Optimality of the Lax-Wendroff Condition

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ABSTRACT

We show that any spectrally dominant vector norm on the vector space of matrices which is invariant under isometries dominates the numerical radius $r(\cdot)$. Thus, the celebrated Lax-Wendroff stability condition, $r(\cdot)\leqslant 1$, defines a maximal isometrically invariant stable set.

1. INTRODUCTION

The study of many iterative procedures involves the question of uniform power-boundedness of elements in a set $\mathbf{F}$ of $n\times n$ complex-valued matrices. That is, one is interested in the existence of a constant $K>0$ such that for every matrix $A\in\mathbf{F}$

$$|A^k|\leqslant K,\qquad k=1,2,\dots. \tag{1.1}$$

Such a set $\mathbf{F}$ is called a *stable set*. Here $|\cdot|$ denotes a *vector norm* on M_n—the algebra of $n\times n$ complex valued matrices. It is customary to call such a norm a generalized matrix norm, to distinguish $|\cdot|$ from a matrix norm, $\|\cdot\|$, which in addition to being a vector norm on M_n is also submultiplicative, i.e., $\|AB\|\leqslant\|A\|\|B\|$. We think, however, that the term "generalized matrix norm" is

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confusing and we prefer to call the norm $|\cdot|$ the vector norm on matrices, or simply the vector norm.

Since M_n is finite dimensional, all vector norms on M_n are equivalent, and hence the stability of **F** does not depend on the particular choice of a vector norm. We would like to study stable sets. A complete characterization of such sets was given by H. O. Kreiss in [3]; his conditions, however, are hard to verify in the general case and hence difficult to apply. One tries, therefore, simplify Kreiss's conditions by putting additional assumptions on the stable set **F**. A natural assumption which is fully justified in applications is

(i) **F** is a convex set.

Next, to simplify the problem in question, we shall suppose

(ii) $e^{i\theta}\mathbf{F}=\mathbf{F}$, i.e., $A\in\mathbf{F}$ iff $e^{i\theta}A\in\mathbf{F}$, for all real θ.
(iii) **F** contains an open neighborhood of the zero matrix.

Finally we note that the power-boundedness (1.1) holds for any A in the closure of **F**, so we may assume

(iv) **F** is a closed set.

We call a vector norm $|\cdot|$ a *stable norm* if its unit ball is a stable set. Assumptions (i)–(iv) imply, therefore, that **F** is the unit ball of some stable norm $|\cdot|$:

$$\mathbf{F}=\{A \mid |A|\leqslant 1\}. \tag{1.2}$$

So we may as well study stable norms. It easily follows that every A in a stable set **F** satisfies the Neumann condition

$$\rho(A)\leqslant 1, \tag{1.3}$$

$\rho(A)$ denoting the *spectral radius* of A. Hence, taking **F** to be in particular the unit ball of a stable norm $|\cdot|$, we find for such a norm

$$\rho(A)\leqslant |A| \qquad \text{for all} \quad A\in M_n. \tag{1.4}$$

A vector norm $|\cdot|$ satisfying the above inequality is called *spectrally dominant*, and we conclude that spectral dominance is necessary for a vector norm to be stable. Thus we arrived at the following

PROBLEM. Which spectrally dominant vector norms on M_n are stable?

Let $(\cdot,\cdot)$ be a given inner product on $\mathbf{C}^n$, the space of n-column complex vectors. Also, any $A \in M_n$ defines an operator $A:\mathbf{C}^n \to \mathbf{C}^n$ in the obvious way. The numerical radius of A is then defined by

$$r(A) = \max_x \left| \frac{(Ax,x)}{(x,x)} \right|. \tag{1.5}$$

It is well known that $r(\cdot)$ is a vector norm. In 1964 Lax and Wendroff [4] showed for the Euclidian inner product $(x,y) = y^*x$ that $r(\cdot)$ is a stable norm, i.e., that the set

$$\{A | r(A) \leqslant 1\} \tag{1.6}$$

is a stable one; their proof proceeds by induction on the dimension n. In fact, the numerical radius $r(\cdot)$ induced by a general inner product—necessarily of the form $(x,y) = y^*Hx$, $H = T^*T > 0$— is a stable norm as well [6]. Indeed, its unit ball is similar to the set (1.6), with T the similarity transformation. The aim of this paper, is to show that the Lax-Wendroff condition, $r(\cdot) \leqslant 1$, is optimal in the following sense.

Main Theorem. *Let $F \subseteq M_n$ be a stable set satisfying assumptions* (i)–(iv). *Assume furthermore that* $\mathbf{F}$ *is invariant under similarity by isometries; that is,*

$$UFU^{-1} = \mathbf{F} \qquad \textit{for all } U \textit{ such that} \quad (Ux,Ux) = (x,x). \tag{1.7}$$

Then $\mathbf{F}$ *is contained in the set* (1.6).

The above result implies

Corollary. *Any spectrally dominant vector norm which is invariant under similarity by isometries is stable.*

We close this section with an interesting conjecture of C. Johnson [2] (which is stated there in an equivalent form).

Conjecture. Any spectrally dominant norm is stable.

2. INVARIANT NORMS

LEMMA 1. *Assume that the vector norm* $|\cdot|$ *is invariant under the similarity by a matrix* U,

$$|UAU^{-1}| = |A| \qquad \textit{for all} \quad A \in M_n. \tag{2.1}$$

Then U *is similar to a diagonal matrix* Λ,

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \qquad |\lambda_i| = |\lambda_j|, \quad 1 \leqslant i, j \leqslant n. \tag{2.2}$$

Proof. Suppose first

$$Ux = \lambda_i x, \quad U^t y = \lambda_j y, \qquad x, y \neq 0,$$

where U^t is the transpose of U. For $A = xy^t$ we then have

$$UAU^{-1} = \lambda_i \lambda_j^{-1} A,$$

so $|\lambda_i| = |\lambda_j|$ by (2.1). It is left to show that U is similar to diagonal matrix. Assume to the contrary that U is similar to an upper triangular matrix $V = (v_{ij})_1^n$:

$$U = TVT^{-1} \qquad \text{with} \quad v_{11} = v_{22} = \lambda, \quad v_{12} = 1.$$

Noting that $\lambda \neq 0$, we choose

$$A = TBT^{-1},$$

where the only nonzero entry of $B = (b_{ij})_1^n$ is $b_{22} = \lambda^{-2}$. A straightforward calculation shows that the (1,2) entry of $T^{-1}(U^k A U^{-k})T$ is k^2. The matrices $U^k A U^{-k}$ are therefore not uniformly bounded. On the other hand, (2.1) yields

$$|U^k A U^{-k}| = |A|$$

and in particular, the matrices $U^k A U^{-k}$ are uniformly bounded. The above contradiction establishes the lemma. ∎

By Lemma 1, the study of invariant norms is reduced to invariance under diagonal matrices of the form (2.2). We continue by considering invariance under such type of similarities.

For each $A=(a_{ij})$ let $\text{diag}(A)$ denote the diagonal matrix $\text{diag}(a_{11},\ldots,a_{nn})$. We have

LEMMA 2. *Let $|\cdot|$ be a vector norm on M_n, invariant under the similarity by a diagonal matrix $\Lambda=\text{diag}(\lambda_1,\ldots,\lambda_n)$, $|\lambda_i|=|\lambda_j|$, $1\leqslant i,j\leqslant n$. Assume that $\lambda_i\neq\lambda_j$ for $i\neq j$. Then*

$$|\text{diag}(A)|\leqslant|A|. \tag{2.3}$$

Proof. From the Λ-invariance it follows that

$$\left|\frac{1}{m+1}\sum_{k=0}^{m}\Lambda^k A\Lambda^{-k}\right|\leqslant\frac{1}{m+1}\sum_{k=0}^{m}|\Lambda^k A\Lambda^{-k}|=|A|.$$

Let A_m denote the matrix on the left:

$$A_m=\left(a_{ij}^{(m)}\right)=\frac{1}{m+1}\sum_{k=0}^{m}\Lambda^k A\Lambda^{-k}.$$

We have

$$a_{ii}^{(m)}=a_{ii},\qquad a_{ij}^{(m)}=a_{ij}\frac{1-(\lambda_i/\lambda_j)^{m+1}}{(m+1)(1-\lambda_i/\lambda_j)}\quad\text{for } i\neq j.$$

Let $m\to\infty$ and obtain (2.3).

Let $(\cdot,\cdot)$ be an inner product on C^n. Then this inner product induces a natural norm on C^n – $\|x\|=(x,x)^{1/2}$. A transformation $U\colon C^n\to C^n$ is called an isometry if $\|Ux\|=\|x\|$.

THEOREM 1. *Let $|\cdot|$ be a vector norm on M_n invariant under similarity by isometries; that is,*

$$|UAU^{-1}|=|A|\qquad \textit{for all } U \textit{ such that}\quad (Ux,Ux)=(x,x). \tag{2.4}$$

Then $|\cdot|$ is spectrally dominant if and only if $|\cdot|$ dominates the numerical radius (1.5), *viz.*

$$r(A)\leqslant|A|. \tag{2.5}$$

Proof. The numerical radius is a stable norm; hence the sufficiency of (2.5) is obvious, since

$$\rho(A) \leqslant r(A). \tag{2.6}$$

We turn now to prove the necessity of (2.5). We consider first the Euclidean inner product $(x, y) = y^*x$; the corresponding numerical radius (1.5) is denoted by $r_I(A)$, and the isometries in this case are unitary matrices. They include in particular any diagonal matrix whose spectrum lies on the unit circle. So we may apply Lemma 2, yielding (2.3).

For $A = (a_{ij})_1^n$ we have

$$\rho(\mathrm{diag}(A)) = \max_{1 \leqslant i \leqslant n} |a_{ii}|.$$

The assumption that $|\cdot|$ is spectrally dominant therefore implies

$$\max_{1 \leqslant i \leqslant n} |a_{ii}| = \rho(\mathrm{diag}(A)) \leqslant |\mathrm{diag}(A)|, \tag{2.7}$$

which combined with (2.3) gives us

$$\max_{1 \leqslant i \leqslant n} |a_{ii}| \leqslant |A|. \tag{2.8}$$

Since $\rho(\cdot)$ and $|\cdot|$ are both unitarily invariant, it follows that (2.8) holds for any matrix unitarily similar to A. Let V be a unitary matrix with first column x, $x^*x = 1$. Employing (2.8) for $V^*AV = (\alpha_{ij})$, we get

$$|x^*Ax| = |\alpha_{11}| \leqslant |V^*AV| = |A|.$$

Since for any x, $x^*x = 1$, we can find a unitary V whose first column is x, we conclude

$$r_I(A) \equiv \max_{x^*x = 1} |x^*Ax| \leqslant |A|. \tag{2.9}$$

Consider now a general inner product $(\cdot, \cdot)$; it is necessarily of the form

$$(x, y) = y^*Hx \tag{2.10}$$

with a positive Hermitian H. For the corresponding numerical radius, $r(\cdot) \equiv$

$r_H(\cdot)$, it is straightforward to show

$$r_H(A) = r_I(H^{1/2}AH^{-1/2}). \tag{2.11}$$

Let $|\cdot|_H$ be another norm on M_n given by

$$|A|_H = |H^{-1/2}AH^{1/2}|. \tag{2.12}$$

Since the isometries in this case consist of matrices U such that $H^{1/2}UH^{-1/2}$ is unitary, we find on account of (2.4) that the new norm $|\cdot|_H$ is unitarily invariant; by (2.9) therefore

$$r_I(B) \leqslant |B|_H.$$

This inequality is equivalent to (2.5) in view of (2.11)–(2.12). The proof of the theorem is completed. ■

We note that in the course of proving necessity in the last theorem, we used only the spectral dominance for diagonal matrices. When combined with the unitary invariance, however, this is equivalent to the spectral dominance (1.4), as easily seen by considering the Schur triangular form.

Proof of Main Theorem. Our assumptions imply that **F** is the unit ball of some vector norm $|\cdot|$, which necessarily satisfies (2.4). Since **F** is stable, the norm $|\cdot|$ is spectrally dominant. By Theorem 1, it dominates the numerical radius as well; whence $r(A) \leqslant |A| \leqslant 1$ and **F** is contained in the set (1.6).

3. AN OPEN PROBLEM

Let $|\cdot|$ be a unitarily invariant vector norm on M_n. For $x \in \mathbf{C}^n$ denote

$$D(x) = \operatorname{diag}(x_1, \dots, x_n), \tag{3.1}$$

and then define a vector norm $\|\cdot\|$ on $\mathbf{C}^n$,

$$\|x\| = |D(x)|. \tag{3.2}$$

Since $|\cdot|$ is unitarily invariant, it follows that

$$\|Px\| = \|x\| \tag{3.3}$$

for all permutation matrices P; that is, $\|x\|$ is a symmetric norm. Vice versa, if $\|\cdot\|$ is a norm on $\mathbf{C}^n$, we can define a unitarily invariant vector norm $|\cdot|$ on M_n, as follows:

$$|A| = \max_{U \in \mathbf{U}} \left|\operatorname{diag}(UAU^{-1})\right|. \tag{3.4}$$

Here $\mathbf{U}$ stands for the set of all unitary matrices, and the norm on the right is the norm of the n-tuple diagonal entries viewed as a vector in $\mathbf{C}^n$. The norm $|\cdot|$ is a minimal invariant in view of (2.3).

Clearly $|\cdot|$ is spectrally dominant iff

$$\|x\| \geqslant \max_{1 \leqslant i \leqslant n} |x_i|. \tag{3.5}$$

In particular, let $\|x\|_p$ be the Holder norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \tag{3.6}$$

and denote by $|\cdot|_p$ the corresponding invariant norm given by (3.4). Since $|\cdot|_p$ is stable, we have the inequality

$$|A^k|_p \leqslant K_{p,n} |A|_p^k, \qquad k = 1,2,\dots. \tag{3.7}$$

For $p = \infty$ equality holds in (3.5), we have $|A|_\infty = r_I(A)$, and hence

$$K_{\infty,n} = 1, \tag{3.8}$$

as

$$r_I(A^k) \leqslant r_I^k(A). \tag{3.9}$$

This was the Halmos conjecture [1]. See [5] for a short proof.

PROBLEM. For which values of p, $1 \leqslant p < \infty$, are the constants $K_{p,n}$ uniformly bounded in n?

Added in proof. The Johnson conjecture is verified in a recent paper, All spectral dominant norms are stable, by S. Friedland and C. Zenger.

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