On the Numerical Radius and Its Applications*

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ABSTRACT

We give a brief account of the numerical radius of a linear bounded operator on a Hilbert space and some of its better-known properties. Both finite- and infinite-dimensional aspects are discussed, as well as applications to stability theory of finite-difference approximations for hyperbolic initial-value problems.

1. DEFINITION, BOUNDS, AND EVALUATION

Let **H** be a Hilbert space over the complex field **C** with inner product $(\mathbf{x},\mathbf{y})$ and norm $\|\mathbf{x}\| = (\mathbf{x},\mathbf{x})^{1/2}$. Let $\mathfrak{B}(\mathbf{H})$ denote the algebra of bounded linear operators on **H**. Then, for any A in $\mathfrak{B}(\mathbf{H})$ we define the *numerical radius* of

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A,

$$r(A) \equiv \sup\{|(Ax, x)| : \|\mathbf{x}\| = 1\}.$$

Three of the most obvious properties of r are:

$$r(A) \geq 0; \tag{1.1}$$

$$r(\alpha A) = |\alpha| r(A) \qquad \forall \alpha \in \mathbb{C}; \tag{1.2}$$

$$r(A+B) \leq r(A) + r(B) \qquad \forall A, B \in \mathfrak{B}(\mathbf{H}). \tag{1.3}$$

In this section we discuss further bounds for r as well as ways to evaluate it.

THEOREM 1.1 (e.g. [11], [18]). *Let $\rho(A)$ be the spectral radius of A. Then*

$$\rho(A) \leq r(A). \tag{1.4}$$

Proof. Denoting the spectrum of A by $\Lambda(A)$ and its *numerical range* by

$$W(A) \equiv \{(A\mathbf{x}, \mathbf{x}) : \|\mathbf{x}\| = 1\},$$

we shall show that

$$\Lambda(A) \subseteq \overline{W(A)}.$$

Since

$$r(A) = \sup\{|z| : z \in W(A)\}, \tag{1.5}$$

this will imply the desired result.

In the finite-dimensional case the proof is simple: if λ is an eigenvalue of A with a corresponding unit eigenvector $\mathbf{x}$, then

$$\lambda = (\lambda\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, \mathbf{x}) \in W(A),$$

so

$$\Lambda(A) \subseteq W(A).$$

In the infinite-dimensional case we have

$$\Lambda(A)=\Pi(A)\cup\Gamma(A),$$

where $\Pi(A)$ and $\Gamma(A)$ are the approximate point spectrum and the compression spectrum, respectively. If $\lambda\in\Pi(A)$ (i.e., λ is an eigenvalue or generalized eigenvalue of A), then there exists a sequence of unit vectors $\{\mathbf{x}_j\}$ such that $(A-\lambda)\mathbf{x}_j\to 0$, $j\to\infty$; hence

$$|(A\mathbf{x}_j,\mathbf{x}_j)-\lambda|=|((A-\lambda)\mathbf{x}_j,\mathbf{x}_j)|\leqslant\|(A-\lambda)\mathbf{x}_j\|\to 0,$$

which implies

$$\Pi(A)\subseteq\overline{W(A)}\,.$$

On the other hand, if $\lambda\in\Gamma(A)$, then $\bar{\lambda}$ is an eigenvalue of the adjoint operator A^*, and as shown in the finite-dimensional case, $\bar{\lambda}\in W(A^*)$. Thus, $\lambda\in W(A)$ and the proof is complete. ■

THEOREM 1.2 (e.g. [10]). *Let*

$$\|A\|=\sup\{\|A\mathbf{x}\|:\|\mathbf{x}\|=1\}$$

be the operator norm induced on $\mathfrak{B}(\mathbf{H})$. *Then*

$$\tfrac{1}{2}\|A\|\leqslant r(A)\leqslant\|A\|. \tag{1.6}$$

Proof. The right-hand side is trivial:

$$r(A)=\sup_{\|\mathbf{x}\|=1}|(A\mathbf{x},\mathbf{x})|\leqslant\sup_{\|\mathbf{x}\|=1}\|A\mathbf{x}\|\cdot\|\mathbf{x}\|=\|A\|.$$

For the left inequality we use the well-known polarization identity

$$4(A\mathbf{x},\mathbf{y})=(A(\mathbf{x}+\mathbf{y}),\mathbf{x}+\mathbf{y})-(A(\mathbf{x}-\mathbf{y}),\mathbf{x}-\mathbf{y})$$
$$+i(A(\mathbf{x}+i\mathbf{y}),\mathbf{x}+i\mathbf{y})-i(A(\mathbf{x}-i\mathbf{y}),\mathbf{x}-i\mathbf{y})$$

and the parallelogram law

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2=2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2,$$

which are easily obtained by expanding their right- and left-hand sides, respectively. We find that

$$4|(A\mathbf{x},\mathbf{y})|\leq r(A)\cdot\left(\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2+\|\mathbf{x}+i\mathbf{y}\|^2+\|\mathbf{x}-i\mathbf{y}\|^2\right)$$
$$\leq 4r(A)\cdot\left(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\right);$$

hence

$$\sup\left\{|(A\mathbf{x},\mathbf{y})|:\|\mathbf{x}\|=\|\mathbf{y}\|=1\right\}\leq 2r(A). \tag{1.7}$$

In particular, if y is a unit vector in the direction of $A\mathbf{x}$, then $|(A\mathbf{x},\mathbf{y})|=\|A\mathbf{x}\|$, and substituting in (1.7), the theorem follows. ■

By (1.4) and (1.6) we have

$$\rho(A)\leq r(A)\leq\|A\| \qquad \forall A\in\mathfrak{B}(\mathbf{H}). \tag{1.8}$$

Since if A is normal then $\rho(A)=\|A\|$ (e.g. [20]), we conclude that

$$\rho(A)=r(A)=\|A\| \qquad \textit{for normal } A. \tag{1.9}$$

Following Halmos [11], we say that A is *spectral* if

$$r(A)=\rho(A).$$

By (1.9), therefore, normal operators are spectral. The converse, however, is false, as shown in Example 2.1 below; the class of normal operators is a proper subclass of the spectral ones.

Characterizations of spectral operators were given by Furuta and Takeda [4]; Goldberg, Tadmor, and Zwas [8]; and Goldberg [5]. Wintner [22] (see also [11]) has shown that operators satisfying $r(A)=\|A\|$ are spectral as well.

Having (1.9), we should perhaps mention a shorter proof of the left-hand side of (1.6): We write A as a sum of normal operators,

$$A=\tfrac{1}{2}(A+A^*)+\tfrac{1}{2}(A-A^*),$$

and observe that

$$r(A)=r(A^*);$$

so by (1.3)

$$\|A\|\leq\tfrac{1}{2}\|A+A^*\|+\tfrac{1}{2}\|A-A^*\|=\tfrac{1}{2}r(A+A^*)+\tfrac{1}{2}r(A-A^*)$$
$$\leq r(A)+r(A^*)=2r(A),$$

which gives the desired result.

Two additional results, often useful in evaluating $r(A)$, are given in the following theorem.

THEOREM 1.3.

(i) *The numerical radius is invariant under unitary similarities, i.e., for any unitary operator* U,

$$r(U^*AU)=r(A). \tag{1.10}$$

(ii) *If* $\mathbf{H}=\mathbf{H}_1\oplus\mathbf{H}_2$ *is a direct sum and* $A_1\in\mathfrak{B}(\mathbf{H}_1)$, $A_2\in\mathfrak{B}(\mathbf{H}_2)$, *then*

$$r(A_1\oplus A_2)=\max\{r(A_1),r(A_2)\}. \tag{1.11}$$

Proof. For (i) we have

$$r(U^*AU)=\sup_{\|\mathbf{x}\|=1}|(U^*AU\mathbf{x},\mathbf{x})|=\sup_{\|\mathbf{x}\|=1}|(AU\mathbf{x},U\mathbf{x})|$$
$$=\sup_{\|\mathbf{y}\|=1}|(A\mathbf{y},\mathbf{y})|=r(A).$$

For (ii), we write $A=A_1\oplus A_2$ and let $W(A_j)$, $j=1,2$, be the numerical range of A_j. Evidently, $W(A_j)\subseteq W(A)$; thus

$$\operatorname{conv}\{W(A_1),W(A_2)\}\subseteq W(A) \qquad \text{(conv for convex hull)}. \tag{1.12}$$

Conversely, for any unit vector $\mathbf{x}$ we write $\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2$, where $\mathbf{x}_1\in\mathbf{H}_1$, $\mathbf{x}_2\in\mathbf{H}_2$, and $\|\mathbf{x}_1\|^2+\|\mathbf{x}_2\|^2=\|\mathbf{x}\|^2=1$. Setting unit vectors $\mathbf{y}_1=\mathbf{x}_1/\|\mathbf{x}_1\|\in$

$\mathbf{H}_1$, $y_2=x_2/\|x_2\|\in\mathbf{H}_2$, we have

$$\begin{aligned}(Ax,x)&=(A(x_1+x_2),x_1+x_2)=(Ax_1,x_1)+(Ax_2,x_2)\\&=\|x_1\|^2(A_1y_1,y_1)+\|x_2\|^2(A_2y_2,y_2)\in\operatorname{conv}\{W(A_1),(W(A_2)\},\end{aligned}$$

so

$$W(A)=\operatorname{conv}\{W(A_1),W(A_2)\}. \tag{1.13}$$

By (1.12), (1.13), therefore,

$$W(A_1\oplus A_2)=\operatorname{conv}\{W(A_1),W(A_2)\},$$

and (1.5) implies (1.11). ■

2. THE FINITE-DIMENSIONAL CASE

We come now to discuss concrete computations of $r(A)$ in the finite-dimensional case where we may of course restrict attention to $\mathbf{H}=\mathbf{C}^n$, the space of complex n-tuples with some inner product, and to $\mathfrak{B}(\mathbf{H})=\mathbf{C}_{n\times n}$, the algebra of complex $n\times n$ matrices.

Indeed, let (x,y) be an inner product on $\mathbf{C}^n$; let $e_1,\dots,e_n$ be the standard basis of $\mathbf{C}^n$; and set $p_{ij}=(e_j,e_i)$, $1\leqslant i$, $j\leqslant n$. Then the matrix $P=(p_{ij})$ is Hermitian, and for any two vectors

$$x=(x_1,\dots,x_n)',\qquad y=(y_1,\dots,y_n)'\in\mathbf{C}^n$$

(prime denoting the transpose), we have

$$(x,y)=\left(\sum_{j=1}^n x_je_j,\ \sum_{i=1}^n y_ie_i\right)=\sum_{i,j}x_j\bar{y}_i(e_j,e_i)=\sum_{i,j}\bar{y}_ip_{ij}x_j.$$

Since $(x,x)>0$ for $x\neq 0$, we obtain the known result that (x,y) is an inner product on $\mathbf{C}^n$ if and only if it is of the form

$$(x,y)\equiv(x,y)_P=y^*Px \tag{2.1}$$

where P is (Hermitian) positive definite and

$$y^* = (\bar{y}_1, \dots, \bar{y}_n)$$

is the conjugate transpose of y.

Temporarily denoting the vector norm and the numerical radius of A, corresponding to the inner product in (2.1), by $\|x\|_P$ and $r_P(A)$, we find that

$$\begin{aligned} r_P(A) &= \sup\left\{|(Ax,x)_P| : \|x\|_P = 1\right\} = \sup\left\{|(PAx,x)_I| : (Px,x)_I = 1\right\} \\ &= \sup\left\{|(P^{1/2}AP^{-1/2}P^{1/2}x,\ P^{1/2}x)_I| : (P^{1/2}x,\ P^{1/2}x)_I = 1\right\} \\ &= \sup\left\{|(P^{1/2}AP^{-1/2}y,y)_I| : \|y\|_I = 1\right\} = r_I(P^{1/2}AP^{-1/2}). \end{aligned}$$

So we see that in the finite-dimensional case it suffices to study the *standard* numerical radius

$$r_I(A) = \sup\left\{|(Ax,x)_I : x \in \mathbf{C}^n,\ \|x\|_I = 1\right\},$$

which by compactness of the unit sphere in $\mathbf{C}^n$ is usually written as

$$r_I(A) = \max\left\{|(Ax,x)_I| : x \in \mathbf{C}^n,\ \|x\|_I = 1\right\}.$$

With this observation in mind we write from now on (x,y), $\|x\|$, and $r(A)$ instead of $(x,y)_I$, $\|x\|_I$, and $r_I(A)$, whenever $x,y \in \mathbf{C}^n$ and $A \in \mathbf{C}_{n \times n}$.

An almost trivial result in the finite-dimensional case is that if M is a principal submatrix of A, then

$$r(M) \leqslant r(A). \tag{2.2}$$

Also, if for $A = (a_{ij})$ we use the notation $A^+ = (|a_{ij}|)$, then

$$r(A) \leqslant r(A^+). \tag{2.3}$$

These two results (an easy exercise), as well as the previous ones, do not relieve us, however, from the need to actually compute $r(A)$ at least for simple matrices. Since most examples in the literature can be written in terms of *positive* matrices, i.e., matrices with nonnegative entries, the following theorem [9] provides a useful tool that often answers our needs.

THEOREM 2.1.

(i) *Let A be a positive matrix. Then*

$$r(A)=\rho(\operatorname{Re}A) \qquad \left[\operatorname{Re}A\equiv\tfrac{1}{2}(A+A')\right].$$

(ii) *Consider the real symmetric matrix*

$$S(\sigma)=\sigma I-\operatorname{Re}A, \qquad \sigma\in\mathbf{R}.$$

If

$$D=\operatorname{diag}(a_1,\dots,a_n)$$

is a diagonal matrix congruent to $S(\sigma)$, *then* $r(A)=\sigma$ *if and only if the* a_i *are nonnegative and at least one of them does not vanish.*

Proof. Since

$$r(A)=\max\{|(A\mathbf{x},\mathbf{x})|:\mathbf{x}\in\mathbf{C}^n,\ (\mathbf{x},\mathbf{x})=1\},$$

then there exists a unit vector $\mathbf{x}_0=(x_1,\dots,x_n)'\in\mathbf{C}^n$ such that $r(A)=|(A\mathbf{x}_0,\mathbf{x}_0)|$. Since A is a positive matrix and $\mathbf{y}=(|x_1|,\dots,|x_n|)'$ is a positive unit vector, it follows that

$$r(A)=|(A\mathbf{x}_0,\mathbf{x}_0)|\leqslant(A\mathbf{y}_0,\mathbf{y}_0)\leqslant r(A),$$

so

$$r(A)=\max\{(A\mathbf{x},\mathbf{x}):\mathbf{x}\in\mathbf{R}^n,\ (\mathbf{x},\mathbf{x})=1\}.$$

Similarly, since $\operatorname{Re}A$ is positive, then

$$r(\operatorname{Re}A)=\max\{(\operatorname{Re}A\mathbf{x},\mathbf{x}):\mathbf{x}\in\mathbf{R}^n,\ (\mathbf{x},\mathbf{x})=1\}.$$

Now, since it is not hard to see that $(\operatorname{Re}A\mathbf{x},\mathbf{x})=(A\mathbf{x},\mathbf{x})$ for all $\mathbf{x}\in\mathbf{R}^n$, then $r(A)=r(\operatorname{Re}A)$, and (1.9) implies (i).

For part (ii) we mention again that

$$r(A)=\max\{(\operatorname{Re}A\mathbf{x},\mathbf{x}):\mathbf{x}\in\mathbf{R}^n,(\mathbf{x},\mathbf{x})=1\}.$$

Hence, $r(A) = \sigma$ for some $\sigma \geqslant 0$, if and only if $(\mathrm{Re}\,A\mathbf{x},\mathbf{x}) \leqslant \sigma(\mathbf{x},\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$ with equality for some $\mathbf{x}_0 \in \mathbf{R}^n$. That is, $r(A) = \sigma$ only if the real symmetric matrix $S(\sigma) = \sigma I - \mathrm{Re}\,A$ is positive semidefinite but not positive definite, or in other words, if the eigenvalues of $S(\sigma)$ are nonnegative and at least one of them vanishes. Thus, part (ii) is now an immediate consequence of Sylvester's law of inertia, and the proof is complete. ■

EXAMPLE 2.1. Often in examples we encounter 2×2 positive matrices of the form

$$A = \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \qquad \left[\text{or} \quad A = \begin{pmatrix} a_1 & 0 \\ b & a_2 \end{pmatrix}\right]. \tag{2.4a}$$

By Theorem 2.1(i), therefore,

$$r(A) = \rho(\mathrm{Re}\,A) = \tfrac{1}{2}(a_1 + a_2) + \tfrac{1}{2}\sqrt{(a_1 - a_2)^2 + b^2}\,. \tag{2.4b}$$

This result also follows from the known fact that if a_1, a_2, and b in (2.4a) are any complex numbers, then the numerical range, $W(A)$, is the (possibly degenerate) elliptic disc $\mathfrak{E}(a_1, a_2, |b|)$ with foci at a_1, a_2 and minor axis $|b|$. As Halmos puts it, however, the proof of this assertion (e.g. Murnaghan [15], Donoghue [3]) is analytic geometry at its worst; hence our direct computation of $r(A)$ in (2.4b) is indeed much shorter.

We can now easily answer a question raised in Section 1: Using (1.11) and (2.4), we find that the $n \times n$ nonnormal matrix

$$A = I_{n-2} \oplus \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \qquad (n \geqslant 3)$$

satisfies $r(A) = \rho(A) = 1$, showing how far from normal a spectral operator can be.

We note that since any matrix is unitarily similar to a triangular matrix and since both the numerical radius and the spectrum are invariant under unitary similarities, then it is easy to see that a 2×2 matrix is spectral if and only if it is normal.

EXAMPLE 2.2. Consider the nilpotent matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and let us compute $r(A^m)$, $m=1,2,3$. Starting with the easiest case, we interchange the first and third rows and columns of A^3 to find that A^3 is unitarily similar to the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

hence by (1.11) and (2.4),

$$r(A^3)=\tfrac{1}{2}.$$

Similarly, interchanging the second and third rows and columns of A^2, we obtain

$$r(A^2)=\tfrac{1}{2}.$$

To find $r(A)$, we may employ Theorem 2.1(i) to verify that

$$r(A)=\rho(\operatorname{Re} A)=\left(1+\sqrt{5}\right)/4.$$

This value of $r(A)$ may be conveniently obtained by alternately operating on the columns and rows of the matrix $S=\sigma I-\operatorname{Re} A$ with elementary operations, to eliminate its off-diagonal entries $s_{1,2}, s_{2,1}, s_{2,3}, s_{3,2}, s_{3,4}, s_{4,3}$ (in that order) and to find that S is congruent to

$$D=\operatorname{diag}(a_1, a_2, a_3, a_4)$$

with

$$a_1=\sigma, \qquad a_j=\sigma-\tfrac{1}{4}a_{j-1}^{-1}, \quad j=2,3,4.$$

Hence $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4=0$ if and only if $\sigma=(1+\sqrt{5})/4$, and Theorem 2.1(ii) yields the above result.

3. SOME NORM PROPERTIES

As usual, we call a mapping $A \to N(A)$ a *seminorm* on $\mathfrak{B}(\mathbf{H})$ if for all $A, B \in \mathfrak{B}(\mathbf{H})$ and $\alpha \in \mathbf{C}$,

$$N(A) \geqslant 0, \tag{3.1a}$$

$$N(\alpha A)=|\alpha| N(A), \tag{3.1b}$$

$$N(A+B) \leqslant N(A)+N(B). \tag{3.1c}$$

If in addition N satisfies

$$N(A)>0 \qquad \text{for} \quad A\neq 0, \tag{3.1d}$$

then N is a *norm* on $\mathfrak{B}(\mathbf{H})$, which may or may not be related to the given operator norm.

Having the above definitions, the relations (1.1)–(1.3) imply that r is a seminorm on $\mathfrak{B}(\mathbf{H})$. In fact we can easily show more:

THEOREM 3.1. *The numerical radius is a norm on* $\mathfrak{B}(\mathbf{H})$.

Proof. We have to check (3.1d), or alternatively to show that $r(A)=0$ implies $A=0$. But then, by (1.6), $\|A\|\leqslant 2r(A)=0$; so $\|A\|=0$ and our claim follows. ■

Theorems 1.2 and 3.1 indicate that r and the operator norm on $\mathfrak{B}(\mathbf{H})$ are certainly related. There is, however, one important feature, namely (*sub-*) *multiplicativity*, which separates the two. More precisely, while

$$\|AB\|\leqslant\|A\|\cdot\|B\| \qquad \forall A,B\in\mathfrak{B}(\mathbf{H}),$$

the inequality

$$r(AB)\leqslant r(A)r(B) \tag{3.2}$$

is disappointingly false. To demonstrate this, take the matrices,

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad B=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

By (2.4), $r(A)=r(B)=\frac{1}{2}$, $r(AB)=1$, and so much for multiplicativity. In fact, Brown and Shields [16] have considered the 4×4 matrix in Example 2.2, for which

$$r(A^3)=\tfrac{1}{2}<\left(1+\sqrt{5}\right)/8=r(A)r(A^2).$$

So in general (2.2) may fail even when A and B commute, or worse yet, when A and B are powers of the same operator.

What *is* true with respect to multiplicativity, however, is the following remarkable result, conjectured by Halmos and first proved by Berger [1].

THEOREM 3.2. *For any* $A \in \mathfrak{B}(\mathbf{H})$,

$$r(A^m) \leqslant r^m(A), \qquad m=1,2,3,\dots; \tag{3.3}$$

or equivalently,

$$r(A) \leqslant 1 \quad \textit{implies} \quad r(A^m) \leqslant 1, \quad m=1,2,3,\dots . \tag{3.4}$$

Proof. Clearly, (3.3) implies (3.4). Conversely, suppose (3.4) holds. If $A=0$ then there is nothing to prove, so assume $A \neq 0$, and consider the operator $B=A/r(A)$. By (1.2), $r(B)=1$; hence, $r(B^m) \leqslant 1$. It follows that $r(A^m/r^m(A)) \leqslant 1$, and by (1.2) again, we obtain (3.3). Consequently, (3.3) and (3.4) are equivalent and it suffices to prove (3.4).

For this purpose let m be a positive integer; let

$$\omega_j = e^{2\pi i j/m}, \qquad j=1,\dots,m,$$

be the mth roots of unity, and consider the polynomial identities

$$1-z^m = \prod_{k=1}^{m} (1-\omega_k z), \tag{3.5}$$

$$1 = \frac{1}{m} \sum_{j=1}^{m} \prod_{\substack{k=1 \\ k \neq j}}^{m} (1-\omega_k z), \tag{3.6}$$

which obviously hold when z is replaced by any operator $B \in \mathfrak{B}(\mathbf{H})$. Now, for an arbitrary unit vector $\mathbf{x} \in \mathbf{H}$ define the vectors

$$\mathbf{x}_j = \left[\prod_{\substack{k=1 \\ k \neq j}}^{m} (1-\omega_k B) \right] \mathbf{x}, \qquad j=1,\dots,m,$$

and use (3.5), (3.6) to find that

$$\frac{1}{m}\sum_{j=1}^{m}\|\mathbf{x}_j\|^2\left[1-\omega_j\left(\frac{B\mathbf{x}_j}{\|\mathbf{x}_j\|},\frac{\mathbf{x}_j}{\|\mathbf{x}_j\|}\right)\right]$$

$$=\frac{1}{m}\sum_{j=1}^{m}\big((1-\omega_j B)\mathbf{x}_j,\mathbf{x}_j\big)=\frac{1}{m}\sum_{j=1}^{m}\left(\left[\prod_{k=1}^{m}(1-\omega_k B)\right]\mathbf{x},\mathbf{x}_j\right)$$

$$=\frac{1}{m}\sum_{j=1}^{m}\big((1-B^m)\mathbf{x},\mathbf{x}_j\big)=\left((1-B^m)\mathbf{x},\frac{1}{m}\sum_{j=1}^{m}\mathbf{x}_j\right)$$

$$=\left((1-B^m)\mathbf{x},\frac{1}{m}\sum_{j=1}^{m}\left[\prod_{\substack{k=1\\k\neq j}}^{m}(1-\omega_k B)\right]\mathbf{x}\right)$$

$$=\big((1-B^m)\mathbf{x},\mathbf{x}\big)=1-(B^m\mathbf{x},\mathbf{x}). \tag{3.7}$$

In particular, setting

$$B=Ae^{i\theta},\qquad \theta\in\mathbf{R},$$

(3.7) yields

$$\frac{1}{m}\sum_{j=1}^{m}\|\mathbf{x}_j\|^2\left[1-\omega_j e^{i\theta}\left(\frac{A\mathbf{x}_j}{\|\mathbf{x}_j\|},\frac{\mathbf{x}_j}{\|\mathbf{x}_j\|}\right)\right]=1-e^{im\theta}(A^m\mathbf{x},\mathbf{x}).$$

Since by hypothesis $r(A)\leqslant 1$, the real part of the expression in the brackets above is nonnegative. So for any unit vector $\mathbf{x}$ and real θ,

$$\operatorname{Re}\left[1-e^{im\theta}(A^m\mathbf{x},\mathbf{x})\right]\geqslant 0;$$

thus

$$|(A^m\mathbf{x},\mathbf{x})|\leqslant 1,\qquad \|\mathbf{x}\|=1,$$

and (3.4) follows. ∎

The above proof is due to Pearcy [16]. We note that the rather interesting evolution of Berger's theorem is described in [2] and [5].

It might be useful to remark that by (1.6) and (3.3),

$$\|A^m\| \leqslant 2r(A^m) \leqslant 2r^m(A),$$

so we have

COROLLARY 3.1. *If $r(A)\leqslant 1$, then the operator norm on $\mathfrak{B}(\mathbf{H})$ satisfies*

$$\|A^m\| \leqslant 2, \qquad m=1,2,3,\dots .$$

Although the numerical radius is nonmultiplicative, a simple multiplication by a scalar may correct this situation, as shown in our next result [6].

THEOREM 3.3. *Let $\nu>0$ be fixed, and consider the function $r_\nu(A)\equiv \nu r(A)$. Then:*

(i) *r_ν is a norm on $\mathfrak{B}(\mathbf{H})$.*
(ii) *r_ν is multiplicative on $\mathfrak{B}(\mathbf{H})$ if and only if $\nu\geqslant 4$.*

Proof. The statement in (i) is trivial and holds for $N_\nu \equiv \nu N$, N being any norm on $\mathfrak{B}(\mathbf{H})$.

To prove (ii), fix some $\nu\geqslant 4$. Then for all $A, B\in\mathfrak{B}(\mathbf{H})$, (1.6) implies

$$\begin{aligned} r_\nu(AB) &= \nu r(AB) \leqslant \nu\|AB\| \leqslant \nu\|A\|\cdot\|B\| \\ &\leqslant 4\nu r(A)r(B) \leqslant \nu^2 r(A)r(B) = r_\nu(A)r_\nu(B), \end{aligned}$$

i.e., r_ν is multiplicative.

Conversely, to show that $\nu=4$ is the least factor for which r_ν is multiplicative, consider the matrices

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad B=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

By (2.4), $r(A)=r(B)=\frac{1}{2}$, $r(AB)=1$. Hence for these matrices r_ν satisfies

$$r_\nu(AB) \leqslant r_\nu(A)r_\nu(B)$$

if and only if $\nu\geqslant 4$, and the theorem follows. ∎

It is worth noting that if N is an arbitrary norm on a finite-dimensional algebra, then *multiplicativity factors*, i.e. constants $\nu>0$ for which $N_\nu \equiv \nu N$ is multiplicative, always exist [6]. This, however, is not always true in the infinite-dimensional case [7].

4. APPLICATIONS: STABLE LAX-WENDROFF SCHEMES

Consider the first-order, linear, 2-space-dimensional hyperbolic system of partial differential equations

$$\mathbf{u}_t = A\mathbf{u}_x + B\mathbf{u}_y, \qquad -\infty < x < \infty, \quad -\infty < y < \infty, \quad t \geqslant 0, \tag{4.1a}$$

where $\mathbf{u} = (u_1(x,y,t),\ldots,u_n(x,y,t))'$ is an unknown vector; A and B fixed $n\times n$ Hermitian coefficient matrices; and $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_t$ the partial derivatives of $\mathbf{u}$ with respect to the independent variables x, y, and t. It is well known (e.g. [17]) that the solution of (4.1a) is uniquely determined and well posed in $L^2(-\infty,\infty)$, if proper initial values

$$\mathbf{u}(x,y,0) = \mathbf{f}(x,y) \in L^2(-\infty,\infty), \qquad -\infty < x < \infty, \quad -\infty < y < \infty, \tag{4.1b}$$

are prescribed.

In order to solve (4.1) by finite-difference approximations we introduce increments $\Delta x>0$, $\Delta y>0$, $\Delta t>0$ with fixed ratios $\lambda = \Delta t/\Delta x$, $\mu = \Delta t/\Delta y$; grid points

$$(x_j, y_k, t_m) = (j\Delta x, k\Delta y, m\Delta t), \qquad j,k = 0, \pm 1, \pm 2, \ldots, \quad m = 0,1,2,\ldots;$$

and the notation

$$\mathbf{u}_{jk}^m = \mathbf{u}(x_j, y_k, t_m).$$

Then a Taylor expansion of a sufficiently smooth solution of (4.1a) about (x_j, y_k, t_m) can be written as

$$\mathbf{u}_{jk}^{m+1} = \mathbf{u}_{jk}^m + \Delta t(\mathbf{u}_t)_{jk}^m + \tfrac{1}{2}(\Delta t)^2(\mathbf{u}_{tt})_{jk}^m + O(\Delta t^3). \tag{4.2}$$

Since by (1.4a),

$$\mathbf{u}_{tt}=(A\mathbf{u}_x+B\mathbf{u}_y)_t=A\mathbf{u}_{tx}+B\mathbf{u}_{ty}$$
$$=A(A\mathbf{u}_x+B\mathbf{u}_y)_x+B(A\mathbf{u}_x+B\mathbf{u}_y)_y=A^2\mathbf{u}_{xx}+B^2\mathbf{u}_{yy}+(AB+BA)\mathbf{u}_{xy},$$

then (4.2) yields

$$\mathbf{u}_{jk}^{m+1}=\mathbf{u}_{jk}^m+\Delta t\left[A\mathbf{u}_x+B\mathbf{u}_y\right]_{jk}^m$$
$$+\tfrac{1}{2}(\Delta t)^2\left[A^2\mathbf{u}_{xx}+B^2\mathbf{u}_{yy}+(AB+BA)\mathbf{u}_{xy}\right]_{jk}^m+O(\Delta t^3);$$

so using the standard difference formulas

$$(\mathbf{u}_x)_{jk}^m=\frac{1}{2\,\Delta x}\left(\mathbf{u}_{j+1,k}^m-\mathbf{u}_{j-1,k}^m\right)+O(\Delta x^2),$$

$$(\mathbf{u}_y)_{jk}^m=\frac{1}{2\,\Delta y}\left(\mathbf{u}_{j,k+1}^m-\mathbf{u}_{j,k-1}^m\right)+O(\Delta y^2),$$

$$(\mathbf{u}_{xx})_{jk}^m=\frac{1}{(\Delta x)^2}\left(\mathbf{u}_{j+1,k}^m-2\mathbf{u}_{jk}^m+\mathbf{u}_{j-1,k}^m\right)+O(\Delta x),$$

$$(\mathbf{u}_{yy})_{jk}^m=\frac{1}{(\Delta y)^2}\left(\mathbf{u}_{j,k+1}-2\mathbf{u}_{jk}^m+\mathbf{u}_{j,k-1}^m\right)+O(\Delta y)$$

$$(\mathbf{u}_{xy})_{jk}^m=\frac{1}{4\,\Delta x\,\Delta y}\left(\mathbf{u}_{j+1,k+1}^m-\mathbf{u}_{j+1,k-1}^m-\mathbf{u}_{j-1,k+1}^m+\mathbf{u}_{j-1,k-1}^m\right)$$
$$+O(\Delta x)+O(\Delta y)+O(\Delta x^2/\Delta y)+O(\Delta y^2/\Delta x)$$

and the relations

$$\Delta x=\lambda\,\Delta t,\quad \Delta y=\mu\,\Delta t \qquad (\lambda,\mu\text{ constants}),$$

we find that

$$\mathbf{u}_{jk}^{m+1}=\mathbf{u}_{jk}^m+\tfrac{1}{2}\lambda A\left(\mathbf{u}_{j+1,k}^m-\mathbf{u}_{j-1,k}^m\right)+\tfrac{1}{2}\mu B\left(\mathbf{u}_{j,k+1}^m-\mathbf{u}_{j,k-1}^m\right)$$
$$+\tfrac{1}{2}\lambda^2A^2\left(\mathbf{u}_{j+1,k}^m-2\mathbf{u}_{jk}^m+\mathbf{u}_{j-1,k}^m\right)+\tfrac{1}{2}\mu^2B^2\left(\mathbf{u}_{j,k+1}^m-2\mathbf{u}_{jk}^m+\mathbf{u}_{j,k-1}^m\right)$$
$$+\tfrac{1}{8}\lambda\mu(AB+BA)\left(\mathbf{u}_{j+1,k+1}^m-\mathbf{u}_{j+1,k-1}^m-\mathbf{u}_{j-1,k+1}^m+\mathbf{u}_{j-1,k-1}^m\right)+O(\Delta t^3)$$

Thus, dropping $O(\Delta t^3)$ terms and replacing $\mathbf{u}$ by an approximation vector $\mathbf{v}$, we finally obtain the celebrated Lax-Wendroff difference scheme for (1.1), [13]:

$$\begin{aligned}\mathbf{v}_{jk}^{m+1}=\mathbf{v}_{jk}^{m}&+\tfrac{1}{2}\lambda A\left(\mathbf{v}_{j+1,k}^{m}-\mathbf{v}_{j-1,k}^{m}\right)+\tfrac{1}{2}\mu B\left(\mathbf{v}_{j,k+1}^{m}-\mathbf{v}_{j,k-1}^{m}\right)\\&+\tfrac{1}{2}\lambda^2A^2\left(\mathbf{v}_{j+1,k}^{m}-2\mathbf{v}_{jk}^{m}+\mathbf{v}_{j-1,k}^{m}\right)+\tfrac{1}{2}\mu^2B^2\left(\mathbf{v}_{j,k+1}^{m}-2\mathbf{v}_{jk}^{m}+\mathbf{v}_{j,k-1}^{m}\right)\\&+\tfrac{1}{8}\lambda\mu(AB+BA)\left(\mathbf{v}_{j+1,k+1}^{m}-\mathbf{v}_{j+1,k-1}^{m}-\mathbf{v}_{j-1,k+1}^{m}+\mathbf{v}_{j-1,k-1}^{m}\right)\end{aligned}\tag{4.3a}$$

which may be solved, time step after time step, if initial values

$$\mathbf{v}_{jk}^{0}=\mathbf{f}_{jk}\equiv\mathbf{f}(x_j,y_k),\qquad j,k=0,\pm1,\pm2,\dots,\tag{4.3b}$$

are set.

The main question concerning the scheme (4.3) is whether it is *convergent*; that is, keeping the ratios λ and μ fixed and letting $\Delta t\to 0$, we ask if the numerical solution $\mathbf{v}$ of (4.3) tends to that of (4.1).

In order to answer this question we introduce the *amplification matrix*

$$\begin{aligned}G=G(\xi,\eta,\lambda,\mu)\equiv I&+\tfrac{1}{2}\lambda A(e^{i\xi}-e^{-i\xi})+\tfrac{1}{2}\mu B(e^{i\eta}-e^{-i\eta})\\&+\tfrac{1}{2}\lambda^2A^2(e^{i\xi}-2+e^{-i\xi})+\tfrac{1}{2}\mu^2B^2(e^{i\eta}-2+e^{-i\eta})\\&+\tfrac{1}{8}\lambda\mu(AB+BA)\left(e^{i(\xi+\eta)}-e^{i(\xi-\eta)}-e^{i(\eta-\xi)}+e^{-i(\xi+\eta)}\right),\end{aligned}\tag{4.4}$$

which is the Fourier transform of the difference operator associated with our scheme, formally obtained by taking the right-hand side of (4.3a) with $\mathbf{v}_{j+p,k+q}^{m}$ replaced by the Fourier component $e^{i(p\xi+q\eta)}$. We say that the scheme (4.3) is *stable* if G is power bounded, i.e., if for some norm N on $\mathbf{C}_{n\times n}$ and a fixed constant $K>0$,

$$N\left[G(\xi,\eta,\lambda,\mu)^m\right]\leqslant K\qquad\forall m=1,2,3,\dots,\quad -\pi<\xi\leqslant\pi,\quad -\pi\leqslant\eta\leqslant\pi.\tag{4.5}$$

Traditionally (e.g. [17]) this definition is stated with the spectral norm rather than an arbitrary N. However, since all norms on $\mathbf{C}_{n\times n}$ are equivalent, it makes no difference with respect to which norm the estimate in (4.5) is taken, and in particular we may use the numerical radius.

Now, it is well known (e.g. [13], [17]) that our Lax-Wendroff scheme is convergent if and only if it is stable; thus the question of convergence is reduced to that of stability. This leads to our final result [19], whose proof employs the numerical radius and some of its previously discussed properties.

THEOREM 4.1. *The Lax-Wendroff scheme* (4.3) *is stable* (*hence convergent*) *if*

$$\lambda^2\rho^2(A)+\mu^2\rho^2(B)\leq\tfrac{1}{4}; \tag{4.6}$$

i.e., if the time step Δt *satisfies*

$$\Delta t\leq\tfrac{1}{2}\left\{\left[\rho(A)/\Delta x\right]^2+\left[\rho(B)/\Delta y\right]^2\right\}^{-1/2}.$$

Proof. Set $\tilde{A}=\lambda A$, $\tilde{B}=\mu B$; then G in (4.4) takes the form

$$G=R+iJ,$$

where

$$R=I-C,$$

$$C=(1-\cos\xi)\tilde{A}^2+(1-\cos\eta)\tilde{B}^2+\tfrac{1}{2}\sin\xi\sin\eta\,(\tilde{A}\tilde{B}+\tilde{B}\tilde{A}),$$

$$J=\sin\xi\,\tilde{A}+\sin\eta\,\tilde{B}.$$

Since A, B are Hermitian, so are R and J; hence (Rx,x) and (Jx,x) are real for all $\mathbf{x}\in\mathbf{C}^n$ and

$$|(G\mathbf{x},\mathbf{x})|^2=|(R\mathbf{x},\mathbf{x})+i(J\mathbf{x},\mathbf{x})|^2=(R\mathbf{x},\mathbf{x})^2+(J\mathbf{x},\mathbf{x})^2$$
$$=\left[(I-C)\mathbf{x},\mathbf{x}\right]^2+(J\mathbf{x},\mathbf{x})=(\mathbf{x},\mathbf{x})^2+(C\mathbf{x},\mathbf{x})^2-2(C\mathbf{x},\mathbf{x})+(J\mathbf{x},\mathbf{x})^2. \tag{4.7}$$

Denoting

$$\alpha\equiv1-\cos\xi\geq0,\qquad\beta\equiv1-\cos\eta\geq0,$$

we easily verify that

$$C=\tfrac{1}{2}\left(\alpha^2\tilde{A}^2+\beta^2\tilde{B}^2+J^2\right).$$

Thus,

$$\begin{aligned}2(C\mathbf{x},\mathbf{x})&=\alpha^2(\tilde{A}^2\mathbf{x},\mathbf{x})+\beta^2(\tilde{B}^2\mathbf{x},\mathbf{x})+(J^2\mathbf{x},\mathbf{x})\\&=\alpha^2(\tilde{A}\mathbf{x},\,\tilde{A}\mathbf{x})+\beta^2(\tilde{B}\mathbf{x},\,\tilde{B}\mathbf{x})+(J\mathbf{x},\,J\mathbf{x})\\&\quad+\alpha^2\|\tilde{A}\mathbf{x}\|^2+\beta^2\|\tilde{B}\mathbf{x}\|^2+\|J\mathbf{x}\|^2.\end{aligned}\tag{4.8}$$

Since also

$$(J\mathbf{x},\mathbf{x})^2\leqslant\|J\mathbf{x}\|^2\|\mathbf{x}\|^2=\|J\mathbf{x}\|^2,\qquad\|\mathbf{x}\|=1,\tag{4.9}$$

then (4.7)–(4.9) yield

$$|(G\mathbf{x},\mathbf{x})|^2\leqslant 1+(C\mathbf{x},\mathbf{x})^2-\alpha^2\|\tilde{A}\mathbf{x}\|^2-\beta^2\|\tilde{B}\mathbf{x}\|^2,\qquad\|\mathbf{x}\|=1.\tag{4.10}$$

Next, we write

$$C=\alpha\tilde{A}^2+\beta\tilde{B}^2+\tfrac{1}{2}\sin\xi\sin\eta\,(\tilde{A}\tilde{B}+\tilde{B}\tilde{A});$$

so

$$|(C\mathbf{x},\mathbf{x})|\leqslant\alpha|(\tilde{A}^2\mathbf{x},\mathbf{x})|+\beta|(\tilde{B}^2\mathbf{x},\mathbf{x})|+\left|\tfrac{1}{2}\sin\xi\sin\eta\left((\tilde{A}\tilde{B}+\tilde{B}\tilde{A})\mathbf{x},\mathbf{x}\right)\right|.\tag{4.11}$$

We have

$$\begin{aligned}&\left|\tfrac{1}{2}\sin\xi\sin\eta\left((\tilde{A}\tilde{B}+\tilde{B}\tilde{A})\mathbf{x},\mathbf{x}\right)\right|\\&\quad\leqslant\tfrac{1}{2}|\sin\xi\sin\eta|\cdot\left\{|(\tilde{A}\tilde{B}\mathbf{x},\mathbf{x})|+|(\tilde{B}\tilde{A}\mathbf{x},\mathbf{x})|\right\}\\&\quad=\tfrac{1}{2}|\sin\xi\sin\eta|\left\{|(\tilde{B}\mathbf{x},\,\tilde{A}\mathbf{x})|+|(\tilde{A}\mathbf{x},\,\tilde{B}\mathbf{x})|\right\}=|\sin\xi\sin\eta(\tilde{A}\mathbf{x},\,\tilde{B}\mathbf{x})|\\&\quad\leqslant|\sin\xi\sin\eta|\cdot\|\tilde{A}\mathbf{x}\|\cdot\|\tilde{B}\mathbf{x}\|\leqslant\tfrac{1}{2}\left\{\sin^2\xi\|\tilde{A}\mathbf{x}\|^2+\sin^2\eta\|\tilde{B}\mathbf{x}\|^2\right\}\\&\quad=\tfrac{1}{2}\left\{(1-\cos^2\xi)\|\tilde{A}\mathbf{x}\|^2+(1-\cos^2\eta)\|\tilde{B}\mathbf{x}\|^2\right\}\\&\quad\leqslant\alpha\|\tilde{A}\mathbf{x}\|^2+\beta\|\tilde{B}\mathbf{x}\|^2.\end{aligned}$$

Therefore (4.11) gives

$$|(Cx,x)| \leqslant 2\alpha \|\tilde{A}x\|^2 + 2\beta \|\tilde{B}x\|^2,$$

and by the Cauchy-Schwarz inequality,

$$(Cx,x)^2 = |(Cx,x)|^2 \leqslant 4(\alpha \|\tilde{A}x\|^2 + \beta \|\tilde{B}x\|^2)^2$$

$$\leqslant 4(\alpha^2 \|\tilde{A}x\|^2 + \beta^2 \|\tilde{B}x\|^2)(\|\tilde{A}x\|^2 + \|\tilde{B}x\|^2). \qquad (4.12)$$

Since $\tilde{A}$, $\tilde{B}$ are Hermitian, then by (1.9),

$$\|\tilde{A}x\| \leqslant \|\tilde{A}\| \cdot \|x\| = \|\tilde{A}\| = \rho(\tilde{A}) = \lambda\rho(A), \qquad \|x\| = 1,$$

and similarly

$$\|\tilde{B}x\| \leqslant \mu\rho(B), \qquad \|x\| = 1.$$

Thus, by (4.12) and the hypothesis (4.6),

$$(Cx,x)^2 \leqslant \alpha^2 \|\tilde{A}x\|^2 + \beta^2 \|\tilde{B}x\|^2,$$

and (4.10) implies

$$|(Gx,x)|^2 \leqslant 1, \qquad \|x\| = 1.$$

Consequently, $r(G) \leqslant 1$; so by Theorem 3.2,

$$r(G^m) \leqslant 1, \qquad m = 1,2,3,\ldots,$$

and the proof is complete. ∎

In their original paper, Lax and Wendroff [13] were the first to utilize numerical-radius techniques for stability purposes, proving that the scheme (4.3) is stable if

$$\max\{\lambda\rho(A), \mu\rho(B)\} \leqslant 1/\sqrt{8}. \qquad (4.13)$$

Evidently, the condition (4.6)—allowing a larger time step—is an improvement over (4.13) unless $\lambda\rho(A)=\mu\rho(B)$, in which case the two conditions coincide.

It is a straightforward matter to follow the construction in (4.2)–(4.5) and obtain a scheme, analogous to (4.3), for the d-dimensional hyperbolic system

$$\mathbf{u}_t=\sum_{j=1}^{d} A_j\mathbf{u}_{x_j}, \qquad -\infty<x_j<\infty, \quad t\geqslant 0, \tag{4.14}$$

where as before, the A_j are fixed $n\times n$ Hermitian matrices. It is not hard to see that for this multidimensional Lax-Wendroff scheme, the conditions (4.6) and (4.13) become

$$\sum_{i=1}^{d}\lambda_j^2\rho^2(A_j)\leqslant\frac{1}{d^2} \qquad (\lambda_j\equiv\Delta t/\Delta x_j) \tag{4.15}$$

and

$$\max_{1\leqslant j\leqslant d}\lambda_j\rho(A_j)\leqslant\frac{1}{d^{3/2}}, \tag{4.16}$$

respectively. Thus as in the 2-dimensional case, the advantage of (4.15) over (4.16) is evident, unless the $\lambda_j\rho(A_j)$ are all equal.

In case A and B in (4.6) are real and symmetric, Turkel [21] has improved (4.6), showing that the scheme (4.3) is stable if

$$\lambda^{2/3}\rho^{2/3}(A)+\mu^{2/3}\rho^{2/3}(B)\leqslant 1,$$

which again coincides with (4.6) and (4.13) when $\lambda\rho(A)=\mu\rho(B)$. It seems, however, that Turkel's interesting result for (4.3) does not go over to the general d-dimensional case.

We remark that Livne [14] and Iusim [12] have used similar techniques to successfully investigate the stability of other difference schemes for hyperbolic systems of type (4.1) and (4.14) with $d=3$.

REFERENCES

1 C. A. Berger, On the numerical range of powers of an operator, Abstract No. 625-152, *Notices Amer. Math. Soc.* 12:590 (1965).

2 C. A. Berger and J. G. Stampfli, Mapping theorems for the numerican range, *Amer. J. Math.* 89:1047–1055 (1967).

3 W. F. Donoghue, On the numerical range of a bounded operator, *Michigan Math. J.* 4:261–263 (1957).

4 T. Furuta and Z. Takeda, A characterization of spectraloid operators and its generalizations, *Proc. Japan Acad.* 43:599–604 (1967).

5 M. Goldberg, On certain finite dimensional numerical ranges and numerical radii, *Linear and Multilinear Algebra* 7:329–342 (1979).

6 M. Goldberg and E. G. Straus, Norm properties of C-numerical radii, *Linear Algebra Appl.* 24:113–131 (1979).

7 M. Goldberg and E. G. Straus, Operator norms, multiplicativity factors, and C-numerical radii, *Linear Algebra Appl.*, to appear.

8 M. Goldberg, E. Tadmor, and G. Zwas, The numerical radius and spectral matrices, *Linear and Multilinear Algebra* 2:317–326 (1975).

9 M. Goldberg, E. Tadmor, and G. Zwas, Numerical radius of positive matrices, *Linear Algebra Appl.* 12:209–214 (1975).

10 P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea, New York, 1951.

11 P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, New York, 1967.

12 R. Iusim, An explicit method for solving three dimensional symmetric hyperbolic equations, M.Sc. Thesis, Dept. of Math., Technion—Israel Inst. of Technology, 1979.

13 P. D. Lax and B. Wendroff, Difference schemes for hyperbolic equations with high order of accuracy, *Comm. Pure Appl. Math.* 17:381–391 (1964).

14 A. Livne, Seven point difference schemes for hyperbolic equations, *Math. Comp.* 29:425–433 (1975).

15 F. D. Murnaghan, On the field of values of a square matrix, *Proc. Nat. Acad. Sci. U.S.A.* 18:246–248 (1932).

16 C. Pearcy, An elementary proof of the power inequality for the numerical radius, *Michigan Math. J.* 13:289–291 (1960).

17 R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-Value Problems*, 2nd ed., Interscience, Wiley, New York, 1967.

18 M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, Amer. Math. Soc. Colloquium Publications, Vol. 15, New York, 1932.

19 E. Tadmor, The numerical radius and power boundedness, M.Sc. Thesis, Dept. of Math. Sci., Tel Aviv Univ., 1975.

20 A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, 2nd ed., Wiley, New York, 1980.

21 E. Turkel, Symmetric hyperbolic difference schemes and matrix problems, *Linear Algebra Appl.* 16:109–129 (1977).

22 A. Wintner, Zur Theorie der beschränkten Bilinearformen, *Math. Z.* 39:228–282 (1932).

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