leq \exp\left\{-2\kappa\int^t \lambda_2(\tau)\mathsf{d}\tau\right\}\delta\mathscr{E}(0). \tag{7}$$

Connectivity, and hence the large time emergence of flocks or swarms, is guaranteed with *long-range* kernels. In many realistic configurations, however, the communication among "social particles" takes place in local neighborhoods induced by *short-range* kernels.

Long- and short-range communication kernels will be the topic of the next two sections. Long-range kernels maintain connectivity which in turn imply decay of fluctuations around an emergent cluster. The large-time dynamics with short-range kernels is considerably more complicated—in particular, algebraically connected initial configurations, $\lambda_2(t=0) > 0$, may break down into two or more disconnected clusters at a finite time so that $\lambda_2(t_c) = 0$. That is, the dynamics with short-range kernels may or may not be stable, which makes it difficult to trace its flocking behavior. Instead, we study here the flocking/swarming behavior with *large crowds*: large-crowd dynamics tends to stabilize the large-time behavior. As already noted by Immanuel Kant in 1784 *"what seems complex and chaotic in the single individual may be seen from the standpoint of the human race as a whole to be a steady and progressive though slow evolution of its original endowment."*

Hydrodynamic description. The large-crowd dynamics of (1) can be encoded in terms of the empirical distribution $f_N(t,\mathbf{x},\mathbf{v}) := \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{x}_i(t)}(\mathbf{x})\otimes\delta_{\mathbf{v}_i(t)}(\mathbf{v})$, which is governed by the kinetic equation in state variables $(t,\mathbf{x},\mathbf{v}) \in \mathbb{R}_+\times\Omega\times\mathbb{R}^d$, e.g., [HT08, HL09, CFTV10],

$$\partial_t f_N + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_N = -\kappa\nabla_{\mathbf{v}}\cdot Q_\phi(f_N, f_N). \tag{8}$$

It is driven according to the pairwise communication protocol[2] on the right of $(1)_2$,

$$Q_\phi(f,f) := \iint \phi(\mathbf{x},\mathbf{x}')(\mathbf{v}'-\mathbf{v})ff'\,\mathsf{d}\mathbf{v}'\,\mathsf{d}\mathbf{x}'. \tag{9}$$

For $N \gg 1$, the dynamics of $f_N(t,\mathbf{x},\mathbf{v})$ is captured by its first two moments which we assume to exist—the density $\rho(t,\mathbf{x}) := \lim_{N\to\infty}\int_{\mathbb{R}^d} f_N(t,\mathbf{x},\mathbf{v})\,\mathsf{d}\mathbf{v}$, and the momentum $\rho\mathbf{u}(t,\mathbf{x}) := \lim_{N\to\infty}\int_{\mathbb{R}^d}\mathbf{v}f_N(t,\mathbf{x},\mathbf{v})\,\mathsf{d}\mathbf{v}$. They admit the *hydrodynamic description* in state variable $(t,\mathbf{x}) \in (\mathbb{R}_+\times\Omega)$,

$$\left\{\begin{array}{c}\rho_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + \mathbf{P}) = \kappa\mathbf{A}_\phi(\rho,\mathbf{u}).\end{array}\right. \tag{10a}$$

Here, the pressure $\mathbf{P}$ on the left of $(10\text{a})_2$ is a symmetric positive-definite stress tensor,

$$\mathbf{P} := \lim_{N\to\infty}\int(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})^\top f_N(t,\mathbf{x},\mathbf{v})\,\mathsf{d}\mathbf{v}, \tag{10b}$$

and $\mathbf{A}_\phi$ on the right of $(10\text{a})_2$ is the communication protocol associated with ϕ, corresponding to (9),

$$\mathbf{A}_\phi(\rho,\mathbf{u})(t,\mathbf{x}) := \int_\Omega \phi(\mathbf{x},\mathbf{x}')(\mathbf{u}'-\mathbf{u})\rho\rho'\,\mathsf{d}\mathbf{x}'. \tag{10c}$$

Observe that system (10) is not a purely hydrodynamic description at the macroscopic scale: while the density and velocity, $\rho = \rho(t,\mathbf{x})$ and $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$, are governed by the macroscopic balances (10a),(10c), the pressure in (10b), $\mathbf{P} = \mathbf{P}(t,\mathbf{x})$, still requires a *closure* of the $\mathbf{v}$-dependent second-order moments of f_N. We recall that in the case of physical particles, one encounters the canonical closure imposed by Maxwellian equilibrium and expressed in terms of the density, velocity, and temperature, $\rho, \mathbf{u}$, and T,

$$M_{\{\rho,\mathbf{u},T\}}(t,\mathbf{x},\mathbf{v}) = \frac{\rho}{(2\pi T)^{d/2}}\exp\left(-\frac{|\mathbf{v}-\mathbf{u}|^2}{2T}\right).$$

We mention in this context the special cases of *monokinetic closure*, $\mathbf{P} \equiv 0$, associated with the vanishing temperature $M_{\{\rho,\mathbf{u},T\downarrow 0\}}(t,\mathbf{x}) = \rho\delta(|\mathbf{v}-\mathbf{u}|)$, [FK19], as well as examples of an entropic-based closure with measured data and the

[2] *We abbreviate* $f = f(t,\mathbf{x},\mathbf{v}), f' = f(t,\mathbf{x}',\mathbf{v}')$ *and likewise* $\square = \square(t,\mathbf{x})$, $\square' = \square(t,\mathbf{x}')$, *etc.*

isothermal closure $\mathbf{P} = \rho\mathbb{I}_{d\times d}$ (corresponding to constant temperature T); see [Tad21].

The general case of "social particles," however, is different: there is no universal closure. The question of closure for the hydrodynamic description of alignment in (10) is therefore left open. We shall revisit this question in the last section of the article. At this stage, we highlight the fact that the decay of energy fluctuations quantified in the next section applies to general mesoscopic pressure stress tensors (10b).

Fluctuations. The total energy of the large-crowd dynamics associated with (1) is given by the second moment (which is assumed to exist)

$$\rho E(t,\mathbf{x}) := \lim_{N\to\infty}\int_{\mathbb{R}^d}\frac{1}{2}|\mathbf{v}|^2 f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}.$$

It satisfies the energy equation

$$\begin{aligned}(\rho E)_t + \nabla_{\mathbf{x}}\cdot(\rho E\mathbf{u} + \mathbf{P}\mathbf{u} + \mathbf{q})\\ = -\kappa\int \phi(\mathbf{x},\mathbf{x}')\big(2E(t,\mathbf{x}) - \mathbf{u}\cdot\mathbf{u}'\big)\rho\rho'\,\mathrm{d}\mathbf{x}'.\end{aligned}\tag{11}$$

The energy flux on the left of (11) is computed as the second moment of (8),

$$\begin{aligned}&\lim_{N\to\infty}\int\frac{|\mathbf{v}|^2}{2}\mathbf{v}f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}\\ &= \mathbf{u}\lim_{N\to\infty}\int\frac{|\mathbf{v}|^2}{2}f_N\,\mathrm{d}\mathbf{v} + \lim_{N\to\infty}\int\frac{|\mathbf{v}|^2}{2}(\mathbf{v}-\mathbf{u})f_N\,\mathrm{d}\mathbf{v}\\ &= \rho E\mathbf{u} + \lim_{N\to\infty}\int\Big[\frac{|\mathbf{u}|^2}{2} + (\mathbf{v}-\mathbf{u})\cdot\mathbf{u}\\ &\qquad + \frac{|\mathbf{v}-\mathbf{u}|^2}{2}\Big](\mathbf{v}-\mathbf{u})f_N\,\mathrm{d}\mathbf{v}\\ &= \rho E\mathbf{u} + \mathbf{P}\mathbf{u} + \mathbf{q},\end{aligned}$$

expressed in terms of the pressure $\mathbf{P}$ and the heat flux $\mathbf{q}(t,\mathbf{x}) := \lim_{N\to\infty}\frac{1}{2}\int(\mathbf{v}-\mathbf{u})|\mathbf{v}-\mathbf{u}|^2 f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}$. The energy production on the right of (11) is driven by the *enstrophy*

$$\begin{aligned}&\iiint\phi(\mathbf{x},\mathbf{x}')\mathbf{v}\cdot(\mathbf{v}-\mathbf{v}')ff'\,\mathrm{d}\mathbf{v}'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x}'\\ &\qquad = \int\phi(\mathbf{x},\mathbf{x}')\big(2E(t,\mathbf{x}) - \mathbf{u}\cdot\mathbf{u}'\big)\rho\rho'\,\mathrm{d}\mathbf{x}'.\end{aligned}$$

The energy can be decomposed as the sum of kinetic and internal energies, $\rho E = \rho e_K + \rho e$, corresponding to the first two terms in the decomposition of kinetic velocity $\frac{1}{2}|\mathbf{v}|^2 \equiv \frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{v}-\mathbf{u}|^2 + (\mathbf{v}-\mathbf{u})\cdot\mathbf{u}$,

$$\begin{aligned}\rho e_K(t,\mathbf{x}) &:= \frac{1}{2}\rho|\mathbf{u}|^2(t,\mathbf{x}),\\ \rho e(t,\mathbf{x}) &:= \lim_{N\to\infty}\frac{1}{2}\int|\mathbf{v}-\mathbf{u}(t,\mathbf{x})|^2 f_N(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}.\end{aligned}$$

Let $\delta\mathscr{E}(t)$ denote the total *energy fluctuations* at time[3] t,

$$\begin{aligned}\delta\mathscr{E}(t) := \frac{1}{2m_0}\iint\Big[\frac{1}{2}|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x}')|^2\\ + e(t,\mathbf{x}) + e(t,\mathbf{x}')\Big]\mathrm{d}m_\rho(\mathbf{x},\mathbf{x}').\end{aligned}$$

The first integrand on the right quantifies local fluctuations of macroscopic velocities, $\mathbf{u}(t,\cdot)$, while the last two integrands quantify microscopic fluctuations, $|\mathbf{v}-\mathbf{u}(t,\cdot)|^2$. Its decay rate is given by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t)\\ &\quad = -\kappa\iint\phi(\mathbf{x},\mathbf{x}')\Big[\frac{1}{2}|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x}')|^2\\ &\qquad + e(t,\mathbf{x}) + e(t,\mathbf{x}')\Big]\mathrm{d}m_\rho(\mathbf{x},\mathbf{x}').\end{aligned}\tag{12}$$

This follows by integration of the energy equation (11) and using the assumed symmetry of $\phi(\cdot,\cdot)$ hence $\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \equiv 0$ on the left, and the symmetric part of the enstrophy on the right,

$$2E - \mathbf{u}\cdot\mathbf{u}' \equiv \frac{1}{2}|\mathbf{u}-\mathbf{u}'|^2 + e + e' + \frac{1}{2}\big(|\mathbf{u}|^2 - |\mathbf{u}'|^2\big) + (e - e').$$

Equation (12) quantifies the decay of energy fluctuations on the left in terms of the total enstrophy on the right and is a key ingredient in studying the emergent behavior of (10).

Flocking/swarming. A main feature of the large-crowd hydrodynamics (10) is *flocking*—modeled after the self-organization of birds, as a prototype for swarming behavior, characterized by the following two properties:

(i) Finite diameter—the "continuum" crowd supported in $\mathcal{S}(t) := \mathrm{supp}\{\rho(\cdot,t)\}$ forms a "flock" with a finite diameter

$$D(t) := \sup_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)}|\mathbf{x}-\mathbf{x}'| \le D_+ < \infty.\tag{13a}$$

(ii) Alignment—velocity fluctuations inside the flock vanish for $t \gg 1$,

$$\iint|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x}')|^2\mathrm{d}m_\rho(\mathbf{x},\mathbf{x}') \stackrel{t\to\infty}{\longrightarrow} 0.\tag{13b}$$

Remark that one can combine (13b) with global time invariant of the momentum $\overline{\rho\mathbf{u}}(t) := \int_\Omega\rho\mathbf{u}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}$ and mass $m(t) := \int_\Omega\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}$, to deduce emergence towards the average velocity $\overline{\mathbf{u}}_0 := \frac{(\overline{\rho\mathbf{u}})_0}{m_0}$,

$$\begin{aligned}\delta\mathscr{E}(t) = \int\Big[\frac{1}{2}|\mathbf{u}(t,\mathbf{x}) - \overline{\mathbf{u}}_0|^2\\ + e(t,\mathbf{x})\Big]\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \stackrel{t\to\infty}{\longrightarrow} 0.\end{aligned}\tag{14}$$

Thus, the large time decay of fluctuations takes place while the dynamics is asymptotically aligned along straight particle paths $\mathbf{x}_c(t) \sim \overline{\mathbf{u}}_0 t$. We mention in passing that the

[3] *Here and below we abbreviate* $\mathrm{d}m_\rho(\mathbf{x},\mathbf{x}') := \rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'$.

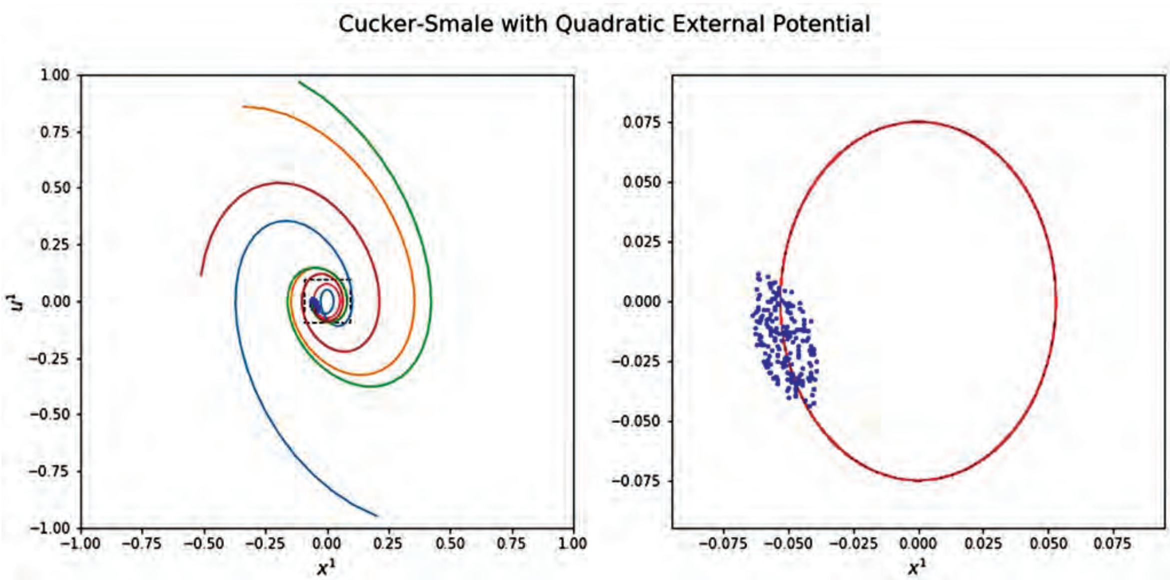

Figure 4. Emergence of flocking as agents concentrate along a harmonic oscillator.

decay of fluctuations in the presence of more general protocol of interactions—repulsion, attraction, and external forcing—leads to emergent behavior with different and more "interesting" patterns. For example, when alignment is augmented by pairwise attraction induced by quadratic potential (5), it implies flocking of agents which are spatially concentrated along a particle path of harmonic oscillators, $(\mathbf{x}_c(t), \mathbf{u}_c(t))$, depicted in Figure 4, as $\rho\mathbf{u}(t,\mathbf{x}) - m_0\mathbf{u}_c(t)\delta(\mathbf{x}-\mathbf{x}_c(t)) \overset{t\to\infty}{\longrightarrow} 0$, [ST21]. A rich gallery of emerging swarming patterns is found in [CFTV10, CDMBC07].

We shall continue with the notion of flocking. It is dictated by the enstrophy on the right of (12), which in turn is determined by the two types of fluctuations—the internal energy and kinetic fluctuations. We shall discuss the large-time behavior of these fluctuations in two separate cases of long-range and short-range communications kernels.

Long-Range Interactions

Long-range kernels maintain global communication so that each part of the crowd with mass distribution $\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x}$ communicates *directly* with every other part with mass distribution $\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}'$. Global communication is quantified in terms of Pareto-type tail[4]

$$\phi(\mathbf{x},\mathbf{x}') \gtrsim \frac{1}{\langle|\mathbf{x}-\mathbf{x}'|\rangle^{\theta}}. \tag{15}$$

According to (12), the decay of energy fluctuations requires that the diameter, $D(t) := \operatorname{diam}\operatorname{supp}\{\rho(\cdot,t)\}$, does not spread too fast.

Corollary 1 (Flocking with long-range kernels). *Let* $(\rho(t,\cdot), \mathbf{u}(t,\cdot), \mathbf{P}(t,\cdot))$ *be a strong solution of* (2),(10), *driven by long-range kernel* (15). *There holds*

$$\delta\mathscr{E}(t) \lesssim \exp\Big\{-\kappa\int_0^t \langle D(\tau)\rangle^{-\theta}\mathrm{d}\tau\Big\}\delta\mathscr{E}(0). \tag{16}$$

The flocking behavior of dynamics driven by long-range kernels is determined by two factors: (i) the assumed "fat-tail" behavior (15); and (ii) the spread of $\operatorname{supp}\{\rho(t,\cdot)\}$. Corollary 1 implies that

$$\text{if } D(t) \lesssim \langle t\rangle^{\beta}, \text{ then flocking follows for } \theta\beta < 1.$$

"Fat-tailed" kernels. A prototype example encountered with uniformly bounded velocity fields, in which case $\langle D(t)\rangle \lesssim 2|\mathbf{u}|_\infty t$ and hence $\beta = 1$, implies unconditional flocking for fat-tailed kernels satisfying (15) with $\theta \in (0,1)$, which in turn implies fluctuations decay rate of order $\lesssim \exp\{-t^{1-\theta}\}$. This is the typical scenario for *monokinetic closure* $\mathbf{P} \equiv 0$. In this case, the momentum equation $(10\mathrm{a})_2$ decouples into d scalar transport equations for the components of $\mathbf{u}$,

$$(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})u_i = \int \phi(\mathbf{x},\mathbf{x}')(u_i' - u_i)\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{17}$$

each satisfying the maximum principle in $\operatorname{supp}\{\rho(\cdot,t)\}$, e.g., [Tad21]. In fact, in this pressureless scenario, (17) implies the uniform decay of velocity fluctuations is tied to the size of $\operatorname{supp}\{\rho(\cdot,t)\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}V_i(t) \le -\kappa\langle D(t)\rangle^{-\theta}V_i(t), \quad V_i(t) := \max_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)}|u_i - u_i'|,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \le \max_i V_i(t).$$

It follows that the functional, [HL09], $H(t) := \kappa\langle D(t)\rangle^{1-\theta} + (1-\theta)\max_i V_i(t)$ is nonincreasing; hence the spread of

[4] *Denoting* $\langle r\rangle := (1+r^2)^{1/2}$.

supp$\{\rho(\cdot,t)\}$ is, in fact, kept uniformly bounded in time,

$$D(t) \leq D_+, \tag{18}$$

which in turn implies exponential flocking $\lesssim \exp\{-\kappa D_+ t\}$. This flocking result for fat-tailed metric-based kernels $\phi(\mathbf{x},\mathbf{x}') = \varphi(|\mathbf{x}-\mathbf{x}'|)$ (with monokinetic closure) goes back to Cucker-Smale [CS07a, CS07b, HT08, HL09], and here we observe that it extends to general fat-tailed symmetric kernels.

Another example for flocking occurs with long-range *matrix-valued* kernels, corresponding to fat-tailed dynamics (16) of order $\theta < 2/3$: in this case, the diameter of supp$\{\rho(\cdot,t)\}$ grows no faster than $D(t) \lesssim \langle t\rangle^{\beta}$ with $\beta < \frac{2}{2-\theta}$, [ST21], leading to flocking decay rate of fractional order $\lesssim \exp\{-t^{1-\theta\beta}\}$.

Remark (Internal energy). Let $\overline{\rho}_\phi$ denote the *averaged density*

$$\overline{\rho}_\phi(t,\mathbf{x}) := \int \phi(\mathbf{x},\mathbf{x}')\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}' \geq m_0\phi_-(t), \tag{19}$$

where $\phi_-(t) := \min_{\mathbf{x},\mathbf{x}'\in\mathcal{S}(t)} \phi(\mathbf{x},\mathbf{x}')$. We observe that the contribution of the internal energy to the decay of fluctuations in (12) admits the lower bound

$$\begin{aligned}\iint \phi(\mathbf{x},\mathbf{x}')(e+e')\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}')\\ \geq \frac{1}{m_0}\min_{\mathbf{x}} \overline{\rho}_\phi(t,\mathbf{x}) \iint (e+e')\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}').\end{aligned} \tag{20}$$

Hence the decay of the internal energy portion of the fluctuations is independent of the specifics of the closure relationship: *any* nonnegative internal energy is dissipated by fat-tailed kernels $\phi_-(t) \gtrsim \langle D(t)\rangle^{-\theta}$ such that $\int\langle D(\tau)\rangle^{-\theta}\mathrm{d}\tau = \infty$.

Short-Range Interactions

We focus our attention on the more realistic scenario of short-range communication kernels, and in particular, when $\phi(\mathbf{x},\mathbf{x}')$ is compactly supported in the vicinity of diagonal $|\mathbf{x}-\mathbf{x}'| < R_0$. This means that alignment takes place in local neighborhoods of size $< R_0$, which is assumed much smaller than the diameter of the ambient space Ω. We consider the case of 2π-periodic torus $\Omega = \mathbb{T}^d$.

Spectral analysis. We revisit the two ingredients involved in the decay rate of the energy fluctuations stated in (12).

(i) **Internal energy**. We replace the lower bound (19) with $\overline{\rho}_\phi(t,\mathbf{x}) \geq \rho_-(t)$ (recall the normalization (2)). The decay bound of the internal energy portion in (12) for *nonvacuous* flows then reads

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')(e+e')\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}')\\ \geq \frac{\rho_-(t)}{m_0}\iint_{\mathbb{T}^d\times\mathbb{T}^d} (e+e')\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}').\end{aligned} \tag{21}$$

(ii) **Kinetic energy**. It remains to bound the contribution of the kinetic energy fluctuations to the enstrophy on the right of (12),

$$\iint_{\mathbb{T}^d\times\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')|\mathbf{u}-\mathbf{u}'|^2\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}').$$

Given the symmetric communication kernel $\phi(\mathbf{x},\mathbf{x}') = \phi(\mathbf{x}',\mathbf{x})$ we set the *weighted Laplacian* as the Hilbert-Schmidt operator $\mathscr{L}_\rho : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$,

$$\begin{aligned}\mathscr{L}_\rho\mathbf{w}(\mathbf{x}) := \int_{\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')\Big[\sqrt{\rho(\mathbf{x}')}\mathbf{w}(\mathbf{x})\\ - \sqrt{\rho(\mathbf{x})}\mathbf{w}(\mathbf{x}')\Big]\sqrt{\rho(\mathbf{x}')}\,\mathrm{d}\mathbf{x}'.\end{aligned}$$

Let $\lambda_k(t)$ be the discrete eigenvalues of $\mathscr{L}_{\rho(t)}$ starting with the eigenpair $\lambda_1 = 0$ (corresponding to eigenfunction $\sqrt{\rho(t,\mathbf{x})}\mathbf{c}$, where $\mathbf{c}$ is any constant vector in $\mathbb{T}^d$). The desired lower bound on the kinetic energy fluctuations part of the enstrophy is given by the *spectral gap*, $\lambda_2(\mathscr{L}_{\rho(t)})$,

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')|\mathbf{u}-\mathbf{u}'|^2\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}')\\ \geq \frac{\lambda_2(\mathscr{L}_{\rho(t)})}{m_0}\iint_{\mathbb{T}^d\times\mathbb{T}^d} |\mathbf{u}-\mathbf{u}'|^2\mathrm{dm}_\rho(\mathbf{x},\mathbf{x}').\end{aligned} \tag{22}$$

This corresponds to the decay of discrete energy fluctuations, in $(6)_2$, quantified in terms of the spectral gap $\lambda_2(\Delta_{\Phi(t)})$. The proof is outlined in the end of this section.

Remark. The spectral gap bound (22) generalizes the spectral bound derived in [ST20, Theorem 1.1] which required the spurious condition $\rho(t,\cdot) \gtrsim \langle t\rangle^{-1/2}$. Instead, (22) encodes the behavior of ρ through the *weighted* Laplacian $\mathscr{L}_{\rho(t)}$.

Inserting (21) and (22) into (12) yields the following.

Theorem 2 (Flocking with positive spectral gap). *Let* $(\rho(t,\cdot),\mathbf{u}(t,\cdot))$ *be a strong solution of the hydrodynamic system* (2), (10) *with a mesoscopic pressure* $\mathbf{P}(t,\cdot)$, *subject to nonvacuous initial data,* $(\rho_0 > 0, \mathbf{u}_0, \mathbf{P}_0)$. *Then the following flocking decay estimate holds:*

$$\delta\mathscr{E}(t) \leq \exp\Big\{-2\kappa\int_0^t \min\{\lambda_2(\mathscr{L}_{\rho(\tau)}),\rho_-(\tau)\}\mathrm{d}\tau\Big\}\delta\mathscr{E}(0).$$

Thus, if $\lambda_2(\mathscr{L}_{\rho(t)})$ and $\rho_-(t)$ have fat-tailed decay in time, then they "communicate" strong enough alignment to imply the flocking behavior sought in (14),

$$\delta\mathscr{E}(t) = \int_{\mathbb{T}^d}\Big[\frac{1}{2}|\mathbf{u}(t,\mathbf{x}) - \overline{\mathbf{u}}_0|^2 + e(t,\mathbf{x})\Big]\rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \to 0.$$

It comes with the additional decay of the internal energy.

Proof of the spectral gap bound (22). Given the symmetric communication kernel $\phi(\mathbf{x},\mathbf{x}') = \phi(\mathbf{x}',\mathbf{x}) \in L^\infty(\mathbb{T}^d\times\mathbb{T}^d)$

and the positive weight function $\rho > 0$, we set the *weighted Laplacian operator*,

$$\mathscr{L}_\rho := \Lambda_\rho - \mathscr{A}_\rho,$$

where $\Lambda_\rho : L^2_\rho(\mathbb{T}^d) \to L^2_\rho(\mathbb{T}^d)$ is a multiplication operator and $\mathscr{A}_\rho : L^2_\rho(\mathbb{T}^d) \to L^2_\rho(\mathbb{T}^d)$ is a Hilbert-Schmidt operator on $L^2_\rho(\mathbb{T}^d) := \left\{\mathbf{w} : \int_{\mathbb{T}^d} |\mathbf{w}|^2 \rho \, \mathrm{d}\mathbf{x} < \infty\right\}$,

$$\begin{aligned}\Lambda_\rho \mathbf{w}(\mathbf{x}) &:= \overline{\rho}_\phi(\mathbf{x})\mathbf{w}(\mathbf{x}),\\ \mathscr{A}_\rho \mathbf{w}(\mathbf{x}) &:= \sqrt{\rho(\mathbf{x})}\int_{\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')\mathbf{w}(\mathbf{x}')\sqrt{\rho(\mathbf{x}')}\,\mathrm{d}\mathbf{x}'.\end{aligned}$$

The Laplacian $\mathscr{L}_\rho$ is a symmetric nonnegative[5] operator in $L^2_\rho(\mathbb{T}^d)$:

$$\begin{aligned}(&\mathscr{L}_\rho\sqrt{\rho}\mathbf{w}, \sqrt{\rho}\mathbf{w})\\ &= \iint_{\mathbb{T}^d\times\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\rho(\mathbf{x})|\mathbf{w}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\\ &\quad - \iint_{\mathbb{T}^d\times\mathbb{T}^d} \sqrt{\rho(\mathbf{x})}\phi(\mathbf{x},\mathbf{x}')\sqrt{\rho(\mathbf{x}')}\\ &\qquad\qquad \times \left\langle\sqrt{\rho(\mathbf{x}')}\mathbf{w}(\mathbf{x}'), \sqrt{\rho(\mathbf{x})}\mathbf{w}(\mathbf{x})\right\rangle \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\\ &\equiv \frac{1}{2}\iint_{\mathbb{T}^d\times\mathbb{T}^d} \phi(\mathbf{x},\mathbf{x}')|\mathbf{w}(\mathbf{x})-\mathbf{w}(\mathbf{x}')|^2 \mathrm{d}m_\rho(\mathbf{x},\mathbf{x}').\end{aligned}$$

Let $(\lambda_k \geq 0, \mathbf{w}_k(\mathbf{x}))$ be the sequence of discrete eigenpairs of $\mathscr{L}_\rho$ starting with the zero eigenvalue $\lambda_1 = 0$ associated with $\mathbf{w}_1(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}\mathbf{c}$, where $\mathbf{c}$ is any constant vector in $\mathbb{T}^d$,

$$\begin{aligned}\mathscr{L}_\rho \mathbf{w}_1 &= \Lambda_\rho(\sqrt{\rho(\mathbf{x})}\mathbf{c}) - \mathscr{A}_\rho(\sqrt{\rho(\mathbf{x})}\mathbf{c})\\ &= \sqrt{\rho(\mathbf{x})}\\ &\quad\times \int_{\mathbb{T}^d}\phi(\mathbf{x},\mathbf{x}')\Big(\rho(\mathbf{x}') - \sqrt{\rho(\mathbf{x}')}\sqrt{\rho(\mathbf{x}')}\Big)\mathrm{d}\mathbf{x}' \times \mathbf{c}\\ &= 0.\end{aligned}$$

We then have

$$\begin{aligned}\lambda_2(\mathscr{L}_\rho) &= \inf_{\sqrt{\rho}\mathbf{w}\perp\sqrt{\rho}\mathbf{c}} \frac{(\mathscr{L}_\rho\sqrt{\rho}\mathbf{w}, \sqrt{\rho}\mathbf{w})}{(\sqrt{\rho}\mathbf{w}, \sqrt{\rho}\mathbf{w})}\\ &= \inf_{\int\rho\mathbf{w}=0} \frac{\frac{1}{2}\iint \phi(\mathbf{x},\mathbf{x}')|\mathbf{w}-\mathbf{w}'|^2\mathrm{d}m_\rho(\mathbf{x},\mathbf{x}')}{\frac{1}{2m_0}\iint |\mathbf{w}-\mathbf{w}'|^2\mathrm{d}m_\rho(\mathbf{x},\mathbf{x}')}.\end{aligned} \tag{23}$$

The desired lower bound of the kinetic energy fluctuations (22) follows. □

[5] *In agreement with the standard convention of keeping positive graph Laplacians, as opposed to the usual Laplacians being negative.*

Metric-based kernels. The main difficulty with Theorem 2 is access to the spectral gap $\lambda_2(\mathscr{L}_{\rho(t)})$. To this end, we restrict attention to the *radial* communication kernel, $\phi(\mathbf{x},\mathbf{x}') = \varphi(|\mathbf{x}-\mathbf{x}'|)$. Short-range communication refers to "thin-tailed" kernels and in particular to kernels with finite support—much smaller than the diameter of the nonvacuous "crowd," $\operatorname{diam}\operatorname{supp}\{\varphi(\cdot)\} < \operatorname{diam}\operatorname{supp}\{\rho(\cdot,t)\}$, and therefore lack direct global communication. Instead, decay of energy fluctuations (and hence flocking) persists for nonvacuous configurations quantified below. We denote

$$c_\rho(t) := \frac{\rho_-(t)}{\rho_+(t)}, \qquad \rho_\pm(t) := {}^{\max_\mathbf{x}}_{\min_\mathbf{x}}\rho(t,\mathbf{x}).$$

Theorem 3 (Flocking with short-range kernels). *Consider the hydrodynamic system* (10) *over the 2π-periodic torus $\mathbb{T}^d$, driven by a nonnegative radial communication kernel, $\phi(\mathbf{x},\mathbf{x}') = \varphi(|\mathbf{x}-\mathbf{x}'|)$, with unit mass $\int_{\mathbb{T}^d}\varphi(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} = 1$. Let $(\rho, \mathbf{u}, \mathbf{P})$ be a strong solution subject to nonvacuous initial data $(\rho_0 > 0, \mathbf{u}_0, \mathbf{P}_0)$. There exists a constant $\sigma_\varphi > 0$,*

$$\sigma_\varphi := 1 - \max_{\mathbf{k}\neq\{\mathbf{0}\}}\int_{\mathbb{T}^d}\varphi(|\mathbf{x}|)\cos\big(\mathbf{k}\cdot\mathbf{x}\big)\,\mathrm{d}\mathbf{x},$$

such that

$$\lambda_2(\mathscr{L}_{\rho(t)}) \geq \frac{1}{2}\sigma_\varphi c_\rho(t)\rho_-(t), \tag{24}$$

and the following bound on the decay of energy fluctuations holds:

$$\delta\mathscr{E}(t) \leq \exp\Big\{-\kappa\,\sigma_\varphi\int_0^t c_\rho(\tau)\rho_-(\tau)\mathrm{d}\tau\Big\}\delta\mathscr{E}(0). \tag{25}$$

Remark (Optimality of the spectral gap bound?). The obvious bound

$$\lambda_2(\mathscr{L}_{\rho(t)}) \geq m_0 \cdot \min\phi(\mathbf{x},\mathbf{x}')$$

implies that when ϕ is a long-range communication kernel, namely, when (15) holds with $\theta < 1$, then it induces an *unconditional flocking*. The question of flocking for short-range kernels is more subtle: Theorem 3 shifts the burden of proving flocking in this case, to a question of nonvacuous bounded density, $\rho_-(t) \gtrsim \langle t\rangle^{-1/2}$ (in which case $c_\rho(t) \gtrsim \langle t\rangle^{-1/2}$ and hence $\delta\mathscr{E}(t) \to 0$). We raise the question whether an improved bound of the spectral gap holds—independent of the aspect ratio c_ρ, $\lambda_2(\mathscr{L}_{\rho(t)}) \gtrsim \rho_-(t)$. This would imply flocking for "fat-tailed" density such that $\rho_-(t) \gtrsim \langle t\rangle^{-1}$.

Proof of Theorem 3. We begin with the following Poincaré inequality, corresponding to the bound of discrete energy fluctuations in (6): for all 2π-periodic $\mathbf{w} \in L^2(\mathbb{T}^d)$ there holds

$$\begin{aligned}\iint_{\mathbb{T}^d\times\mathbb{T}^d} &\varphi(|\mathbf{x}-\mathbf{x}'|)|\mathbf{w}(\mathbf{x})-\mathbf{w}(\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'\\ &\geq \frac{\sigma_\varphi}{(2\pi)^d}\iint_{\mathbb{T}^d\times\mathbb{T}^d}|\mathbf{w}(\mathbf{x})-\mathbf{w}(\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}'.\end{aligned} \tag{26}$$

Indeed, expressed in terms of the Fourier expansion

$$\mathbf{w}(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \sum_{\mathbf{k}} \widehat{\mathbf{w}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $\widehat{\mathbf{w}}(\mathbf{k}) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} \mathbf{w}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}$, the integral on the right of (26) amounts to

$$\begin{aligned}
&\frac{1}{2}\iint_{\mathbb{T}^d\times\mathbb{T}^d} |\mathbf{w}(\mathbf{x}) - \mathbf{w}(\mathbf{x}')|^2\, d\mathbf{x}\, d\mathbf{x}' \\
&\quad = (2\pi)^d \int_{\mathbb{T}^d} |\mathbf{w}(\mathbf{x})|^2\, d\mathbf{x} - \Big|\int_{\mathbb{T}^d} \mathbf{w}(\mathbf{x})\, d\mathbf{x}\Big|^2 \\
&\quad = (2\pi)^d \sum_{\mathbf{k}\neq\mathbf{0}} |\widehat{\mathbf{w}}(\mathbf{k})|^2.
\end{aligned} \tag{27}$$

Computing the convolution terms on the left of (26), $\widehat{\langle\varphi * \mathbf{w}\rangle}(\mathbf{k}) = (2\pi)^{\frac{d}{2}} \langle\widehat{\varphi}(\mathbf{k}), \widehat{\mathbf{w}}(\mathbf{k})\rangle$, and using the assumed unit mass $\widehat{\varphi}(0) = (2\pi)^{-\frac{d}{2}}$, yields for the left of (26),

$$\begin{aligned}
&\frac{1}{2}\iint_{\mathbb{T}^d\times\mathbb{T}^d} \varphi(|\mathbf{x}-\mathbf{x}'|)|\mathbf{w}-\mathbf{w}'|^2\, d\mathbf{x}\, d\mathbf{x}' \\
&\quad = \int_{\mathbb{T}^d} |\mathbf{w}(\mathbf{x})|^2\, d\mathbf{x} - \mathrm{Re}\int_{\mathbb{T}^d} \big\langle \mathbf{w}(\mathbf{x}), (\varphi * \mathbf{w})(\mathbf{x})\big\rangle\, d\mathbf{x} \\
&\quad = \sum_{\mathbf{k}\neq\mathbf{0}} \Big(1 - \mathrm{Re}\int_{\mathbb{T}^d} \varphi(|\mathbf{x}|) e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}\Big) |\widehat{\mathbf{w}}(\mathbf{k})|^2 \\
&\quad \geq \sigma_\varphi \sum_{\mathbf{k}\neq\mathbf{0}} |\widehat{\mathbf{w}}(\mathbf{k})|^2.
\end{aligned}$$

Using (26) we compute the lower bound

$$\begin{aligned}
&\iint_{\mathbb{T}^d\times\mathbb{T}^d} \varphi(|\mathbf{x}-\mathbf{x}'|)|\mathbf{u}-\mathbf{u}'|^2 dm_\rho \\
&\quad \geq \rho_-^2(t) \iint_{\mathbb{T}^d\times\mathbb{T}^d} \varphi(|\mathbf{x}-\mathbf{x}'|)|\mathbf{u}-\mathbf{u}'|^2\, d\mathbf{x}\, d\mathbf{x}' \\
&\quad \geq \frac{\sigma_\varphi}{(2\pi)^d}\rho_-^2(t) \iint_{\mathbb{T}^d\times\mathbb{T}^d} |\mathbf{u}-\mathbf{u}'|^2\, d\mathbf{x}\, d\mathbf{x}' \\
&\quad \geq \frac{\sigma_\varphi}{(2\pi)^d}\frac{\rho_-^2(t)}{\rho_+(t)} \iint_{\mathbb{T}^d\times\mathbb{T}^d} |\mathbf{u}-\mathbf{u}'|^2 \rho(t,\mathbf{x})\, d\mathbf{x}\, d\mathbf{x}' \\
&\quad \geq \frac{\sigma_\varphi}{(2\pi)^d}\frac{\rho_-^2(t)}{\rho_+(t)} (2\pi)^d \int_{\mathbb{T}^d} |\mathbf{u}-\overline{\mathbf{u}}|^2 \rho(t,\mathbf{x})\, d\mathbf{x} \\
&\quad \geq \sigma_\varphi \frac{\rho_-^2(t)}{\rho_+(t)}\frac{1}{m_0} \iint_{\mathbb{T}^d\times\mathbb{T}^d} |\mathbf{u}-\overline{\mathbf{u}}|^2 dm_\rho \\
&\quad = \sigma_\varphi \frac{\rho_-^2(t)}{\rho_+(t)}\frac{1}{2m_0} \iint_{\mathbb{T}^d\times\mathbb{T}^d} |\mathbf{u}-\mathbf{u}'|^2 dm_\rho.
\end{aligned}$$

(Recall $\overline{\mathbf{u}} = \frac{\overline{\rho\mathbf{u}}}{m_0}$ so the fourth inequality follows from $\int |\mathbf{u}-\mathbf{c}|^2\rho \geq \int |\mathbf{u}-\overline{\mathbf{u}}|^2\rho$ for all constant vectors $\mathbf{c}$.) We now deduce (24) from the optimality of $\lambda_2(\mathscr{L}_\rho)$ in (23); observe that the eigenspace associated with $\lambda_2(\mathscr{L}_{\rho(t)})$ remains uniformly bounded away from the eigenspace of constants associated with $\lambda_1(\mathscr{L}_{\rho(t)}) = 0$. Moreover, $\frac{1}{2}\sigma_\varphi c_\rho \rho_-(t) \leq \rho_-(t) \leq \overline{\rho}_\phi(t)$ and (25) follows from Theorem 2. □

Let $\overline{m}_0$ denote the average mass $\overline{m}_0 := \frac{m_0}{(2\pi)^d}$. Theorem 3 tells us that as long as the density fluctuations remain below the threshold,

$$\rho_+(t) - \rho_-(t) \leq (1-c)\overline{m}_0, \qquad c < 1, \tag{28}$$

then $c_\rho(t) \geq c$ and hence $\rho_-(t) \geq \overline{m}_0 c$. We end up with the exponential flocking bound

$$\delta\mathscr{E}(t) \leq \exp\{-\delta\sigma_\varphi t\}\delta\mathscr{E}(0), \qquad \delta := \kappa\overline{m}_0 c^2. \tag{29}$$

This echoes a similar result for *first-order* consensus dynamics encoded in terms of positions, $\{\mathbf{x}_i\}$, corresponding to transported density ρ: if the variation of the density remains below a specified φ-dependent threshold, then smooth solutions approach a consensus, [GPY17]. In both cases, the threshold, quantified in terms of the Fourier transform of φ, dictates flocking/consensus for short-range kernels.

We close the section with two examples.

Example 1. Consider the 1D dynamics over the 2π-torus $\mathbb{T}$ driven by the communication kernel[6] $\phi(x,x') = \frac{1}{2}\mathbb{1}_1(|x-x'|)$. The corresponding threshold is given by $\sigma_\varphi = 1 - \sin(1) \sim 0.158$. It follows that if the density has a finite variation so that (28) holds, then the 1D CS dynamics (10) admits exponentially converging flocking $\lesssim e^{-0.158\delta t}$. In fact, in a recent work of Dietert and Shvydkoy (cf. [Tad21]) it was shown that in the special 1D case, *any* discrete CS dynamics (1) with nontrivial communication kernel admits flocking rate of order $\lesssim (\ln(t)/t)^{1/5}$. Here, by restricting attention to large-crowd dynamics with slowly varying density, we improve the flocking result to exponential rate.

Example 2. Consider the 2D dynamics over the 2π-periodic torus $\mathbb{T}^2$ driven by the communication kernel $\phi(\mathbf{x},\mathbf{x}') = \frac{1}{\pi}\mathbb{1}_1(|\mathbf{x}-\mathbf{x}'|)$. The Fourier coefficients of the radial ϕ are given by $2\pi\widehat{\varphi}(\mathbf{k}) = \frac{2}{|\mathbf{k}|}J_1(|\mathbf{k}|)$ and hence

$$\sigma_\varphi = 1 - \frac{J_1(1)}{\pi} \sim 0.86.$$

It follows that if the density variation remains with the range (28), then the 2D CS dynamics (10) admits exponentially converging flocking $\lesssim e^{-0.86\delta t}$.

Global Smooth Solutions

The hydrodynamics (10) is driven by two competing mechanisms: a generic effect in Eulerian dynamics of *steepening local fluctuations* which may lead to finite-time blow-up when $\lim_{(t,\mathbf{x})\uparrow(t_c,\mathbf{x}_c)} \nabla_\mathbf{x}\cdot\mathbf{u}(t,\mathbf{x}) = -\infty$, and alignment which prevents the formation of shock discontinuities,

$$\nabla_\mathbf{x}\cdot\mathbf{u}(t,\cdot) \geq -C_0 > -\infty, \tag{30}$$

[6] *With $\mathbb{1}_R$ denoting the characteristic function* $\mathbb{1}_R(r) := \begin{cases} 1, & 0 \leq r \leq R, \\ 0, & r > R. \end{cases}$

as $\iint |\mathbf{u}-\mathbf{u}'|^2 \mathrm{d}m_\rho(\mathbf{x},\mathbf{x}') \stackrel{t\to\infty}{\longrightarrow} 0$. The outcome of this competition determines whether (10) admits strong solutions sought in Theorems 2 and 3. The global existence results available in current literature are almost exclusively devoted to monokinetic closure $\mathbf{P}\equiv 0$,

$$\begin{aligned} &\rho_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0,\\ &(\rho\mathbf{u})_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u})\\ &\qquad = \kappa\int \phi(\mathbf{x},\mathbf{x}')\big(\mathbf{u}'-\mathbf{u}\big)\rho\rho'\,\mathrm{d}\mathbf{x}'. \end{aligned} \tag{31}$$

Although the repulsive forcing of pressure is missing, system (31) is still driven by a competition between nonlinear advection and alignment. Indeed, the alignment hydrodynamics with or without pressure, (10) or (31), may form finite-time shock-discontinuities, coupled with the emergence of Dirac masses, which requires their interpretation as *weak solutions*. A proper notion of weak solutions which enforces uniqueness within an admissible class of solutions is still missing.[7] Existence of global strong solutions, on the other hand, depends on certain *critical thresholds* in the space of initial configurations.

Critical thresholds. To motivate our choice for initial thresholds, we turn to discuss the *thermodynamics* of the general alignment system (10). The entities governed by collective description (1) are fundamentally different than physical particles. While physical particles are driven by forces induced by the environment of other particles, the "social particles" we consider here are driven by *probing* the environment—living organisms, human interactions, and sensor-based agents have senses and sensors with which they actively probe the environment. In particular, such social agents are often driven by outside processes. This is particularly apparent in self-organization of biological agents that receive energy from the outside, thus forming thermodynamically open systems. Accordingly, the mesoscopic description for flocking cannot be expected to provide a self-contained notion of thermodynamic closure sought in (10), and as such, there is no universal Maxwellian for thermal equilibrium. As noted in [VZ12, §1.1], *"The source of energy making the motion possible ... are not relevant."* Nevertheless, we argue that lack of thermal equilibrium in the form of certain closure *equalities* can be substituted with certain *inequalities*, which are compatible with the decay of internal energy fluctuations in (12). To this end, we trace the separate contributions of the kinetic and internal energies. Multiplying the momentum $(10a)_2$ by $\mathbf{u}$, we find

$$\begin{aligned} \partial_t(\rho e_K) + \nabla_{\mathbf{x}}\cdot\Big(\rho\mathbf{u}\frac{|\mathbf{u}|^2}{2} + \mathbf{P}\mathbf{u}\Big) - \sum_{i,j} P_{ij}\frac{\partial u_i}{\partial x_j}\\ = -\kappa\int \phi(\mathbf{x},\mathbf{x}')\big(|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{u}'\big)\rho\rho'\,\mathrm{d}\mathbf{x}'. \end{aligned}$$

Subtracting the portion of kinetic energy from the balance of total energy in (11) we find the dynamics for the internal energy, expressed in terms of the average density (19), $\overline{\rho}_\phi = \int \phi(\mathbf{x},\mathbf{x}')\rho(t,\mathbf{x}')\,\mathrm{d}\mathbf{x}'$, and the $d\times d$ velocity gradient matrix $\nabla\mathbf{u} := \Big\{\frac{\partial u_i}{\partial x_j}\Big\}$,

$$\partial_t(\rho e) + \nabla_{\mathbf{x}}\cdot(\mathbf{u}\rho e + \mathbf{q}) = -\mathrm{trace}\big(\mathbf{P}\nabla\mathbf{u}\big) - 2\kappa\overline{\rho}e\rho_\phi(t,\mathbf{x}).$$

Since $\mathrm{trace}(\mathbf{P}) = \int |\mathbf{v}-\mathbf{u}|^2 f(\mathbf{v})\,\mathrm{d}\mathbf{v} = 2\rho e > 0$, we find that the equation governing the internal energy can be put into a relaxation form,

$$\partial_t(\rho e) + \nabla_{\mathbf{x}}\cdot(\mathbf{u}\rho e + \mathbf{q}) = -2\mathcal{J}\rho e, \tag{32}$$

with $\mathcal{J} = \mathcal{J}(t,\mathbf{x})$ given in terms of the normalized pressure $\overline{\mathbf{P}} = \frac{1}{\mathrm{trace}(\mathbf{P})}\mathbf{P}$,

$$\mathcal{J}(t,\mathbf{x}) := \mathrm{trace}\big(\overline{\mathbf{P}}\nabla\mathbf{u}\big)(t,\mathbf{x}) + \kappa\overline{\rho}_\phi(t,\mathbf{x}). \tag{33}$$

We do not enforce any specific form for closure of the internal energy. Instead, we explore the flocking dynamics subject to a rather general set of thermodynamic configurations with the minimal assumption that the total amount of internal energy is nonincreasing, in agreement with (12). Integration yields

$$\int \rho e(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \le \exp\Big\{-2\int^t \mathcal{J}_-(\tau)\mathrm{d}\tau\Big\}\int(\rho e)_0(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

where $\mathcal{J}_-(t) := \min_{\mathbf{x}}\mathcal{J}(t,\mathbf{x})$. Thus, a nonincreasing total energy is tied to the inequality $\mathcal{J}_-(t)\ge 0$ (and in fact, if there is no heat flux, $\mathbf{q}\equiv 0$, then (32) would yield a *uniform* decay of internal energy in this case). A simple exercise shows that since $\overline{\mathbf{P}}$ is a symmetric positive definite matrix with trace 1, then $\mathrm{trace}(\overline{\mathbf{P}}M)\ge\lambda_{\min}(M_S)$ for any matrix M with symmetric part $M_S := \frac{1}{2}(M+M^\top)$. In particular we have the following lower bound in terms of the symmetric gradient $\nabla_S\mathbf{u} := \frac{1}{2}(\partial_j u_i + \partial_i u_j)$:

$$\mathrm{trace}\big(\overline{\mathbf{P}}\nabla\mathbf{u}\big) \ge \lambda_{\min}(\nabla_S\mathbf{u}). \tag{34}$$

In summary, in view of (33),(34) we are motivated to postulate the following critical threshold requirement: there exists $\eta_c\ge 0$ such that

$$\eta(\rho,\mathbf{u})(t,\mathbf{x}) := \lambda_{\min}(\nabla_S\mathbf{u})(t,\mathbf{x}) + \kappa\overline{\rho}_\phi(t,\mathbf{x}) \ge \eta_c. \tag{35}$$

Observe that the threshold (35) is independent of the thermodynamic state of the system and it will guarantee the decay $\int\rho e(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \lesssim e^{-2\eta_c t}$. The key question is whether

[7] *We mention in this context related works (see* [Tad21]*) on dissipative and measure-valued solutions with a weak-strong uniqueness principle.*

such threshold persists in time. At this point it is instructive to compare (35) with the known results of global regularity in dimension $d = 1, 2$. A global smooth solution in the 1D case, and in the more general setup of *unidirectional* flows, exists if and only if the initial configuration satisfies the threshold $u_0'(x) + \kappa\varphi * \rho_0(x) \geq 0$, [LS20]; this corresponds to $(35)_{t=0}$ with $\eta_c = 0$. A sufficient threshold for 2D regularity (consult [Tad21]) requires a lower bound on the initial divergence *and* an upper bound on the spectral gap (recall (18)),

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}_0 + \kappa\varphi * \rho_0 > 0,$$

$$(\lambda_2 - \lambda_1)(\nabla_S \mathbf{u}_0) \leq \delta_0, \qquad \delta_0 =: \frac{1}{2} m_0 \phi(D_+),$$

which imply that $(35)_{t=0}$ holds with

$$\eta_c = \frac{1}{2}(\min_{\mathbf{x}} \varphi * \rho_0 - \delta_0).$$

Existence of strong solutions in $d \geq 3$ dimensions for "small data" can be found in [Shv19]. The next result settles the open question of existence of strong solutions in $d \geq 3$ dimensions.

Theorem 4 ([Tad21] Existence of global strong solutions of multiD Euler alignment system). *Consider the Euler alignment system* (31) *subject to nonvacuous initial data,* $(\rho_0, \mathbf{u}_0) \in H^m \times H^{m+1}$, *with initial velocity of finite variation* $\max_{\mathbf{x}} |\mathbf{u}_0' - \mathbf{u}_0| \leq C_\phi$. *If the initial conditions satisfy the threshold condition*

$$\lambda_{\min}(\nabla_S \mathbf{u}_0)(\mathbf{x}) + \kappa(\overline{\rho_0})_\phi(\mathbf{x}) \geq \eta_c, \;\; \eta_c = \frac{1}{2}(\rho_0)_- > 0,$$

then this threshold persists in time,

$$\lambda_{\min}(\nabla_S \mathbf{u})(t, \mathbf{x}) + \kappa\overline{\rho}_\phi(t, \mathbf{x}) \geq \eta_c, \qquad t > 0,$$

and (31) *admits global smooth solution* $(\rho, \mathbf{u}) \in C([0, T]; H^m \times H^{m+1})$.

The main feature here is that the initial threshold, $\eta(\rho_0, \mathbf{u}_0)(\mathbf{x}) \geq \eta_c$, forms an invariant region in $(\rho, \mathbf{u})$-configuration space,

$$\eta(\rho_0, \mathbf{u}_0)(\mathbf{x}) \geq \eta_c \;\rightsquigarrow\; \eta(\rho, \mathbf{u})(t, \mathbf{x}) \geq \eta_c,$$

and therefore

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot \mathbf{u}(t, \mathbf{x}) &= \sum_\lambda \lambda(\nabla_S \mathbf{u}) \geq d\lambda_{\min}(\nabla_S \mathbf{u})(t, \mathbf{x}) \\ &\geq d\big(\eta(\rho, \mathbf{u})(t, \mathbf{x}) - \kappa\overline{\rho}_\phi(t, \mathbf{x})\big).\end{aligned}$$

It follows that (30) holds with a constant lower bound $C_0 = d\Big(\kappa(\overline{\rho_0})_\phi - \eta_c\Big)$ which in turn implies the Sobolev regularity of $\big(\rho(t, \cdot), \mathbf{u}(t, \cdot)\big)$ by standard energy estimates.

Singular kernels. We conclude by mentioning existence results for the important class of singular kernels $\varphi_\beta(r) := r^{-\beta}$ with $\beta < d+2$: in this case, the communication framework emphasizes short-range interactions over long-range interactions, yet their global support still reflects global communication. For the regularity for 1D weakly singular kernels, φ_β with $\beta < 1$, and strongly singular kernels with $1 \leq \beta < 3$ we refer to [MMPZ19] and the references therein. Here, alignment is structured as fractional diffusion which was shown, at least in the one-dimensional case, to enforce *unconditional* flocking behavior, independent of any initial threshold. A typical result asserts that (31) with strongly singular kernel, $\varphi_{1+\alpha}$ with $0 < \alpha < 2$ on $\mathbb{T}$, any non-vacuous initial data evolves into a unique global solution, $(\rho, u) \in L^\infty([0, \infty); H^{s+\alpha} \times H^{s+1})$, $s \geq 3$, which converges to a flocking traveling wave,

$$\|u(t, \cdot) - \overline{u}_0\|_{H^s} + \|\rho(t, \cdot) - \rho_\infty(\cdot - t\overline{u}_0)\|_{H^{s-1}} \lesssim e^{-\eta t}.$$

Existence of strong solutions for multiD problems with strongly singular kernels $\varphi(r) = r^{-(d+\alpha)}$, $1 < \alpha < 2$, in $d \geq 2$ dimensions is open, except for "small data" results for small initial data results for Hölder spaces, $|\mathbf{u}_0 - \mathbf{u}_\infty|_\infty \lesssim (1 + \|\rho_0\|_{W^{3,\infty}} + \|\mathbf{u}_0\|_{W^{3,\infty}})^{-d}$ with $0 < \alpha < 2$, and for Besov data

$$\|\mathbf{u}_0\|_{B^{2-\alpha}_{d,1}} + \|\rho_0 - 1\|_{B^1_{d,1}} \leq \epsilon, \qquad \alpha \in (1, 2).$$

There are fewer results on the existence of strong solutions with *short-range* interactions and in particular, compactly supported φ's. This include the class of strongly singular kernels, φ_β, $d < \beta < d+2$, with "thin tails," $\int_1^\infty \varphi_\beta(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} < \infty$—thinner than those sought for long-range communication (16). We mention the 1D dynamics driven by short-range *topological kernels* with singular pairwise interactions restricted to finite balls, (4), where communication within the balls is dictated by the density in the intermediate communication range, $\mu_\rho(\mathbf{x}, \mathbf{x}')$, a continuum analogue of the discrete case (3), [ST20],

$$\phi(\mathbf{x}, \mathbf{x}') = \frac{\mathbb{1}_{R_0}(|\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|^{\beta-\gamma}} \times \frac{1}{\mu_\rho^\gamma(\mathbf{x}, \mathbf{x}')}, \quad 0 < \gamma < \beta,$$

where $\mu_\rho(\mathbf{x}, \mathbf{x}') := \left(\int_{\mathcal{C}(\mathbf{x}, \mathbf{x}')} \rho(t, \mathbf{z})\,\mathrm{d}\mathbf{z}\right)^{1/d}$ is the rescaled mass in a communication region $\mathcal{C}(\mathbf{x}, \mathbf{x}')$ enclosed between $\mathbf{x}$ and $\mathbf{x}'$.

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References

[Bal08] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic. *Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study*, Proc. Natl. Acad. Sci. USA **105** (2008), 1232–1237.

[CFTV10] José A. Carrillo, Massimo Fornasier, Giuseppe Toscani, and Francesco Vecil, *Particle, kinetic, and hydrodynamic models of swarming*, Mathematical modeling of collective behavior in socio-economic and life sciences, Model. Simul. Sci. Eng. Technol., Birkhäuser Boston, Boston, MA, 2010, pp. 297–336, DOI 10.1007/978-0-8176-4946-3_12. MR2744704

[CDMBC07] Yao-li Chuang, Maria R. D'Orsogna, Daniel Marthaler, Andrea L. Bertozzi, and Lincoln S. Chayes, *State transitions and the continuum limit for a 2D interacting, self-propelled particle system*, Phys. D **232** (2007), no. 1, 33–47, DOI 10.1016/j.physd.2007.05.007. MR2369988

[CS07a] Felipe Cucker and Steve Smale, *Emergent behavior in flocks*, IEEE Trans. Automat. Control **52** (2007), no. 5, 852–862, DOI 10.1109/TAC.2007.895842. MR2324245

[CS07b] Felipe Cucker and Steve Smale, *On the mathematics of emergence*, Jpn. J. Math. **2** (2007), no. 1, 197–227, DOI 10.1007/s11537-007-0647-x. MR2295620

[FK19] Alessio Figalli and Moon-Jin Kang, *A rigorous derivation from the kinetic Cucker-Smale model to the pressureless Euler system with nonlocal alignment*, Anal. PDE **12** (2019), no. 3, 843–866, DOI 10.2140/apde.2019.12.843. MR3864212

[GPY17] Josselin Garnier, George Papanicolaou, and Tzu-Wei Yang, *Consensus convergence with stochastic effects*, Vietnam J. Math. **45** (2017), no. 1-2, 51–75, DOI 10.1007/s10013-016-0190-2. MR3600416

[HL09] Seung-Yeal Ha and Jian-Guo Liu, *A simple proof of the Cucker-Smale flocking dynamics and mean-field limit*, Commun. Math. Sci. **7** (2009), no. 2, 297–325. MR2536440

[HT08] Seung-Yeal Ha and Eitan Tadmor, *From particle to kinetic and hydrodynamic descriptions of flocking*, Kinet. Relat. Models **1** (2008), no. 3, 415–435, DOI 10.3934/krm.2008.1.415. MR2425606

[KTIHC11] Y. Katz, K. Tunstrøm, C.C. Ioannou, C. Huepe and I.D. Couzin, *Inferring the structure and dynamics of interactions in schooling fish*, Proc. Natl. Acad. Sci. USA **108** (2011), 18720–18725.

[LS20] D. Lear and R. Shvydkoy, *Existence and stability of unidirectional flocks in hydrodynamic Euler alignment systems*, Analysis & PDE, in press, `arXiv:1911.10661`, 2019.

[MMPZ19] Piotr Minakowski, Piotr B. Mucha, Jan Peszek, and Ewelina Zatorska, *Singular Cucker-Smale dynamics*, Active particles, Vol. 2., Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2019, pp. 201–243. MR3932462

[MT11] Sebastien Motsch and Eitan Tadmor, *A new model for self-organized dynamics and its flocking behavior*, J. Stat. Phys. **144** (2011), no. 5, 923–947, DOI 10.1007/s10955-011-0285-9. MR2836613

[Rey87] C. Reynolds, *Flocks, herds and schools: A distributed behavioral model*, ACM SIGGRAPH **21** (1987), 25–34.

[ST21] R. Shu and E. Tadmor, *Anticipation breeds alignment*, Arch. Rat. Mech. Anal. (2021), DOI 10.1007/s00205-021-01609-8.

[Shv19] Roman Shvydkoy, *Global existence and stability of nearly aligned flocks*, J. Dynam. Differential Equations **31** (2019), no. 4, 2165–2175, DOI 10.1007/s10884-018-9693-8. MR4028569

[ST20] Roman Shvydkoy and Eitan Tadmor, *Topologically based fractional diffusion and emergent dynamics with short-range interactions*, SIAM J. Math. Anal. **52** (2020), no. 6, 5792–5839, DOI 10.1137/19M1292412. MR4176893

[Tad21] E. Tadmor, *On the mathematics of swarming: emergent behavior in alignment dynamics*, `arXiv:2102.09134` (2021).

[VCBCS95] Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett. **75** (1995), no. 6, 1226–1229, DOI 10.1103/PhysRevLett.75.1226. MR3363421

[VZ12] T. Vicsek and A. Zefeiris, *Collective motion*, Physics Reprints **517** (2012), 71–140.

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