Numerical Radius of Positive Matrices

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ABSTRACT

We study some properties of the numerical radius of matrices with non-negative entries, and explicit ways to compute it. We also characterize positive matrices with equal spectral and numerical radii, i.e., positive *spectral* matrices.

Let A be an $n\times n$ complex matrix with numerical radius

$$r(A)=\max_{|x|=1}|(Ax,x)|.$$

Here (x,y) is the unitary inner product and, $|x|=(x,x)^{1/2}$. We shall study $r(A)$ for positive matrices, i.e., for matrices with non-negative entries, which we denote by $A\geqslant 0$.

LEMMA 1. *If* $A\geqslant 0$, *then*

$$r(A)=\max_{|x|=1}\{(Ax,x),\ x\in\mathbf{R}^n\}. \tag{1}$$

Proof. There exists a unit vector $x_0=(\xi_1,\dots,\xi_n)^t$ such that $r(A)=|(Ax_0,x_0)|$. Since $A\geqslant 0$, and $y_0\equiv(|\xi_1|,\dots,|\xi_n|)^t$ has norm 1, we have

$$r(A)\leqslant|(Ax_0,x_0)|\leqslant(Ay_0,y_0)\leqslant r(A), \tag{2}$$

and the lemma follows. ■

Using the notation

$$\operatorname{Re} A = \tfrac{1}{2}(A + A^t), \tag{3}$$

we prove the following theorem.

THEOREM 1. *If $A \geqslant 0$, then*

$$r(A) = r(\operatorname{Re} A). \tag{4}$$

Proof.

$$r(\operatorname{Re} A) = \max_{|x|=1} |(\operatorname{Re} Ax, x)| = \max_{|x|=1} \tfrac{1}{2}\left|(Ax,x) + \overline{(Ax,x)}\,\right|$$

$$= \max_{|x|=1} |\operatorname{Re}(Ax,x)| \leqslant \max_{|x|=1} |(Ax,x)| = r(A). \tag{5}$$

On the other hand, if $y_0 \in \mathbf{R}^n$ is the positive vector of Lemma 1, then

$$r(\operatorname{Re} A) = \max|\operatorname{Re}(Ax,x)| \geqslant (Ay_0, y_0) = r(A), \tag{6}$$

and the proof is complete. ■

Since $\operatorname{Re} A$ is symmetric, $\rho(\operatorname{Re} A) = r(\operatorname{Re} A)$, where ρ denotes the spectral radius. Therefore, Theorem 1 states that if $A \geqslant 0$, then

$$r(A) = \rho(\operatorname{Re} A). \tag{7}$$

In general, $\rho(A) \leqslant r(A)$. A matrix for which $\rho(A) = r(A)$ we call *spectral.* This definition and (7) yield the next result.

COROLLARY 1. *If $A \geqslant 0$, then A is spectral if and only if*

$$\rho(A) = \rho(\operatorname{Re} A). \tag{8}$$

Another simple result is the following.

COROLLARY 2. *If $A \geqslant 0$ is spectral, then*

$$\rho(A^k) = \rho(\operatorname{Re} A^k), \qquad k = 1, 2, 3, \ldots. \tag{9}$$

Proof. Clearly $A^k \geqslant 0$, and in Theorem 2 of [1] we proved that A^k is spectral if A is. Corollary 1 completes the proof. ■

In [1] we studied the problem whether, in general, an equality of the form $r(A^m) = r^m(A)$ for some integer m implies the spectrality of A. Theorem 3 of

[1] shows that if A is an n-square matrix with minimal polynomial of degree p, and m is some integer with $m \geqslant p$, then A is spectral if and only if $r(A^m) = r^m(A)$. Since generally it is not true that $p < n$, we raised the question whether, in general, an equality of the form $r(A^m) = r^m(A)$ for some $m < n$ implies spectrality. An example for $n = 3$, given in [1], excluded this possibility even for the case $m = n - 1 = 2$. Now we are able to answer the above question in the negative for $m = n - 1$, for any order n. Before introducing our example we need the following results.

THEOREM 2. *If* $A \geqslant 0$, *then* $r(A) = s$ *if and only if the matrix*

$$S = sI - \operatorname{Re} A \tag{10}$$

is positive semi-definite but not positive definite.

Proof. By Lemma 1, $s = r(A)$ if and only if $s(x,x) \geqslant (Ax,x)$ for every $x \in \mathbf{R}^n$, with equality holding for some $x_0 \neq 0$. Clearly for all $x \in \mathbf{R}^n$, $(Ax,x) = (\operatorname{Re} Ax, x)$. Therefore $s = r(A)$ if and only if $(Sx,x) \geqslant 0$ for all $x \in \mathbf{R}^n$, with $(Sx_0, x_0) = 0$, where S is the matrix in (10). This completes the proof. ∎

A consequence of Theorem 2 is the following.

COROLLARY 3. *If* $A \geqslant 0$ *and if*

$$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \tag{11}$$

is congruent to the matrix S *in* (10), *then* $r(A) = s$ *if and only if all the* λ_i*'s are non-negative and at least one of them vanishes.*

Proof. By Theorem 2, $r(A) = s$ if and only if the eigenvalues of the symmetric matrix S are non-negative and at least one of them vanishes. By Sylvester's law of inertia the corollary follows. ∎

We are ready now for the above mentioned example.

EXAMPLE. *For each* $n > 1$ *there exists an* $n \times n$ *matrix which is not spectral but satisfies* $r^{n-1}(A) = r(A^{n-1})$.

Proof. For $n = 2$ there is nothing to prove. For $n \geqslant 3$ consider the $n \times n$ matrix

$$A = \operatorname{diag}(0, \sqrt{2}, 1, 1, \dots, 1, \sqrt{2})E, \qquad \text{where } E_{ij} = \delta_{i-1,j}. \tag{12}$$

Clearly, $r(A)>0=\rho(A)$, i.e., A is not spectral. To see that $r^{n-1}(A)=r(A^{n-1})$, note first that $A^{n-1}=2E^{n-1}$, so that $r(A^{n-1})=1$. All that remains is to show that $r(A)=1$. In order to do so we consider the matrix $S=I-\operatorname{Re}A$ and operate on its rows and columns by elementary operations, to eliminate its off-diagonal elements $S_{1,2}, S_{2,1}, S_{2,3}, S_{3,2},\dots,S_{n-1,n}, S_{n,n-1}$, in that order. We find that S is congruent to the diagonal matrix

$$D=\operatorname{diag}(1,\tfrac{1}{2},\tfrac{1}{2},\dots,\tfrac{1}{2},0), \tag{13}$$

and by Corollary 3 the example is established. ■

We continue by proving a simple mapping rule for the numerical radius of positive matrices.

THEOREM 3. *If A is a positive spectral matrix and $P(z)=\sum_j\alpha_j z^j$ is a polynomial with non-negative coefficients, then*

$$r(P(A))=P(r(A)). \tag{14}$$

Proof. For any matrix C and scalar γ, $r(\gamma C)=|\gamma|r(C)$. We also have the Berger-Halmos inequality ([2], p. 176),

$$r(C^j)\leqslant r^j(C), \qquad j=1,2,3,\dots. \tag{15}$$

Therefore, by the sub-additivity of the numerical radius it follows that

$$r(P(A))=r\Big(\sum_j\alpha_jA^j\Big)\leqslant\sum_j\alpha_jr(A^j)\leqslant\sum_j\alpha_jr^j(A)=P(r(A)). \tag{16}$$

Note that (16) is valid even if A is not positive. Now, by the Perron-Frobenius theorem, the positive matrix A has a positive eigenvalue λ with $\lambda=\rho(A)$. Thus, if λ_i, $1\leqslant i\leqslant n$, are the eigenvalues of A, then the eigenvalues $P(\lambda_i)$ of $P(A)$ satisfy

$$|P(\lambda_i)|=\Big|\sum_j\alpha_j\lambda_i^j\Big|\leqslant p(|\lambda_i|)\leqslant P(\lambda)=P(\rho(A)), \tag{17}$$

with equality for $\lambda_i=\lambda$. Therefore

$$\rho(P(A))=P(\rho(A)). \tag{18}$$

By the spectrality of A, by (17), and since in general $\rho(\cdot)\leqslant r(\cdot)$, we finally obtain

$$P(r(A)) = \sum_j \alpha_j r^j(A) = \sum_j \alpha_j \rho^j(A) = P(\rho(A)) = \rho(P(A)) \leqslant r(P(A)). \quad (19)$$

The inequalities (16) and (19) complete the proof. ∎

We are able now to generalize the characterization of spectral matrices given in Theorem 3 of [1], in the case of positive matrices.

THEOREM 4. *Let A be a positive matrix with minimal polynomial of degree p, and let*

$$P_m(z) = \sum_{j=0}^{m} \alpha_j z^j \quad (20)$$

be any polynomial of degree $m \geqslant p$ with non-negative coefficients. Then A is spectral if and only if

$$P_m(r(A)) = r(P_m(A)). \quad (21)$$

Proof. If A is spectral, then (21) holds by Theorem 3. Conversely, assume that (21) holds. By the Halmos inequality in (15) we have

$$\begin{aligned} r(P_m(A)) = r\left(\sum_j \alpha_j A^j\right) &\leqslant \sum_j \alpha_j r(A^j) \leqslant \sum_j \alpha_j r^j(A) \\ &= P_m(r(A)) = r(P_m(A)). \end{aligned} \quad (22)$$

Hence, we have equalities in (22) and consequently

$$\sum_{j=0}^{m} \alpha_j [r^j(A) - r(A^j)] = 0. \quad (23)$$

Each summand in (23) is non-negative, and therefore must vanish. In particular, since $\alpha_m > 0$, we obtain

$$r(A^m) = r^m(A), \quad (24)$$

where by assumption $m > p$. By Theorem 3 of [1] this is a necessary and sufficient condition for the spectrality of A, and the theorem follows. ∎

Since the degree p of the minimal polynomial of an $n \times n$ matrix A satisfies $p \leqslant n$, we have the following immediate consequence of Theorem 4.

COROLLARY 4. *Let A be a positive n-square matrix, and let $P_n(z)$ be some polynomial of degree n with non-negative coefficients. Then A is spectral if and only if*

$$P_n(r(A)) = r(P_n(A)). \tag{25}$$

REFERENCES

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2 P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, 1967.

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