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COMMUTATOR METHOD FOR AVERAGING LEMMAS

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We introduce a commutator method with multiplier to prove averaging lemmas, the regularizing effect for the velocity average of solutions for kinetic equations. Our method requires only elementary techniques in Fourier analysis and highlights a new range of assumptions that are sufficient for the velocity average to be in $L^2([0,T], H_x^{1/2})$. Our result provides a direct proof (without interpolation) and improves the regularizing result for the measure-valued solutions to scalar conservation laws in space dimension 1.

1. Introduction

1.1. ***Brief overview for averaging lemmas.*** Our goal in this paper is to introduce a *commutator method* for the kinetic transport equations in the form

$$\varepsilon\partial_t f + a(v)\cdot\nabla_x f = (-\Delta_v)^{\frac{\alpha}{2}} g, \tag{1}$$

where $\varepsilon > 0$, $\alpha \geq 0$, $a : \mathbb{R}_v^n \to \mathbb{R}^n$ and $g : \mathbb{R}_t \times \mathbb{R}_v^n \times \mathbb{R}_x^n \to \mathbb{R}$ are given functions; ε is the macroscopic scale often introduced when a hydrodynamic limit is considered. The nonlinear coefficient $a(v)$ in our setting appears for instance in the relativistic, quantum kinetic models [Escobedo et al. 2003; Golse and Poupaud 1992] or the kinetic formulation of scalar conservation laws [Lions et al. 1994a].

We shall utilize a commutator method as a new approach to derive *averaging lemmas*, which state that the velocity average ρ_ϕ of f in (1), defined by

$$\rho_\phi(t,x) := \int f(t,x,v)\phi(v)\,dv, \quad \phi \in C_c^\infty,$$

has a better regularity than f and g in the x-variable.

There is a vast literature of averaging lemmas and we only mention few of them that are relatively closer to our discussion here. Averaging lemmas are famous for getting the compactness for kinetic models, such as the Vlasov–Maxwell system [DiPerna and Lions 1989a], the renormalized solutions [DiPerna and Lions 1989b] and the hydrodynamic limits for the Boltzmann equation [Golse and Saint-Raymond 2005], and the renormalized solutions to the semiconductor Boltzmann–Poisson system [Masmoudi and Tayeb 2007]. Averaging lemmas also contribute to the regularizing effect of solutions when kinetic formulations

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exist. They were applied for this purpose to, for instance, the isentropic gas dynamics [Lions et al. 1994b], the Ginzburg–Landau model [Jabin and Perthame 2001], and scalar conservation laws [Lions et al. 1994a].

The classical averaging lemma in the L^2 framework was first introduced independently by [Agoshkov 1984] and [Golse et al. 1988]. The derivation in [Golse et al. 1988] involves the decomposition in the Fourier space according to the order of $a(v)\cdot\xi$, and the singular part $|a(v)\cdot\xi| < c$ is controlled by the nondegeneracy conditions for all $c > 0$. Combining with interpolation arguments, the averaging lemma was later extended to general L^p spaces with $1 < p < \infty$ by [Bézard 1994; DiPerna et al. 1991]. The optimal Besov result was then proved in [DeVore and Petrova 2001] with wavelet decomposition. More recently, the regularity result for L^p spaces has been further improved in the one-dimensional case, precisely from $\frac{1}{p}$ to $1-\frac{1}{p}$ when $p > 2$ by [Arsénio and Masmoudi 2019] with the dispersive property of the kinetic transport operator.

The averaging lemmas under different assumptions on f and g were further investigated. For example, in [Westdickenberg 2002] the author considered f and g in the same Besov space in x but with possibly different integrability in v. The results for general mixed norm assumptions were obtained in [Jabin and Vega 2003; 2004]. Their work inspired the work in [Arsénio and Masmoudi 2014] to consider the case when f and g have less integrability in x than v. Besides the explorations in the direction of general conditions, averaging results for a larger class of operators in the form $a(v)\cdot\nabla_x - \nabla_x^{\perp}\cdot b(v)\nabla_x$ were obtained by [Tadmor and Tao 2007], where several applications of their results were presented. In particular, they improved the regularity of solutions to scalar conservation laws with general nondegeneracy conditions.

The limiting L^1 case for classical averaging lemmas in general is not true and a counterexample was given in [Golse et al. 1988]. However, the L^1 compactness can be shown when assuming the equi-integrability condition in the v-variable [Golse and Saint-Raymond 2002] and this type of results has been extended to the transport equations in more general forms in [Arsénio and Saint-Raymond 2011; Han-Kwan 2010].

1.2. ***Commutator method with multipliers.*** We use a commutator method with multipliers to transform the dispersion of transport operator in Fourier space into a gain of regularity in the x-variable. Let us introduce the commutator method in a general setting and narrow down to our case shortly. Assume

$$\varepsilon\partial_t f + Bf = g, \tag{2}$$

where B is a skew-adjoint operator, $\varepsilon \le 1$ and g are given. For a time-independent operator Q, we consider

$$\varepsilon\partial_t \int f\,\overline{Qf}\,dx\,dv = \int [B,Q]\,f\,\bar{f}\,dx\,dv + \int g\,\overline{Qf}\,dx\,dv + \int f\,\overline{Qg}\,dx\,dv.$$

By the fundamental theorem of calculus, we have

$$\operatorname{Re}\int_0^T [B,Q]\,f\,\bar{f}\,dx\,dv\,dt \le \sup_{t=0,T}\left|\int f\,\overline{Qf}\,dx\,dv\right| + \left|\int g\,\overline{Qf}\,dx\,dv\,dt\right| + \left|\int f\,\overline{Qg}\,dx\,dv\,dt\right|. \tag{3}$$

The idea is to find an operator Q which is bounded in some L^p spaces such that the commutator $[B,Q]$ of B and Q is positive-definite and gains extra derivatives. Hence by applying these conditions on (3), we get a desired bound on f.

This method has been utilized, for example, for the local smoothing property of the Schrödinger equations when B is taken to be of Schrödinger type, when commutators appear naturally from some Hamilton vector field. Frequently, it involves constructing a proper symbol such that the Poisson bracket implies a spacetime bound on f by Gårding's inequality; See for example [Colliander et al. 2006; Doi 1994; Kajitani 1998; Staffilani and Tataru 2002].

The operator B of our interest is the kinetic transport operator and Q is a bounded multiplier operator. That is, we consider

$$\varepsilon\partial_t f + a(v)\cdot\nabla_x f = g, \tag{4}$$

and

$$\mathcal{F}_{\xi,\zeta}(Qf) := m(\xi,\zeta)\mathcal{F}_{\xi,\zeta}(f),$$

where m is a bounded function and $\mathcal{F}_{\xi,\zeta}$ is the Fourier transform in the x- and v-variables, with the Fourier dual variables ξ and ζ respectively. The subscripts indicate the target variables.

As m is bounded, there is a tempered distribution $K(x,v)$ such that $Qf = K\star_{x,v} f$, with $\mathcal{F}_{\xi,\zeta}(K) = m$. The commutator then can be written as

$$\int [a(v)\cdot\nabla_x, K\star_{x,v}]\, f\bar{f}\,dx\,dv = \int (a(v)-a(w))\cdot\nabla_x K(x-y,v-w)f(y,w)\,dy\,dw\; f(x,v)\,dx\,dv.$$

When $a(v)=v$, it is simply the quadratic form with the multiplier $\xi\cdot\nabla_\zeta m$. We shall take advantage of this simple formula and show that the velocity average of f would gain regularity $\frac{1}{2}$ in x when $a(v)=v$ and g is in some L^p space.

The multiplier we select for this purpose is

$$m_0(\xi,\zeta) = \frac{\xi}{|\xi|}\cdot\frac{\zeta}{(1+|\zeta|^2)^{\frac{1}{2}}}. \tag{5}$$

Notice that our multiplier corresponds to the inner product of Riesz transform in x and the convolution with the gradient of Bessel potential of order 1 in v in physical space. Let us recall that the Riesz transform in dimension n can be defined weakly as a convolution operator with

$$R(x) := \frac{1}{\Gamma((n+1)/2)}\frac{x}{|x|^{n+1}}, \tag{6}$$

where Γ is the gamma function. On the other hand, the Bessel potential G^n_β of order β in dimension n is defined as

$$G^n_\beta(x) := \frac{1}{\Gamma(\beta/2)}\int_0^\infty \exp\!\left(-\frac{|x|^2}{\delta}+\frac{\delta}{4\pi}\right)\delta^{\frac{-n+\beta}{2}-1}\,d\delta. \tag{7}$$

With this notation, the corresponding kernel K_0 in physical space for m_0 is

$$K_0 = R\cdot\nabla_v G^n_1.$$

From the results in Calderón–Zygmund theory (see for example [Stein 1970]), the convolution operator with K_0 is bounded on L^p spaces for all $1<p<\infty$. Therefore, the right-hand side of (3) is bounded as long as f is in $L^\infty([0,T], L^2(\mathbb{R}^n_x\times\mathbb{R}^n_v))$ and the dual space of g.

Moreover, by the Plancherel identity

$$\begin{aligned}\int [v\cdot\nabla_x, K_0\star_{x,v}] f\bar{f}\,dx\,dv\,dt &= \int \xi\cdot\nabla_\zeta m_0 |\mathcal{F}_{\xi,\zeta}(f)|^2\,d\xi\,d\zeta\,dt\\ &= \iint\Big[\frac{1}{(1+|\zeta|^2)^{\frac12}} - \frac{|(\xi/|\xi|)\cdot\zeta|^2}{(1+|\zeta|^2)^{\frac32}}\Big]|\xi|\,|\mathcal{F}_{\xi,\zeta}(f)|^2\,d\zeta\,d\xi\,dt\\ &\geq \iint \frac{|\xi|}{(1+|\zeta|^2)^{\frac32}}|\mathcal{F}_{\xi,\zeta}(f)|^2\,d\zeta\,d\xi\,dt\\ &= \|f\|^2_{L^2([0,T],\dot{H}^{1/2}(\mathbb{R}^n_x,H^{-3/2}(\mathbb{R}^n_v)))}.\end{aligned}$$

For convenience, the conjugate index of p is denoted by p' (where $\frac{1}{p}+\frac{1}{p}'=1$). From the discussion above, we have shown:

Theorem 1. *Let* $\varepsilon\leq 1$. *If* $f\in L^\infty([0,T],(L^2\cap L^p)(\mathbb{R}^n_x\times\mathbb{R}^n_v))$ *solves* (4) *with* $a(v)=v$ *for some* $g\in L^1([0,T],L^{p'}(\mathbb{R}^n_x\times\mathbb{R}^n_v))$, *where* $1<p<\infty$, *then, for all* $\phi\in H^{3/2}(\mathbb{R}^n_v)$, *we have* $\rho_\phi\in L^2([0,T],H^{1/2}(\mathbb{R}^n_x))$. *Moreover*,

$$\tfrac12\|f\|^2_{L^2_t\dot{H}^{1/2}_x H^{-3/2}_v}\leq \|f\|^2_{L^\infty_t L^2_{x,v}}+\|f\|^2_{L^\infty_t L^p_{x,v}}+\|g\|^2_{L^1_t L^{p'}_{x,v}}.$$

Remark 2. By the Wigner transform, this result with $p=2$ connects to the local smoothing effect for the Schrödinger equation.

Remark 3. The exchange of regularity between the x- and v-variables is visible through the calculation of the commutator, which shares its similarity with the hypoellipticity phenomenon. Roughly speaking, it is a phenomenon that the degenerate directions can be recovered by commutators and was developed systematically in [Hörmander 1967] for Fokker–Planck-type operators. For the hypoellipticity of kinetic transport equations, we refer to [Bouchut 2002].

The difference here is that we introduced a homogeneous zero multiplier m_0 as a buffer, which takes on the direct impact from the transport operator. Therefore, the request for extra regularity in v goes to the test function ϕ, unlike the results in [Bouchut 2002], where an extra regularity in v was required for f.

Our setting is reminiscent of the multiplier method in [Gasser et al. 1999], where the moment and trace lemmas for kinetic equations were proven. Their results were derived employing the dispersive nature of solutions acquired by integrating along the characteristics in physical space. Here we utilize a similar technique in the frequency space, and hence instead of moments, it results in a gain of regularity.

For the rest of this paper, we shall extend this commutator method to (1) with the variable coefficient $a(v)$ and the singular source term $(-\Delta_v)^{\alpha/2}g$, which introduce nontrivial technical issues. The advantage of this approach is that the integrability of f and g can be of assistance to each other. This feature distinguishes our results from the others in the previous literature and provides averaging results for a new type of mixed integrability assumptions, which fits nicely for the conditions that the kinetic formulation of scalar conservation law naturally attains.

Let us conclude with a brief introduction on our ideas for the extension. To include variable coefficients in this method, the idea is to make an appropriate change of variables. Different change of variables gives distinct conditions on $a(v)$. Our main result will require conditions on the Jacobian matrix of a^{-1}.

With the same procedure but a different change of variables, we also recover the averaging results for the nondegeneracy condition when $\nu = 1$ (see (12)) in the L^2 setting.

On the other hand, we make this approach compatible with the singular term $(-\Delta_v)^{\alpha/2} g$ with a regularization for (1). Our regularization contains a rescaling in the v-variable and is inspired by the regularization process utilized in [DiPerna and Lions 1989c], where the scaling is a constant going to zero in a limiting process. Differently, our scaling depends on the frequency $|\xi|$. Because of this dependence, the order α of singularity contributes to the resulting gain of regularity of the velocity average ρ_ϕ.

This paper is organized as follows: We present our main theorems in Section 2 and an example of application to scalar conservation laws in Section 3. Finally, the proofs of theorems are provided in Section 4.

Our focus in this article is on dealing with a broad range of L^p exponents and fluxes $a(v)$. We will further elaborate in a coming work by considering nonhomogeneous fluxes $a(x, v)$ that also depend on the position. A summary of our results on the commutator method can be found in [Jabin et al. 2020], announced in the Séminaire Laurent Schwartz in March 2020. Related interesting results, derived by a different energy method approach, were later announced in [Arsénio and Lerner 2021].

2. Main results

2.1. ***Notation and functional framework.*** Our work relates to various topics in classical Fourier theory, including the Littlewood–Paley decomposition and Besov spaces. This subsection briefly recalls some definitions that will be used in our later discussion.

The *Littlewood–Paley decomposition* of f is described as follows: Consider a smooth radial bump function $\chi(\xi)$ with support on $\frac{1}{2} < |\xi| < 2$ such that $\chi_0(\xi) + \sum_{k\in\mathbb{N}} \chi(2^{-k}\xi) \equiv 1$ for all $\xi \neq 0$. We denote by P_k the convolution operator defined by

$$P_k f = 2^{nk}\Psi(2^k\,\cdot) \star f, \quad P_0 f = \Psi_0 \star f,$$

where χ and χ_0 are the Fourier transforms of Ψ and Ψ_0 respectively. Then we have $f = \sum_{k=0}^{\infty} P_k f$. Note that this is a decomposition in the Fourier space, and each P_k restricts f to the places where its frequency is of order 2^k.

One way to define the *Besov spaces* is through the above decomposition. The norm of the Besov spaces $B^s_{p,q}$ of a function f is defined by

$$\|f\|_{B^s_{p,q}} := \left(\sum_{k=0}^{\infty} 2^{ksq} \|P_k f\|^q_{L^p}\right)^{\frac{1}{q}}.$$

There are many excellent references for these classical materials. We refer to for instance [Klainerman 2011; Stein 1970].

2.2. ***Our main velocity averaging result.*** Our results make use of the dispersion of the kinetic transport operator $a(v) \cdot \nabla_x$ in the Fourier space. In order to have the dispersive property, one needs conditions on the variable coefficient $a(v)$. Indeed, there is no gain of regularity if a is only constant for example.

In this section, we assume $a(v) \in \mathrm{Lip}(\mathbb{R}^n)$ with the conditions

$$a(v) \text{ is one-to-one}, \quad \text{and} \quad J_{a^{-1}} \in L^\gamma, \tag{8}$$

where $J_{a^{-1}} = \det(Da^{-1})$. The conditions in (8) quantify the nonlinearity of $a(v)$ with the index γ and allow us to control the integrability of functions after the change of variables $v \mapsto w = a(v)$. Note that (8) will be utilized locally due to the compactly supported test functions ϕ.

Our proof involves a regularization of (1) through various embeddings. The interaction between the embedding and the singular term $(-\Delta_v)^{\alpha/2} g$ will affect the resulting gain of regularity. This process introduces several exponents and indices in the formulas. To simplify the notation, we denote $\max\{C, 0\}$ by C_+ for all $C \in \mathbb{R}$. The exponents and indices are collected below:

$$d_1 = n\left(\frac{1}{p_2} + \frac{1}{q_2} - \ell\right)_+, \quad d_2 = n\left(\frac{2}{p_2} - \ell\right)_+, \quad \text{with } \ell = \frac{\gamma - 2}{\gamma - 1}, \tag{9}$$

$$d_3 = n\left(\frac{1}{p_1} + \frac{1}{q_1} - 1\right)_+ \quad \text{and} \quad d_4 = n\left(\frac{2}{p_1} - 1\right)_+. \tag{10}$$

We denote the ball with center x_0 and radius R by $B(x_0, R)$. Our result is as follows:

Theorem 4. *Given* $\alpha \geq 0$, $T > 0$ *and* $0 < \varepsilon \leq 1$. *Assume* $a \in \mathrm{Lip}(\mathbb{R}^n)$ *satisfy* (8) *with* $\gamma \geq 2$. *Let*

$$f \in L^\infty([0, T], L^{p_1}(\mathbb{R}^n_x, L^{p_2}(\mathbb{R}^n_v)))$$

solve (1) *for some*

$$g \in L^1([0, T], L^{q_1}(\mathbb{R}^n_x, L^{q_2}(\mathbb{R}^n_v))),$$

with $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ *and*

$$\frac{1 - (d_3 - d_4)}{\alpha + 1 + (d_1 - d_2)} \geq 0.$$

Then for any $B(x_0, R) \subset \mathbb{R}^n_x$ *and* $\phi \in C^\infty_c(\mathbb{R}^n_v)$, *one has*

$$\rho_\phi(t, x) \in L^2([0, T], H^s(B(x_0, R)))$$

for all $s < S = \frac{1}{2}[(1 - d_2)\theta - d_4]$, *where*

$$\theta = \left[\min\left\{\frac{1 - (d_3 - d_4)}{\alpha + 1 + (d_1 - d_2)}, 1\right\}\right],$$

and d_i *are defined in* (9) *and* (10) *for* $i = 1, 2, 3, 4$. *Moreover*,

$$\|\rho_\phi\|^2_{L^2([0,T],H^s(B(x_0,R)))} \leq C(\|f\|^2_{L^\infty([0,T],L^{p_1}(\mathbb{R}^n_x,L^{p_2}(\mathbb{R}^n_v)))} + \|g\|^2_{L^1([0,T],L^{q_1}(\mathbb{R}^n_x,L^{q_2}(\mathbb{R}^n_v)))}),$$

where C *only depends on* R, $\|\phi\|_\infty$ $\|J_{a^{-1}}\|_{L^\gamma}$, *and* $\mathrm{Lip}(a)$.

Remark 5. The restriction for γ can be relaxed to $\gamma \geq 1$, but the formula of S for $1 \leq \gamma < 2$ would be changed to $\frac{1}{2}\{[1 - n(2/p_2 + 2/\gamma - 1)]\theta - d_4\}$, with

$$\theta = \min\left\{\frac{1 - (d_3 - d_4)}{\alpha + 1 + n(1/q_2 - 1/p_2)}, 1\right\}.$$

Remark 6. The end point $s = S$ can be included, when $f \in L^\infty([0, T], B^0_{p_1,2}(\mathbb{R}^n_x, L^{p_2}(\mathbb{R}^n_v)))$, $g \in L^1([0, T], B^0_{q_1,2}(\mathbb{R}^n_x, L^{q_2}(\mathbb{R}^n_v)))$ and $1 < p_1, p_2, q_1, q_2 < \infty$.

Remark 7. Because of the quadratic form in our method, our result bounds the velocity average in L^2 and the upper bound always has the same weight on the norms of f and g, independent of p_1, p_2, q_1, q_2.

When $a(v) = v$, one has that $\gamma = \infty$. In this case, we have a simpler formula for Theorem 4 when f and g are in the dual space of each other:

Corollary 8. *Given $\alpha \geq 0$, $T > 0$ and $0 < \varepsilon \leq 1$. If f belongs to the space $L^\infty([0,T], L^{p_1}(\mathbb{R}^n_x, L^{p_2}(\mathbb{R}^n_v)))$ and solves* (1) *with $a(v) = v$ for some $g \in L^1([0,T], L^{p_1'}(\mathbb{R}^n_x, L^{p_2'}(\mathbb{R}^n_v)))$, where $2 \leq p_1, p_2 \leq \infty$, then, for any $B(x_0, R) \subset \mathbb{R}^n_x$ and $\phi \in C_c^\infty(\mathbb{R}^n_v)$, we have $\rho_\phi \in L^2([0,T], H^s(B(x_0,R)))$ for all $s < 1/(2(\alpha+1))$.*

2.3. ***Comparison with previous literature.*** There is already a huge literature on averaging lemmas and in some situations the existing results have been proven optimal. In order to provide the readers an idea of when our method becomes effective and the potential advantages of our method, a comparison of the regularity in x shall be presented between our result and the theorems in [Arsénio and Masmoudi 2019; DiPerna et al. 1991; Westdickenberg 2002].

Because our resulting space has a different integrability from the previous results except for in the L^2 case, our method may render a more appropriate tool under certain circumstances. We will point out the regions where one theorem can imply the other through embedding or interpolation. The interpolation is applied between the resulting space of ρ_ϕ and the assumption space of f, because ρ_ϕ has the same integrability in x as f when $\phi \in C_c^\infty$.

Notice that some theorems we quote here apply to more general conditions in the original papers, but for simplicity we will only state the parts that concern our discussion and restrict to the special case $a(v) = v$. We also assume for convenience that f and g are compactly supported in both the x- and v-variables and $\phi \in C_c^\infty$ for this entire discussion.

Let us begin with the classical averaging result in [DiPerna et al. 1991], where the case with different integrabilities for f and g and $\alpha > 0$ is available.

Theorem 9 [DiPerna et al. 1991]. *If $f \in L^p(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v)$ and $g \in L^q(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v)$ satisfy* (1) *with $a(v) = v$, then $\rho_\phi \in B^s_{r,\infty}(\mathbb{R}_t \times \mathbb{R}^n_x)$, where*

$$s = \frac{1}{\bar{p}}\left(\alpha + \frac{1}{\bar{p}} + \frac{1}{\underline{q}}\right)^{-1}, \quad \bar{p} = \max\{p, p'\}, \quad \underline{q} = \min\{q, q'\}, \quad \frac{1}{r} = \frac{s}{q} + \frac{1-s}{p}.$$

Moreover, if $p = q \in (1,\infty)$, then $\rho_\phi \in B^s_{r,t}(\mathbb{R}_t \times \mathbb{R}^n_x)$, where $t = \max\{p, 2\}$.

Under the assumption of Theorem 9, we start our discussions for the cases when $p = q$.

- *When $p = q = 2$, both Theorems 4 and 9 reach the same regularity $H^{1/(2(1+\alpha))}$.*
- *When $p = q \in (1,2)$, Theorem 9 implies Theorem 4*: Indeed, Theorem 9 reaches $B^{1/(p'(1+\alpha))}_{p,2}$, while Theorem 4 gives H^s for all

$$s < S = \frac{1}{2(1+\alpha)}\left[1 - n(2+\alpha)\left(\frac{2}{p} - 1\right)\right].$$

By the embedding theorem for Besov spaces, $B_{p,2}^{1/(p'(1+\alpha))} \subset H^{\tilde{s}}$, with

$$\tilde{s} = \frac{1}{p'(1+\alpha)} + n\left(\frac{1}{2} - \frac{1}{p}\right),$$

which is greater than or equal to S for all $n \geq 1$.

• *When* $p = q \in (2, \infty)$, *the result by Theorem 4 has more differentiability but less integrability than Theorem 9. Furthermore, when* $n = 1$ *and* $\alpha = 0$, *Theorem 4 implies Theorem 9*: Theorem 9 reaches $B_{p,p}^{1/(p(1+\alpha))}$, while Theorem 4 has $H^{1/(2(1+\alpha))}$. By embedding $H^{1/(2(1+\alpha))} \subset B_{p,2}^{\tilde{s}}$, where

$$\tilde{s} = \frac{1}{2(1+\alpha)} + n\left(\frac{1}{p} - \frac{1}{2}\right).$$

and $\tilde{s} < 1/(p(1+\alpha))$ except when $n = 1$ and $\alpha = 0$.

Because of the quadratic form in our method, the more favorable type of conditions for our method is when $p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. We therefore compare Theorems 4 and 9 under this assumption:

• *Under the assumption of Theorem 9 with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \in (2, \infty)$, *the result by Theorem 4 has more differentiability but less integrability in* x. *Moreover, Theorem 4 implies Theorem 9 when* $\alpha = 0$ *by interpolation or when* $0 \leq \alpha < \frac{1}{n}$ *and* $2 < p < 2n/(n(1+\alpha)-1)$ *by embedding*: Under these conditions, Theorem 4 results in $H^{1/(2(1+\alpha))}(\mathbb{R}^n_x)$, while Theorem 9 reaches $B_{r,\infty}^{1/(p(1+\alpha))}(\mathbb{R}^n_x)$, where

$$\frac{1}{r} = \frac{1}{p(1+\alpha)}\left(1 - \frac{2}{p}\right) + \frac{1}{p}.$$

By the interpolation between $H^{1/(2(1+\alpha))}$ and L^p, we derive $\rho_\phi \in W^{1/(p(1+\alpha)^2),r}$. Hence when $\alpha = 0$, Theorem 4 implies Theorem 9.

On the other hand, by embedding $H^{1/(2(1+\alpha))} \subset B_{r,2}^{\tilde{s}}$, where

$$\tilde{s} = \frac{1}{2(1+\alpha)} + n\left(\frac{1}{r} - \frac{1}{2}\right).$$

Even with the dimension dependence, there are regions that embedding gives a better regularity than interpolation. For example when $n = 1$,

$$\tilde{s} \geq \frac{1}{p(1+\alpha)^2}, \quad \text{when } p \leq 2 + \frac{2}{\alpha}.$$

We compare $\tilde{s}$ with the regularity obtained by Theorem 9. In general for each fixed n,

$$\tilde{s} \geq \frac{1}{p(1+\alpha)}, \quad \text{when } p \leq \frac{2n}{n(1+\alpha)-1},$$

which is compatible with $p > 2$ only when $\alpha < \frac{1}{n}$. Hence Theorem 4 implies Theorem 9 when $0 \leq \alpha < \frac{1}{n}$ and $2 < p < 2n/(n(1+\alpha)-1)$.

We now compare our result with [Arsénio and Masmoudi 2019] and [Westdickenberg 2002], where mixed norm conditions in general dimensions were considered for the stationary transport equation

$$v \cdot \nabla_x f = g. \tag{11}$$

We shall take $\varepsilon = 0$, in order to compare our theorem with results for (11).

Theorem 10 [Westdickenberg 2002]. *For $1 < p < n/(n-1)$, if $f \in B^0_{p,q}(\mathbb{R}^n_x, L^{p_2}(\mathbb{R}^n_v))$ and $g \in B^0_{p,q}(\mathbb{R}^n_x, L^{q_2}(\mathbb{R}^n_v))$ satisfy* (11), *then $\rho_\phi \in B^S_{P,q}(\mathbb{R}^n_x)$, where*

$$S = -n+1+\frac{1}{p_2'}\left[1+\frac{1}{q_2}-\frac{1}{p_2}\right]^{-1} \quad \text{and} \quad P = \left[\frac{1}{p}-\frac{n-1}{n}\right]^{-1}.$$

Theorem 11 [Arsénio and Masmoudi 2019]. *When $\frac{4}{3} \le p \le 2$, if $f, g \in L^p(\mathbb{R}^n_x, L^2(\mathbb{R}^n_v))$ satisfy* (11), *then $\rho_\phi \in W^{s,p}(\mathbb{R}^n)$ for all $s < S$, where $S = \frac{1}{2}$ when $n = 1, 2$, and*

$$S = \frac{1}{2}\left(3-\frac{4}{p}\right)+\frac{n}{4(n-1)}\left(\frac{4}{p}-2\right), \quad \text{when } n \ge 3.$$

For the comparison with Theorem 10, we take $q = 2$ for an easier discussion with our H^s result. And since Theorem 10 allows general integrabilities in v, let us consider $p_2 = q_2' \ge 2$, which is the most favorable condition for our method.

- *Under the assumption of Theorem 10 with $n = 1$, $q = 2$ and $p_2 = q_2' \ge 2$. Both Theorems 4 and 10 reach the same regularity when $p = 2$. Theorem 10 implies Theorem 4 when $p \ne 2$*: Here Theorem 10 reaches $B^{1/2}_{p,2}$, while Theorem 4 has $H^{1/p'}$ when $p \le 2$ and $H^{1/2}$ when $p > 2$, as was mentioned in Remark 6. When $p = 2$, the two results are exactly the same. When $p < 2$, $B^{1/2}_{p,2} \subset H^{1/p'}$ by embedding and for $p > 2$, $B^{1/2}_{p,2} \subset H^{1/2}$ locally.

Notice, for $n \ge 2$, Theorem 10 no longer applies to $p > 2$. The restriction $p < 2$ is not ideal for our method, but the comparison is still interesting under these mixed norm conditions.

- *Under the assumption of Theorem 10 with $n \ge 2$ (which forces $1 < p < 2$), $q = 2$ and $p_2 = q_2' \ge 2$, our result implies Theorem 10*: In this case, Theorem 10 reaches $B^{3/2-n}_{P,2}$ with

$$P = \left[\frac{1}{p}-\frac{n-1}{n}\right]^{-1}$$

and our method reaches $H^{(1/2)[1-2n/p+n]}$ as was stated in Remark 6. Our result has more differentiability but less integrability. Moreover, by the embedding $H^{(1/2)[1-2n/p+n]} \subset B^{3/2-n}_{P,2}$.

- *Under the assumption of Theorem 11, the result by Theorem 4 has more integrability but less differentiability than Theorem 11. Furthermore, Theorem 11 implies Theorem 4 when $n = 1$ and 2, but the implication does not hold for $n \ge 3$*: Under this assumption, we again have H^s with

$$s < \frac{1}{2}\left[1-\frac{2n}{p}+n\right].$$

For both $n = 1$ and 2, $W^{1/2,p}_x \subset H^{(1/2)[1-2n/p+n]}_x$ by Sobolev embedding. As for $n \ge 3$, we have $W^{s,p}_x \subset H^{\tilde{s}}_x$, where

$$s = \frac{1}{2}\left(3-\frac{4}{p}\right)+\frac{n}{4(n-1)}\left(\frac{4}{p}-2\right),$$
$$\tilde{s} = \frac{1}{2}\left(3-\frac{4}{p}\right)+\frac{n}{4(n-1)}\left(\frac{4}{p}-2\right)+n\left(\frac{1}{2}-\frac{1}{p}\right).$$

Hence Theorem 11 cannot imply Theorem 4 in this case.

In addition to the new regularity results, our method also renders the following properties:

- Our velocity-averaging result is independent of small ε. This could have applications to the compactness of solutions for rescaled kinetic equations, which frequently appear in the discussions of hydrodynamic limits. We refer to for example [Golse 2014; Saint-Raymond 2009] for more details in this direction.

- Our argument does not perform the Fourier transform in time variable. Therefore, this method has possible extensions to time-discretized and stochastic kinetic equations.

2.4. ***On the nondegeneracy conditions.*** The assumption (8) we imposed on $a(v)$ is different from the classical conditions in the previous literature, which are called the nondegeneracy conditions:

Definition 12. We say $a \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m)$ satisfies the *nondegeneracy condition of order* $\nu \in (0, 1]$ if there exists $c_0 > 0$ such that for every compact set $D \subset \mathbb{R}^n$, $\sigma \in \mathbb{S}^{m-1}$ and $\tau \in \mathbb{R}$ we have

$$\mathcal{L}^n(\{v \in D : |a(v)\cdot\sigma - \tau| \le \alpha/2\}) \le c_0\alpha^\nu, \tag{12}$$

where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$.

Our assumption (8) is stronger than (12) with $\nu = 1 - \frac{1}{\gamma}$. Indeed, when $n = m$, the assumption $J_{a^{-1}} \in L^\gamma_v$ implies (12) with $\nu = 1 - \frac{1}{\gamma}$, but the other direction holds only when $n = \nu = 1$. When $n > 1$, (12) only gives restrictions on the preimages of bands and when $\nu < 1$, one can construct a Lipschitz function a_ν on $\mathbb{R}$ satisfying (12) and a sequence of measurable sets $\mathcal{O}^i$ such that

$$\frac{|a_\nu^{-1}(\mathcal{O}^i)|}{|\mathcal{O}^i|^\nu} \to \infty \quad \text{as } i \to \infty,$$

which shows $J_{a^{-1}} \notin L^\gamma$. An example of such a construction can be found in the Appendix.

The dimension of interests is $n \le m$ for applications, especially when $n = 1$ for scalar conservation laws. In an attempt to weaken our assumption to the nondegeneracy conditions with general $n \le m$ cases, a different change of variables $v \mapsto \lambda = a(v)\cdot\xi/|\xi|$ is performed, and the traditional result in the L^2 setting for $\nu = 1$ is recovered with our method.

Theorem 13. *Given $n \le m$, $\alpha \ge 0$, $T > 0$ and $0 < \varepsilon \le 1$. Assume $a \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m)$ satisfies the nondegeneracy condition* (12) *with $\nu = 1$. Let $f \in L^\infty([0,T], L^2(\mathbb{R}^m_x \times \mathbb{R}^n_v))$ solve* (1) *for some $g \in L^1([0,T], L^2(\mathbb{R}^m_x \times \mathbb{R}^n_v))$. Then, for any $\phi \in C_c^\infty(\mathbb{R}^n_v)$, one has $\rho_\phi(t,x) \in L^2([0,T], H^{1/(2(\alpha+1))}(\mathbb{R}^m_x))$ and*

$$\|\rho_\phi\|^2_{L^2([0,T],H^{1/(2(\alpha+1))}(\mathbb{R}^m_x))} \le C(\|f\|^2_{L^\infty([0,T],L^2(\mathbb{R}^m_x\times\mathbb{R}^n_v))} + \|g\|^2_{L^1([0,T],L^2(\mathbb{R}^m_x\times\mathbb{R}^n_v))}),$$

where C only depends on c_0, $\|\phi\|_\infty$ and $\mathrm{Lip}(a)$.

This L^2 theorem recovers the same regularity $H^{1/(2(\alpha+1))}$ in x as in [DiPerna and Lions 1989a; DiPerna et al. 1991]. Even though this regularity result is not new, we provide a different approach for proving this theorem. As we mentioned in the discussion after Corollary 8, some interesting features which are also inherited by Theorem 9 include:

- Due to the independence of ε, our results have potential applications to the hydrodynamic limit type of problems.

- The absence of the Fourier transform in the time variable enables potential extensions of our method to time-discretized or stochastic kinetic equations.

Remark 14. We were unable to obtain an L^p statement as we did in Theorem 4, because our natural choice of multiplier for the alternate proof is not a Calderón–Zygmund operator and we lose the bounds in general L^p spaces. In fact when $a(v) = v$, due to the change of variables $\lambda \mapsto v \cdot \xi/|\xi|$, our natural multiplier would be in the form of $S(\zeta \cdot \xi/|\xi|)$, where S is a smooth function and ζ is the Fourier dual variable of v. Its inverse Fourier transform in dimension 2 is in the form of $(v \cdot x/|x|^3)\widetilde{S}(v \cdot x^\perp/|x|)$, which is not bounded on $L^p_{x,v}$.

Remark 15. Our proof is not directly applicable when $\nu < 1$. The nondegeneracy condition in our proof is employed as a constraint on the determinant of Jacobian matrices. We were not able to derive this connection for $\nu < 1$ and so the same proof was not extended immediately.

3. An example of future perspective: regularizing effects for measure-valued solutions to scalar conservation law

Among several potential applications of the new method for averaging lemmas presented here, this section focuses on the regularity of so-called measure-valued solutions to conservation laws and in particular scalar conservation laws.

Scalar conservation laws can be viewed as a simplified model of hyperbolic systems which still captures some of the basic singular structure. They read

$$\begin{cases} \partial_t u + \sum_{i=1}^n \partial_{x_i} A_i(u) = 0, \\ u(t=0,x) = u_0(x), \end{cases} \tag{13}$$

where $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is the scalar unknown and $A : \mathbb{R} \to \mathbb{R}^n$ is a given flux.

The concept of measure-valued solutions to hyperbolic systems such as (13) was introduced in [DiPerna 1985]. It has recently seen a significant revival of interest as measure-valued solutions offer a more statistical description of the dynamics; see in particular [Fjordholm et al. 2016; 2017].

It is convenient to define a measure-valued solution through the kinetic formulation of (13), which also allows for a straightforward application of our results. A scalar function $u(t,x) \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^n))$ corresponds to a measure-valued solution if there exists $f(t,x,v) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R})$ with the constraint

$$u(t,x) = \int_{\mathbb{R}} f(t,x,v)\,dv, \quad -1 \le f \le 1, \tag{14}$$

and if f solves the kinetic equation

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v m \tag{15}$$

for $a(v) = A'(v)$ and any finite Radon measure m. If u is obtained as a weak-limit of a sequence u_n then f includes some information on the oscillations of u_n since it can directly be obtained from the Young measure μ of the sequence

$$f(t,x,v) = \int_0^v \mu(t,x,dz).$$

The system (14)–(15) is hence immediately connected to the notion of kinetic formulation for scalar conservation laws introduced in the seminal article [Lions et al. 1994a] and extended to isentropic gas dynamics in [Lions et al. 1994b]. If u is an entropy solution to (13), then one may define

$$f(t,x,v)=\begin{cases} 1 & \text{if } 0\le v\le u(t,x),\\ -1 & \text{if } u(t,x)\le v<0,\\ 0 & \text{otherwise,}\end{cases} \tag{16}$$

and f solves the kinetic equation (15) with the additional constraint that $m\ge 0$, which corresponds to the entropy inequality.

We refer for example to [Perthame 2002] for a thorough discussion of kinetic formulations and their usefulness, such as recovering the uniqueness of the entropy solution first obtained in [Kruzhkov 1970].

The use of kinetic formulations has proved effective in particular in obtaining regularizing effects for scalar conservation laws. In one dimension and for strictly convex flux, early on it was proven in [Oleinik 1957] that entropy solutions are regularized in BV. In more than one dimension and for more complex flux that are still nonlinear in the sense of (12) with $\nu=1$, a first regularizing effect was obtained in [Lions et al. 1994a], yielding $u\in W^{s,p}$ for all $s<\frac{1}{3}$ and some $p>1$.

Such regularizing effects actually do not use the sign of m and for this reason hold for any weak solution to (13) with bounded entropy production. Among that wider class a counterexample constructed in [De Lellis and Westdickenberg 2003] proves that solutions cannot in general be expected to have more than $\frac{1}{3}$ derivative. The optimal space $(B^{1/3}_{3,\infty})_{x,\text{loc}}$ was eventually derived in [Golse and Perthame 2013]. Whether a higher regularity actually holds for entropy solutions (instead of only bounded entropy production) remains a major open problem though.

It was observed in [Jabin and Perthame 2002] that the regularizing effect for the kinetic formulation relies in part on the regularity of the function f defined by (16): for example such an f belongs to $L^\infty(\mathbb{R}_+\times\mathbb{R}^n,\mathrm{BV}(\mathbb{R}))$. Unfortunately such additional regularity is lost for measure-valued solutions since we only have $f\in L^1\cap L^\infty$ by (14).

A priori, one may hence only apply the standard averaging result from [DiPerna et al. 1991] directly on (15). Assuming nondegeneracy of the flux, i.e., (12) with $\nu=1$, we may apply Theorem 9 for any $\alpha>1$, $g\in L^1$ and $f\in L^2$ (the optimal space for this theorem). One then deduces that if u corresponds to a measure-valued solution with f compactly supported in v then $u\in B^s_{5/3,\infty}$ for any $s<\frac{1}{5}$.

However we are then making no use of the additional integrability of f. Instead one may also apply our new result Theorem 4 to (15):

Theorem 16. *Let f satisfy* (14) *and solve* (15) *for some finite Radon measure m and some $a:\mathbb{R}^n\to\mathbb{R}^n$ with* (8) *for $\gamma=\infty$. Assume moreover that $f\in L^\infty([0,T],L^1(\mathbb{R}^n\times\mathbb{R}^n))$ and is compactly supported in velocity. Then for any $B(x_0,R)\subset\mathbb{R}^n$, we have $u\in L^2([0,T],H^s(B(x_0,R)))$ for any $s<\frac{1}{4}$.*

In dimension 1, Theorem 16 directly applies to measure-valued solutions and improve the regularity from almost $B^{1/5}_{5/3,\infty}$ in x to almost $H^{1/4}$. In higher dimensions, as we observed, we cannot directly replace (8) with (12). Therefore a better understanding of the regularity of measure-valued solutions is directly connected to further investigations of what should replace (8) if $a:\mathbb{R}^m\to\mathbb{R}^n$ with $m<n$.

4. Proofs

4.1. *Proof of Theorem 4.*

4.1.1. *Main proof.* The Fourier transform of f in x will be denoted by $\hat{f}$ for simplification in this proof as it appears quit often.

The proof contains mainly three steps as follows:

Step 1: *Preparations — localization, regularization and change of variables.* Without loss of generality, assume $\mathrm{supp}(f) \subset B(0,1)$ in x and fix $\phi \in C_c^\infty$ with $\mathrm{supp}(\phi) \subseteq B(0,1)$ in v.

Take another smooth function $\Phi(v)$ with $\mathrm{supp}(\Phi) \subseteq B(0,1)$. Consider

$$F_\theta = (\hat{f}\phi) \star_v \Phi_{\text{rescale}},$$

where

$$\Phi_{\text{rescale}}(v) = \begin{cases} 2^{nk\theta}\Phi(v2^{k\theta}) & \text{when } 2^{k-1} \le |\xi| < 2^{k+1} \text{ for all } k \in \mathbb{N}, \\ \Phi(v) & \text{when } |\xi| < 1, \end{cases} \tag{17}$$

and $\theta \ge 0$ will be decided later.

Notice that

$$\mathrm{supp}(F_\theta) \subseteq \overline{\mathrm{supp}(\phi) + \mathrm{supp}(\Phi_{\text{rescale}})} \subseteq \overline{B(0,2)}$$

for all ξ and F_θ satisfies

$$\varepsilon\partial_t F_\theta + i a(v)\cdot\xi F_\theta = ((-\Delta_v)^{\frac{\alpha}{2}}\hat{g}\phi) \star_v \Phi_{\text{rescale}} + \mathrm{Com}, \tag{18}$$

where

$$\mathrm{Com}(v) = i\int (a(v) - a(w))\cdot\xi \hat{f}(w)\phi(w)\Phi_{\text{rescale}}(v-w)\,dw.$$

By the change of variables $v \mapsto w = a(v)$, (18) can be rewritten as

$$\varepsilon\partial_t h + i w\cdot\xi h = k^1 + k^2 \tag{19}$$

in the sense of distribution, where h, k^1 and k^2 are defined as

$$\int h(w)\psi(w)\,dw = \int F_\theta(v)\psi(a(v))\,dv.$$
$$\int k^1(w)\psi(w)\,dw = \int [((-\Delta_v)^{\alpha/2}\hat{g}\phi) \star_v \Phi_{\text{rescale}}](v)\psi(a(v))\,dv,$$
$$\int k^2(w)\psi(w)\,dw = \int \mathrm{Com}(v)\psi(a(v))\,dv.$$

Step 2: *Commutator method with m_0.* Consider a smooth radial bump function $\chi(\xi)$ with support on $\frac{1}{2} < |\xi| < 2$ such that $\sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi) \equiv 1$ for all $\xi \neq 0$. Define

$$\chi_0(\xi) = \sum_{k\in\mathbb{Z}\setminus\mathbb{N}} \chi(2^{-k}\xi) \quad \text{and} \quad \chi_k(\xi) = \chi(2^{-k}\xi) \quad \text{for all } k \in \mathbb{N}.$$

For each $k \in \mathbb{N} \cup \{0\}$, we apply the commutator method with the multiplier m_0 defined in (5) to $h(w)\chi_k(\xi)$. With the same structure as the proof of Theorem 1,

$$
\begin{aligned}
&\iint_0^T \int |\mathcal{F}_\eta(h)|^2(\eta) \frac{2^k}{(1+|\eta|^2)^{3/2}}\, d\eta\, dt\, \chi_k(\xi)\, d\xi \\
&\lesssim \int_0^T \iint \xi \cdot \nabla_\eta m_0(\xi,\eta) |\mathcal{F}_\eta(h)|^2 \chi_k(\xi)\, d\xi\, d\eta\, dt \\
&= \iint \bar{h}(w)\Big(\frac{1}{i}\frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w h\Big)\, dw\, \chi_k(\xi)\, d\xi \Big|_{t=0}^{t=T} + \operatorname{Re}\int_0^T \iint \bar{h}(w) \frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w (k^1+k^2)\, dw\, \chi_k(\xi)\, d\xi\, dt \\
&:= A_k.
\end{aligned} \tag{20}
$$

Recall that G_1^n is the Bessel potential of order 1 defined in (7). A_k is estimated as follows:

Lemma 17. *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$. For each fixed $k \in \mathbb{N} \cup \{0\}$ and any small $\delta > 0$,*

$$
\begin{aligned}
|A_k| &\lesssim 2^{k(d_4+\theta d_2)} \|f\|^2_{L_x^{p_1} L_v^{p_2}}\Big|_{t=0}^{t=T} \\
&\quad + 2^{k(d_3+\theta d_1+\alpha\theta+\delta)} \int_0^T \|f\|_{L_x^{p_1} L_v^{p_2}} \|g\|_{L_x^{q_1} L_v^{q_2}}\, dt + 2^{k(d_4+\theta d_2+1-\theta)} \int_0^T \|f\|^2_{L_x^{p_1} L_v^{p_2}}\, dt,
\end{aligned} \tag{21}
$$

where

$$
d_1 = n\Big(\frac{1}{p_2}+\frac{1}{q_2}-\ell\Big)_+, \quad d_2 = n\Big(\frac{2}{p_2}-\ell\Big)_+, \quad \text{with } \ell = \frac{\gamma-2}{\gamma-1}.
$$
$$
d_3 = n\Big(\frac{1}{p_1}+\frac{1}{q_1}-1\Big)_+, \quad d_4 = n\Big(\frac{2}{p_1}-1\Big)_+,
$$

To minimize the highest order of ξ in (21), we choose

$$
\theta = \min\Big\{\frac{1-(d_3-d_4)}{\alpha+1+(d_1-d_2)}, 1\Big\}
$$

and the highest order becomes $1-S$, where $S = \theta(1-d_2) - d_4$.

Dividing the whole inequality (20) by $2^{k(1-S+2\delta)}$, we attain

$$
\begin{aligned}
&\iint_0^T \int |\mathcal{F}_\eta(h)|^2 \frac{|\xi|^{(S-\delta)}}{(1+|\eta|^2)^{\frac{3}{2}}}\, d\eta\, dt\, \chi_k(\xi) d\xi \\
&\qquad \lesssim 2^{-k\delta}\Big[\|f\|^2_{L_x^{p_1} L_v^{p_2}}\Big|_{t=0}^{t=T} + \int_0^T \|f\|_{L_x^{p_1} L_v^{p_2}} \|g\|_{L_x^{q_1} L_v^{q_2}}\, dt + \int_0^T \|f\|^2_{L_x^{p_1} L_v^{p_2}}\, dt\Big]
\end{aligned} \tag{22}
$$

for all $k \in \mathbb{N} \cup \{0\}$.

Notice that the additional 2^δ is added in case either of the pairs (p_1, q_1) or (p_2, p_2) equals $(1, \infty)$ or $(\infty, 1)$ and the additional logarithm growth appears from the weak boundedness of Calderón–Zygmund operators.

Summing (22) over all $k \in \mathbb{N} \cup \{0\}$, we obtain

$$\int_0^T \iiint |\xi|^s \bar{h}(v) G_3^n(v-w) h(w)\, dw\, dv\, d\xi\, dt \lesssim \|f\|^2_{L^\infty([0,T],L_x^{p_1} L_v^{p_2})} + \|g\|^2_{L^1([0,T],L_x^{q_1} L_v^{q_2})},$$

with

$$s < S = (1-d_2)\min\left\{\frac{1-(d_3-d_4)}{\alpha+1+(d_1-d_2)}, 1\right\} - d_4.$$

Step 3: *Derive result back to* f. The last step is to translate the quadratic form of h back to a norm of velocity average of f.

By the change of variables again,

$$\begin{aligned}\int_0^T \int \left| \int F_\theta(v)\psi(a(v))\, dv\right|^2 |\xi|^s\, d\xi\, dt &= \int_0^T \int \left| \int h(w)\psi(w)\, dw\right|^2 |\xi|^s\, d\xi\, dt,\\ &\lesssim \int_0^T \iiint |\xi|^s \bar{h}(v) G_3^n(v-w) h(w)\, dw\, dv\, d\xi\, dt\\ &\lesssim \|f\|^2_{L^\infty([0,T],L_x^{p_1} L_v^{p_2})} + \|g\|^2_{L^1([0,T],L_x^{q_1} L_v^{q_2})} \end{aligned} \tag{23}$$

for all $\psi \in H^{3/2}$ and $s < S$.

By the assumption that ϕ and Φ are compactly supported in v, we show that:

Lemma 18. *There exists* $\psi \in H^{3/2}$ *such that*

$$\int_0^T \int \left| \int \hat{f}\phi\, dv\right|^2 |\xi|^s\, d\xi\, dt \lesssim \int_0^T \int \left| \int F_\theta(v)\psi(a(v))\, dv\right|^2 |\xi|^s\, d\xi\, dt. \tag{24}$$

With (23), (24) and the Poincaré inequality, we attain

$$\|\rho_\phi\|^2_{L^2([0,T],H_x^{s/2}(B(0,1)))} \lesssim \|f\|^2_{L^\infty([0,T],L_x^{p_1} L_v^{p_2})} + \|g\|^2_{L^1([0,T],L_x^{q_1} L_v^{q_2})}$$

for all $s < S$, which concludes our proof. □

Remark 19. Note that $m(\xi,\zeta)$ being homogeneous zero in ζ is essential for the commutator to be positive-definite after interacting with the transport operator. In fact, if we consider

$$m(\xi,\zeta) = \frac{\xi}{|\xi|}\cdot\frac{\zeta}{(1+|\zeta|^2)^{\frac{\beta}{2}}},$$

with $\beta > 1$,

$$\xi\cdot\nabla_\zeta m = \frac{|\xi|[(1+|\zeta|^2) - \beta|(\xi/|\xi|)\cdot\zeta|^2]}{(1+|\zeta|^2)^{\frac{\beta}{2}+1}}.$$

When ζ is parallel to ξ and $|\zeta|$ is large, $\xi\cdot\nabla_\zeta m$ becomes negative and our argument does not work.

Remark 20. Our regularization recollects the regularization process in [DiPerna and Lions 1989c]. Here the interaction between $\Phi_{|\xi|^{-\theta}}$ and $(-\Delta_v)^{\alpha/2} g$ shows explicitly the exchange of regularity between x and v.

4.1.2. *Proof of Lemma 17.* Our estimation of A_k will use the following proposition:

Lemma 21. *Let $a \in \mathrm{Lip}(\mathbb{R}^n)$. If $J_{a^{-1}} \in L^\gamma$, the change of variables is bounded from L^p to $L^{(p'\gamma')'}$. Precisely, if $\int \ell(w)\psi(w)\,dw = \int L(v)\psi(a(v))\,dv$,*

$$\|\ell\|_{L^{(p'\gamma')'}} \lesssim \|L\|_{L^p}.$$

Proof of Lemma 21. By Hölder's inequality,

$$\int |\psi(a(v))|^{p'}\,dv = \int |\psi(w)|^{p'} J_{a^{-1}}(w)\,dw \le \|J_{a^{-1}}\|_{L^\gamma}\left(\int |\psi(w)|^{p'\gamma'}\,dw\right)^{\frac{1}{\gamma'}}.$$

Hence

$$\begin{aligned}\|\ell\|_{L^{(p'\gamma')'}} &= \sup_{\|\psi\|_{L^{p'\gamma'}}=1}\left|\int \ell(w)\psi(w)\,dw\right| = \sup_{\|\psi\|_{L^{p'\gamma'}}=1}\left|\int L(v)\psi(a(v))\,dv\right| \\ &\le \sup_{\|\psi\|_{L^{p'\gamma'}}=1}\|L\|_{L^p}\|\psi(a(v))\|_{L^{p'}} \le \|J_{a^{-1}}\|_{L^\gamma}\|L\|_{L^p}.\end{aligned}$$

□

We shall estimate A_k term by term for the case when $p_i \ge q_i$, $\gamma \ge 2$ and $d_j \ge 0$ for all $i = 1, 2$ and $j = 1, 2, 3, 4$. All the results for the other cases can be derived by the same calculations and hence are omitted here.

- For the first term: By the Cauchy–Schwarz inequality and since $R \cdot \nabla_v G_1^n$ is a Calderón–Zygmund operator:

$$\iint \bar{h}\left(\frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w h\right) dw\, \chi_k(\xi) d\xi\Big|_{t=0}^{t=T} \le \|\mathcal{F}_x^{-1}(h\chi_k(\xi))\|^2_{L^2_{xw}}\Big|_{t=0}^{t=T}. \tag{25}$$

Denote by S the inverse Fourier transform $\mathcal{F}_x^{-1}(\chi)$ in ξ of χ. For notation simplification, we further write the rescaled functions $S_k := 2^{nk}S(x2^k)$ and $\Phi_{k,\theta} := 2^{nk\theta}\Phi(v2^{k\theta})$. By Lemma 21,

$$\|\mathcal{F}_x^{-1}(h\chi_k(\xi))\|_{L^2_{xw}} \lesssim \|S_k \star_x f \star_v \Phi_{k,\theta}\|_{L^2_x L_v^{2(\gamma-1)/(\gamma-2)}} \lesssim 2^{kn(\frac{1}{p_1}-\frac{1}{2})+k\theta(\frac{1}{p_2}-\frac{\gamma-2}{2\gamma-2})}\|f\|_{L_x^{p_1}L_v^{p_2}}.$$

Hence

$$\iint \bar{h}\left(\frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w h\right) dw\, \chi_k(\xi) d\xi\Big|_{t=0}^{t=T} \lesssim 2^{kn(\frac{2}{p_1}-1)+k\theta(\frac{2}{p_2}-\frac{\gamma-2}{\gamma-1})}\|f\|^2_{L_x^{p_1}L_v^{p_2}}\Big|_{t=0}^{t=T}.$$

- For the second term:

$$\begin{aligned}&\int_0^T\iint \bar{h}\left(\frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w k^1\right) dw\, \chi_k(\xi)\, d\xi\, dt \\ &\quad\lesssim |\log 2^k||\log 2^{k\theta}|\int_0^T \|S_k \star_x f \star_v \Phi_{k,\theta}\|_{L_x^{q_1'}L_v^{q_2'(\gamma-1)/(\gamma-q_2')}} \\ &\qquad\qquad \times \|2^{k\alpha\theta}S_k \star_x g \star_v ((-\Delta_v)^{\alpha/2}\Phi)_{k,\theta}\|_{L_x^{q_1}L_v^{q_2(\gamma-1)/(\gamma-q_2)}}\,dt, \\ &\quad\lesssim 2^{kn(\frac{1}{p_1}+\frac{1}{q_1}-1)+k\theta n(\frac{1}{p_2}+\frac{1}{q_2}-\frac{\gamma-2}{\gamma-1})+k\alpha\theta+k\delta}\int_0^T \|f\|_{L_x^{p_1}L_v^{p_2}}\|g\|_{L_x^{q_1}L_v^{q_2}}\,dt\end{aligned}$$

for any $\delta > 0$. It is only necessary to have $|\log 2^k|$ when $q_1 = 1$ and $|\log 2^{k\theta}|$ when $q_2(\gamma-1)/(\gamma-q_2) = 1$, due to the weak boundedness of R and ∇G_1^n respectively.

- For the last term:

$$\int_0^T \iint \bar{h}\Big(\frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w k^2\Big)\, dw\, \chi_k(\xi)\, d\xi\, dt \lesssim \int_0^T \|S_k \star_x f \star_v \Phi_{k,\theta}\|_{L_x^2 L_v^{2(\gamma-1)/(\gamma-2)}} \|S_k \star_x \operatorname{Com} \star_v \Phi_{k,\theta}\|_{L_x^2 L_v^{2(\gamma-1)/(\gamma-2)}}\, dt.$$

Because Φ is compactly supported, $\Phi_{k,\theta}(v-w)$ forces $|v-w| \lesssim 2^{-k\theta}$. As a is Lipschitz, we have $|a(v)-a(w)| \lesssim 2^{-k\theta}$. So

$$\begin{aligned}
\|S_k \star_x \operatorname{Com} \star_v \Phi_{k,\theta}\|_{L_x^2 L_v^{2(\gamma-1)/(\gamma-2)}} &= \Big\| \int 2^k (a(v)-a(w))\cdot (f \star_x (\nabla_x S)_k)(w)\phi(w)\Phi_{k,\theta}(v-w)\, dw \Big\|_{L_x^2 L_v^{2(\gamma-1)/(\gamma-2)}} \\
&\lesssim 2^{k-k\theta} \| |f \star_x (\nabla_x S)_k| \star_v |\Phi_{k,\theta}| \|_{L_x^2 L_v^{2(\gamma-1)/(\gamma-2)}} \\
&\lesssim 2^{kn\theta(\frac{1}{p_2}-\frac{\gamma-2}{2(\gamma-1)})+kn(\frac{1}{p_1}-\frac{1}{2})+k(1-\theta)} \|f\|_{L_x^{p_1} L_v^{p_2}}.
\end{aligned}$$

Therefore,

$$\int_0^T \iint \bar{h}\Big(\frac{\xi}{|\xi|}\cdot \nabla G_1^n \star_w k^2\Big)\, dw\, \chi_k(\xi)\, d\xi\, dt \lesssim 2^{kn(\frac{2}{p_1}-1)+k\theta n(\frac{2}{p_2}-\frac{\gamma-2}{\gamma-1})+k(1-\theta)} \int_0^T \|f\|^2_{L_x^{p_1} L_v^{p_2}}\, dt.$$

Combining all estimates,

$$\begin{aligned}
|A_k| &\lesssim 2^{kn(\frac{2}{p_1}-1)+k\theta n(\frac{2}{p_2}-\frac{\gamma-2}{\gamma-1})} \|f\|^2_{L_x^{p_1} L_v^{p_2}}\Big|_{t=0}^{t=T} \\
&\quad + 2^{kn(\frac{1}{p_1}+\frac{1}{q_1}-1)+k\theta n(\frac{1}{p_2}+\frac{1}{q_2}-\frac{\gamma-2}{\gamma-1})+k\alpha\theta+k\delta} \int_0^T \|f\|_{L_x^{p_1} L_v^{p_2}} \|g\|_{L_x^{q_1} L_v^{q_2}}\, dt \\
&\quad + 2^{kn(\frac{2}{p_1}-1)+k\theta n(\frac{2}{p_2}-\frac{\gamma-2}{\gamma-1})+k(1-\theta)} \int_0^T \|f\|^2_{L_x^{p_1} L_v^{p_2}}\, dt. \qquad \square
\end{aligned}$$

4.1.3. *Proof of Lemma 18.* Choose two smooth functions ψ and $\tilde{\psi}$ such that $\psi(a(v)) \equiv 1$ and $\tilde{\psi}(v) \equiv 1$ on $v \in B(0,3)$. The function $\tilde{\psi}$ serves as an auxiliary function and can replace $\psi(a(v))$ since both their values are 1 on the support of ϕ. Recall that ϕ is the compact function used for localization in v. Then

$$\begin{aligned}
\int_0^T \int \Big| \int F_\theta(v)\psi(a(v))\, dv \Big|^2 |\xi|^s\, d\xi\, dt &= \int_0^T \int \Big| \int F_\theta(v)\tilde{\psi}(v)\, dv \Big|^2 |\xi|^s\, d\xi\, dt \\
&= \int_0^T \int \Big| \int \mathcal{F}_\zeta(\hat{f}\phi)(\zeta)\mathcal{F}_\zeta(\Phi_{\text{rescale}})(\zeta)\mathcal{F}_\zeta(\tilde{\psi})(\zeta)\, d\zeta \Big|^2 |\xi|^s\, d\xi\, dt \\
&= \int_0^T \int \Big| \int (\hat{f}\phi)(\Phi_{\text{rescale}} \star_v \tilde{\psi})\, dv \Big|^2 |\xi|^s\, d\xi\, dt.
\end{aligned}$$

Since $\tilde{\psi} \equiv 1$ on $B(0,3)$ and $\operatorname{supp}(\Phi) \subset B(0,1)$,

$$(\Phi_{\text{rescale}} \star_v \tilde{\psi})(v) = \int 2^{nk\theta}\Phi(w2^{k\theta})\tilde{\psi}(v-w)\, dw = \int 2^{nk\theta}\Phi(w2^{k\theta})\, dw = \|\Phi\|_{L^1} \quad \text{for all } \xi \text{ and } |v| \le 1.$$

Therefore,

$$\int_0^T \int \left| \int F_\theta(v)\psi(a(v))\,dv \right|^2 |\xi|^s \,d\xi\,dt = \|\Phi\|_{L^1}^2 \int_0^T \int \left| \int \hat{f}\phi\,dv \right|^2 |\xi|^s\,d\xi\,dt. \qquad \square$$

4.2. ***Proof of Theorem 13.*** This proof is essentially the same as Theorem 4, but with a different change of variable. After Step 1, instead of $v \mapsto w = a(v)$, we make $v \mapsto \lambda = a(v)\cdot(\xi/|\xi|)$ for each fixed ξ. Parallel to (19), we have

$$\varepsilon \partial_t h + i\lambda|\xi|h = k^1 + k^2 \tag{26}$$

in the sense of distribution, where h, k^1 and k^2 are defined as

$$\int F_\theta(v)\psi\left(a(v)\cdot\frac{\xi}{|\xi|}\right)dv = \int h(\lambda)\psi(\lambda)\,d\lambda.$$

$$\int k^1(\lambda)\psi(\lambda)\,d\lambda = \int [((-\Delta_v)^{\frac{\alpha}{2}}\hat{g}\phi) \star_v \Phi_{\text{rescale}}](v)\psi\left(a(v)\cdot\frac{\xi}{|\xi|}\right)dv,$$

$$\int k^2(\lambda)\psi(\lambda)\,d\lambda = \int \text{Com}(v)\psi\left(a(v)\cdot\frac{\xi}{|\xi|}\right)dv.$$

Owning to the nondegeneracy condition with $\nu = 1$, this change of variables preserves L^p norms:

Proposition 22. *Let a be Lipschitz and satisfy* (12) *with $\nu = 1$ and $\psi : \mathbb{R} \to \mathbb{R}$. Then for all $\sigma \in \mathbb{S}^{m-1}$ and $1 \le p \le \infty$,*

$$\|\psi(a(v)\cdot\sigma)\|_{L_v^p} \le c_0^{\frac{1}{p}}\|\psi\|_{L_\lambda^p},$$

and hence if $\int L(v)\psi(a(v)\cdot\sigma)\,dv = \int \ell^\sigma(\lambda)\psi(\lambda)\,d\lambda$,

$$\|\ell^\sigma\|_{L_\lambda^p} \lesssim \|L\|_{L_v^p}.$$

Proof of Proposition 22. When $p = \infty$, the result is straightforward. For $1 \le p < \infty$, (12) with $\nu = 1$ implies that for any interval I,

$$m(\{v \in B(0,1) : a(v)\cdot\sigma \in I\}) \le c_0 m(I).$$

By the standard approximation from intervals to measurable sets, we have that, for any measurable set A,

$$m(\{v \in B(0,1) : a(v)\cdot\sigma \in A\}) \le c_0 m(A).$$

Therefore,

$$\begin{aligned} d_{\psi(a(v)\cdot\sigma)}(s) &= m(\{v \in B : a(v)\cdot\sigma \in \{\lambda : |\psi(\lambda)| > s\}\}) \\ &\le c_0 m(\{\lambda : |\psi(\lambda)| > s\}) = c_0 d_\psi(s) \end{aligned}$$

and hence

$$\begin{aligned} \|\psi(a(v)\cdot\sigma)\|_{L_v^p} &= p^{\frac{1}{p}}\left(\int_0^\infty [d_{\psi(a(v)\cdot\sigma)}(s)^{\frac{1}{p}}s]^p\,\frac{ds}{s}\right)^{\frac{1}{p}} \\ &\le p^{\frac{1}{p}}\left(\int_0^\infty [c_0^{\frac{1}{p}}d_\psi(s)^{\frac{1}{p}}s]^p\,\frac{ds}{s}\right)^{\frac{1}{p}} = c_0^{\frac{1}{p}}\|\psi\|_{L_\lambda^p}. \end{aligned} \tag{27}$$

By the duality of L^p spaces,

$$\|\ell^\sigma\|_{L^p_\lambda} = \sup_{\|\psi\|_{L^{p'}_\lambda}=1} \left| \int \ell^\sigma \psi \, d\lambda \right| = \sup_{\|\psi\|_{L^{p'}_\lambda}=1} \left| \int L(v)\psi(a(v)\cdot\sigma)\, dv \right|$$
$$\le \sup_{\|\psi\|_{L^{p'}_\lambda}=1} \|L\|_{L^p_v} \|\psi(a(v)\cdot\sigma)\|_{L^{p'}_v} \le c_0^{\frac{1}{p}} \sup_{\|\psi\|_{L^{p'}_\lambda}=1} \|L\|_{L^p_v} \|\psi\|_{L^{p'}_\lambda} = c_0^{\frac{1}{p}} \|L\|_{L^p_v},$$

where the first inequality is by Hölder's inequality and the second by (27). This concludes our proof of Proposition 22. □

Let us come back to the proof of Theorem 13 by considering

$$\iint \bar{h}(\lambda) \frac{1}{i} (\partial_\lambda G_1^1)(\lambda - \alpha) h(\alpha) \, d\alpha \, d\lambda.$$

Procedures similar to those in Step 2 lead us to

$$\int_0^T \iiint |\xi|^{1/(\alpha+1)} \bar{h}(\lambda) G_3^1(\lambda-\alpha) h(\alpha) \, d\alpha \, d\lambda \, d\xi \, dt < \infty.$$

We can then conclude the proof of Theorem 13 from here by following the computation in Step 3. □

Appendix: Example for the nondegeneracy condition

We say $a(v) \in \mathrm{Lip}(\mathbb{R})$ satisfies (12) with $\nu \in (0,1]$ on intervals if

$$|\{v : a(v) \in I\}| \le C|I|^\nu \quad \text{for all intervals } I, \tag{28}$$

and $a(v)$ satisfies the nondegeneracy condition on open sets with $\nu \in (0,1]$,

$$|\{v : a(v) \in \mathcal{O}\}| \le C|\mathcal{O}|^\nu \quad \text{for all open set } \mathcal{O}. \tag{29}$$

Here we give an example to show (28) cannot imply (29) with the same ν when $\nu = \frac{1}{2}$. In fact the construction can be adapted to produce examples for all $\nu < 1$. Notice (28) and (29) are equivalent when $\nu = 1$.

Define $a : \left[0, \sum_{i=0}^\infty 1/3^i\right] \to \left[0, \sum_{i=0}^\infty 1/3^{2i}\right] \subset \mathbb{R}$ as

$$a(v) = a_1(v) = 1 - (1-v)^2 \qquad \text{on } [0,1] = D_1,$$
$$a(v) = a_2(v) = 1 + \frac{1}{3^2} a_1((v-1)3) \quad \text{on } \left[1, 1+\tfrac{1}{3}\right] = D_2,$$
$$\vdots$$

The general formula is

$$a(v) = a_n(v) = \sum_{i=0}^{n-2} \frac{1}{3^{2i}} + \frac{1}{3^{2(n-1)}} a_1\left(\left(v - \sum_{i=0}^{n-2} \frac{1}{3^i}\right) 3^{n-1}\right) \quad \text{on } \left[\sum_{i=0}^{n-2} \frac{1}{3^i}, \sum_{i=0}^{n-1} \frac{1}{3^i}\right] = D_n.$$

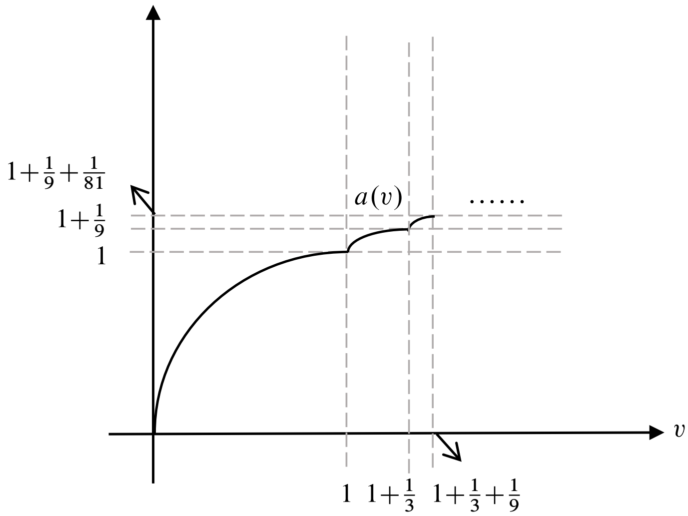

Figure 1. Graph of $a(v)$.

We shall prove that a satisfies condition (28) with $\nu = \frac{1}{2}$, but it fails (29) with the same ν.

Proposition 23. *There exists $C > 0$ such that, for any interval I,*

$$|a^{-1}(I)| = |\{v : a(v) \in I\}| \leq C|I|^{\frac{1}{2}}. \tag{30}$$

Proof. Consider an interval

$$I = \left[\sum_{i=0}^{n-1} \frac{1}{3^{2i}} - p_2, \sum_{i=0}^{n-1} \frac{1}{3^{2i}} - p_1\right] = [c, d]$$

inside some $a(D_n)$, where $0 \leq p_1 < p_2 \leq 1/3^{n-1}$. So $|I| = p_2 - p_1$. Denote the preimages of c and d by v_2 and v_1 respectively. Then we have, for each $k = 1, 2$,

$$a_n(v_k) = \sum_{i=0}^{n-2} \frac{1}{3^{2i}} + \frac{1}{3^{2(n-1)}} a_1\left(\left(v_k - \sum_{i=0}^{n-2} \frac{1}{3^i}\right) 3^{n-1}\right) = \sum_{i=0}^{n-1} \frac{1}{3^{2i}} - p_k.$$

So

$$a_n^{-1}\left(\sum_{i=0}^{n-1} \frac{1}{3^{2i}} - p_1\right) = v_k = \sum_{i=0}^{n-1} \frac{1}{3^i} - \sqrt{p_k}.$$

We therefore have

$$|a^{-1}(I)| = \sqrt{p_2} - \sqrt{p_1} \leq \sqrt{p_2 - p_1} = |I|^{\frac{1}{2}}.$$

If $I = [c, d] \subset a\left(\bigcup_{i=m_1}^{m_2} D_i\right)$, separate I into three subintervals: $I = I_1 \cup I_2 \cup I_3$, where

$$I_1 = \left[c, \sum_{i=0}^{m_1-1} \frac{1}{3^{2i}}\right], \quad I_2 = \left[\sum_{i=0}^{m_1-1} \frac{1}{3^{2i}}, \sum_{i=0}^{m_2-2} \frac{1}{3^{2i}}\right], \quad I_3 = \left[\sum_{i=0}^{m_2-2} \frac{1}{3^{2i}}, d\right].$$

The above case applies to I_1 and I_3, so $|a^{-1}(I_1)| \leq |I_1|^{1/2}$ and $|a^{-1}(I_3)| \leq |I_3|^{1/2}$.

For I_2, we have

$$|I_2| = \sum_{m_1}^{m_2-2} \frac{1}{3^{2i}} = \frac{9}{8}\frac{1}{3^{2m_1}}\left[1-\left(\frac{1}{9}\right)^{m_2-m_1-1}\right].$$

and

$$|a^{-1}(I_2)|^2 = \left(\sum_{m_1}^{m_2-2} \frac{1}{3^{i}}\right)^2 = \frac{9}{4}\frac{1}{3^{2m_1}}\left[1-\left(\frac{1}{3}\right)^{m_2-m_1-1}\right]^2$$
$$\leq \frac{9}{4}\frac{1}{3^{2m_1}}\left[1-2\left(\frac{1}{9}\right)^{m_2-m_1-1}+\left(\frac{1}{9}\right)^{m_2-m_1-1}\right] = 2|I_2|.$$

So

$$|a^{-1}(I_2)| \leq 2^{\frac{1}{2}}|I_2|^{\frac{1}{2}}.$$

Notice that this inequality is still true when m_2 goes to infinity, so there are no issues near the right end point.

Combining the three inequalities we get

$$|a^{-1}(I)| = \sum_{i=1}^{3}|a^{-1}(I_i)| \leq 2^{\frac{1}{2}}\sum_{i=1}^{3}|I_i|^{\frac{1}{2}} \leq 6^{\frac{1}{2}}\left(\sum_{i=1}^{3}|I_i|\right)^{\frac{1}{2}} = 6^{\frac{1}{2}}|I|^{\frac{1}{2}}. \qquad \square$$

Proposition 24. *There exists a sequence of set $\mathcal{O}^m$ such that*

$$\frac{|a^{-1}(\mathcal{O}^m)|}{|\mathcal{O}^m|^{\frac{1}{2}}} \to \infty \quad \textit{as } m \to \infty.$$

Proof. Let

$$\mathcal{O}^m = \bigcup_{n=1}^{m} I_n,$$

where

$$I_n = \left[\sum_{i=0}^{n-1}\frac{1}{3^{2i}} - \frac{1}{3^{2(m-1)}}, \sum_{i=0}^{n-1}\frac{1}{3^{2i}}\right] \quad \text{for all } 1 \leq n \leq m.$$

So

$$|I_n| = \frac{1}{3^{2(m-1)}} \quad \text{for all } 1 \leq n \leq m,$$

and

$$|a^{-1}(I_n)| = |I_n|^{\frac{1}{2}} = \frac{1}{3^{m-1}} \quad \text{for all } 1 \leq n \leq m.$$

Therefore,

$$\frac{|a^{-1}(\mathcal{O}^m)|}{|\mathcal{O}^m|^{\frac{1}{2}}} = \frac{m/3^{m-1}}{(m/3^{2(m-1)})^{\frac{1}{2}}} = \sqrt{m} \to \infty \quad \text{as } m \to \infty. \qquad \square$$

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