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The Numerical Radius and Spectral Matrices

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In this paper we investigate *spectral* matrices, i.e., matrices with equal spectral and numerical radii. Various characterizations and properties of these matrices are given.

1. INTRODUCTION

Let A be an n-square complex matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and let

$$\rho(A) = \max_{1 \leqslant j \leqslant n} |\lambda_j| \tag{1.1}$$

be the spectral radius of A. Let

$$r(A) = \max_{|x|=1} |(Ax, x)| \tag{1.2}$$

be the numerical radius of A, and

$$\|A\| = \max_{|x|=1} |Ax| \tag{1.3}$$

the spectral norm of A. Here (x, y) is the unitary inner product of the vectors x and y, and $|x| = (x, x)^{\frac{1}{2}}$.

It is well known that

$$\rho(A) \leqslant r(A) \leqslant \|A\| \leqslant 2r(A). \tag{1.4}$$

In this paper we investigate matrices for which

$$\rho(A) = r(A). \tag{1.5}$$

Following Halmos ([3] p. 115), we call matrices which satisfy (1.5), *spectral* matrices. Our main purpose is to characterize the spectral matrices and find some of their properties.

Before turning to some new results, we recall a few known results which we shall use later on. It is known that

$$r(A) = 0 \text{ if and only if } A = 0. \tag{1.6}$$

$$r(\alpha A) = |\alpha| r(A), \text{ for every scalar } \alpha. \tag{1.7}$$

$$r(A + B) \leqslant r(A) + r(B). \tag{1.8}$$

However, the numerical radius is not a matrix-norm, since in general it is not true that $r(AB) \leqslant r(A) \cdot r(B)$ even if A and B are powers of the same matrix [8]. On the other hand, we always have the Halmos inequality

$$r(A^k) \leqslant r^k(A), \qquad k = 1, 2, 3, \ldots. \tag{1.9}$$

This power inequality was conjectured by Halmos and proved by Berger. The proof was simplified by Pearcy [8]. Generalizations of the power inequality were given by Kato [4], and by Berger and Stampfli [1].

It is also known that

$$r(A_1 \oplus \cdots \oplus A_m) = \max_{1 \leqslant j \leqslant m} r(A_j). \tag{1.10}$$

Another concept associated with the numerical radius of a matrix is the numerical range $F(A)$, defined by

$$F(A) = \{(Ax, x), |x| = 1\}. \tag{1.11}$$

The numerical range, known also as the field of values of A, is a convex set in the complex plane. If U is a unitary transformation, then

$$F(U^*AU) = F(A), \qquad r(U^*AU) = r(A). \tag{1.12}$$

If M is any principle sub-matrix of A, then

$$F(M) \leqslant F(A), \qquad r(M) \leqslant r(A). \tag{1.13}$$

For a 2×2 matrix it is known that $F(A)$ is an ellipse whose foci are the eigenvalues λ_1 and λ_2 of A. In particular, if A is of the form

$$A = \begin{pmatrix} \lambda_1 & 0 \\ \sigma & \lambda_2 \end{pmatrix}, \tag{1.14}$$

then $|\sigma|/2$ is the semi-minor axis of the ellipse $F(A)$. We shall refer to this result as the Elliptic Range Theorem (see for example [6]).

Most of the above mentioned results can be found in [3]. A survey of properties of the numerical range and the numerical radius, some of which were proven by Parker, is given in [7].

The investigation of spectral matrices is motivated by stability problems related to finite difference schemes, where the uniform boundedness of $\|A^k\|$, $k = 1, 2, 3, \ldots$ plays a central role. In general we have

$$\rho^k(A) = \rho(A^k) \leqslant \|A^k\| \leqslant 2r(A^k) \leqslant 2r^k(A). \tag{1.15}$$

Therefore, $\rho(A) \leqslant 1$ is a necessary condition for uniform boundedness of the powers of A. However, if A is spectral, this condition is sufficient as well, and implies that $\|A^k\| \leqslant 2$ for all k. Such an idea was applied first by Lax and Wendroff [5].

2. STRUCTURE-CHARACTERIZATION OF SPECTRAL MATRICES

Before we start characterizing the class of spectral matrices we note that this class is wider than the class of normal matrices. For, if A is normal, then it is unitarily similar to a diagonal matrix, and by (1.10) and (1.12)

$$r(A) = \max|\lambda_j| = \rho(A), \tag{2.1}$$

so A is spectral. However, not every spectral matrix is normal. To see that, take the non-normal matrix

$$A = I \oplus B, \qquad I = I_{n-2}, \qquad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad n \geqslant 3. \tag{2.2}$$

We have $\rho(A) = 1$, where by (1.10) and the Elliptic Range Theorem $r(A) = 1$ too. Thus for $n \geqslant 3$ the class of normal matrices is a proper subclass of the class of spectral matrices.

The example just given shows that the spectrality of a direct sum does not imply the spectrality of each of the summands. On the other hand, it is clear that a direct sum of spectral matrices is spectral.

Let us now order the eigenvalues of an arbitrary n-square matrix A such that

$$\rho(A) = |\lambda_1| = \cdots = |\lambda_s| > |\lambda_{s+1}| \geqslant \cdots \geqslant |\lambda_n|, \tag{2.3}$$

where $s = s(A)$ is the number of eigenvalues of A on the spectral circle $|z| = \rho(A)$.

THEOREM 1 *The matrix A is spectral if and only if A is unitarily similar to a triangular matrix of the form*

$$\begin{pmatrix} \Lambda & 0 \\ 0 & B \end{pmatrix}, \tag{2.4a}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & \cdot & 0 \\ & \cdot & \\ 0 & \cdot & \lambda_s \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda_{s+1} & \cdot & 0 \\ & \cdot & \\ (B_{ij}) & \cdot & \lambda_n \end{pmatrix}, \tag{2.4b}$$

and where

$$r(B) \leqslant \rho(A). \tag{2.5}$$

Proof It is known that A is unitarily similar to a triangular matrix T, where the eigenvalues λ_j, are ordered along its diagonal as in (2.3). Since $\rho(A) = \rho(T)$ and $r(A) = r(T)$, we may assume that A is already triangular.

Suppose that A is spectral and take j and k with $1 \leqslant j \leqslant s$ and $j \leqslant k \leqslant n$. By (1.13)

$$\rho(A) = r(A) \geqslant r\begin{pmatrix} \lambda_j & 0 \\ a_{kj} & \lambda_k \end{pmatrix} \geqslant r(\lambda_j) = |\lambda_j| = \rho(A), \tag{2.6}$$

and therefore

$$r\begin{pmatrix} \lambda_j & 0 \\ a_{kj} & \lambda_k \end{pmatrix} = |\lambda_j|. \tag{2.7}$$

Using the Elliptic Range Theorem, it is clear that (2.7) is satisfied if and only if $a_{kj} = 0$. Thus $A = \Lambda + B$ as in (2.4). Now by (1.10)

$$\rho(A) = r(A) = \max\{\rho(A), r(B)\}, \tag{2.8}$$

hence (2.5) holds.

If (2.4) and (2.5) are satisfied, then by (1.10)

$$r(A) = \max\{r(\Lambda), r(B)\}. \tag{2.9}$$

Since $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_s)$ we have $\rho(\Lambda) = r(\Lambda)$, and by (2.5) $\rho(A) = r(A)$. Thus A is spectral.

COROLLARY 1 *If $s = s(A) \geqslant n - 1$, then A is spectral if and only if A is normal.*

Proof Normality implies spectrality. If A is spectral and $s \geqslant n - 1$, then by Theorem 1 it is unitarily similar to a diagonal matrix and A is normal.

In particular we obtain the following:

COROLLARY 2 *If $n = 2$, then A is a spectral if and only if it is normal.*

Corollary 2 follows also directly from the Elliptic Range Theorem. For if A is not normal then, without restriction, it is of the form (1.14) with $\sigma \neq 0$. Therefore, the ellipse $F(A)$ includes points z with $r(A) \geqslant |z| > \max\{|\lambda_1|, |\lambda_2|\} = \rho(A)$, and A is not spectral.

For $A = [a_{ij}]$, denote $A^+ = [|a_{ij}|]$. By the definition of the numerical radius we find that

$$r(A^+) \geqslant r(A). \tag{2.10}$$

Therefore, Theorem 1 yields the following:

COROLLARY 3 *If $s(A) = n - 2$, then a sufficient condition for A to be spectral is that A is unitarily similar to a matrix of the form* (2.4), *where*

$$B = \begin{pmatrix} \lambda_{n-1} & 0 \\ \beta & \lambda_n \end{pmatrix}, \tag{2.11}$$

and

$$|\beta| \leqslant 2[\rho(A) - |\lambda_{n-1}|]^{\frac{1}{2}}[\rho(A) - |\lambda_n|]^{\frac{1}{2}}. \tag{2.12}$$

Proof In order to satisfy (2.5), it is sufficient, by (2.10), to require that

$$r\begin{pmatrix} |\lambda_{n-1}| & 0 \\ |\beta| & |\lambda_n| \end{pmatrix} \leqslant \rho(A). \tag{2.13}$$

By the Elliptic Range Theorem, (2.13) means that the circle $|z| = \rho(A)$ contains the ellipse with the non-negative foci $|\lambda_{n-1}|$, $|\lambda_n|$, and minor axis $|\beta|$. This clearly holds if and only if (2.12) is satisfied.

In the case $s(A) = n - 2$ we remark that if $\arg(\lambda_{n-1}) = \arg(\lambda_n)$, then (2.12) is also necessary for the spectrality of A. However, for general λ_{n-1} and λ_n, finding a condition on the size of $|\beta|$ in (2.11) which is necessary as well as sufficient for the spectrality of A, involves the solution of a general quadric equation.

3. CRITICAL POWER CHARACTERIZATION

We start with the following theorem.

THEOREM 2 *The matrix A is spectral if and only if*

$$r(A^k) = r^k(A), \qquad k = 1, 2, 3, \ldots. \tag{3.1}$$

Proof If A is spectral, then by (1.9)

$$\rho(A^k) \leqslant r(A^k) \leqslant r^k(A) = \rho^k(A) = \rho(A^k), \qquad k = 1, 2, 3, \ldots, \tag{3.2}$$

and (3.1) holds. Conversely, we know that

$$\|A^k\|^{1/k} \underset{k\to\infty}{\rightarrow} \rho(A). \tag{3.3}$$

Therefore, if (3.1) is satisfied, then using (1.4) we have

$$\rho(A) = \rho^{1/k}(A^k) \leqslant r^{1/k}(A^k) = r(A) \leqslant \|A^k\|^{1/k} \underset{k\to\infty}{\rightarrow} \rho(A), \tag{3.4}$$

and the theorem follows.

Theorem 2 leads to the following conclusion.

COROLLARY 4 *If A is spectral, then any power of A is spectral.*

Proof Consider A^m. By Theorem 2 we have, for all k,

$$r((A^m)^k) = r(A^{mk}) = r^{mk}(A) = (r^m(A))^k = r^k(A^m). \tag{3.5}$$

Hence, using Theorem 2 once again, A^m is spectral.

Note that if a power of A is spectral, then A is not necessarily spectral. To see this, take

$$A = I_{n-2} \oplus B, \qquad B = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}, \qquad |\alpha| > 2,\ n \geqslant 3. \tag{3.6}$$

By (1.10) and the Elliptic Range Theorem, A is not spectral. On the other hand, all the powers, $A^m = I_{n-2} \oplus O_{2\times 2}$, $m \geqslant 2$, are normal and hence spectral.

Equation (3.1) of Theorem 2 provides infinitely many conditions, whose simultaneous satisfaction is equivalent to spectrality. However, the finite nature of a matrix leads us to conjecture the existence of a finite integer $k_0 = k_0(A)$ such that the validity of (3.1) for $k = k_0$ only, is sufficient as well as necessary for A to be spectral. The remainder of this section deals with this question.

Let m be a positive integer and let $\omega_j = e^{2\pi ij/m}$, $1 \leqslant j \leqslant m$, be the mth roots of unity. The following polynomial identities are well known:

$$1 - z^m = \prod_{k=1}^{m} (1 - \omega_k z); \tag{3.7}$$

$$m = \sum_{j=1}^{m} \prod_{\substack{k=1 \\ k\neq j}}^{m} (1 - \omega_k z). \tag{3.8}$$

Using these identities, which hold also when z is replaced by any square matrix B, Pearcy [8] proved the following lemma.

LEMMA (Pearcy) *Let B be a square matrix, m a positive integer, and x a unit vector. Then*

$$1 - (B^m x, x) = \frac{1}{m}\left[\sum_{j=1}^{m} |x_j|^2 - \sum_{j=1}^{m} \omega_j (Bx_j, x_j)\right], \tag{3.9}$$

where the vectors x_j are defined by

$$x_j = \left[\prod_{\substack{k=j \\ k\neq j}}^{m} (1 - \omega_k B)\right] x, \qquad 1 \leqslant j \leqslant m. \tag{3.10}$$

By the known identity

$$\prod_{\substack{k=1 \\ k\neq j}}^{m} (1 - \omega_k z) = \sum_{k=0}^{m-1} \omega_j^k z^k, \tag{3.11}$$

the vectors x_j in Pearcy's Lemma, may be rewritten in the form

$$x_j = \sum_{k=0}^{m-1} \omega_j^k B^k x, \qquad 1 \leqslant j \leqslant m. \tag{3.12}$$

From (3.7) and (3.11) we obtain $(1 - B^m)x = (1 - \omega_j B)x_j$; thus

$$\omega_j B x_j = x_j + B^m x - x, \qquad 1 \leqslant j \leqslant m. \tag{3.13}$$

Now let $A \neq 0$ be an n-square matrix, m a positive integer, and $x = x(m)$ a unit vector such that

$$|(A^m x, x)| = r(A^m) \tag{3.14}$$

Define the matrix

$$B = \frac{e^{i\theta}}{r(A)} A, \tag{3.15a}$$

where

$$\theta = \theta(m) = -\frac{1}{m} \arg(A^m x, x). \tag{3.15b}$$

Note that $r(B) = 1$; moreover $B = B(m)$ is spectral if and only if A is spectral. We are now in a position to prove the following lemma.

LEMMA 1 *Let $A \neq 0$ be a square matrix, $x = x(m)$ a unit vector satisfying* (3.14), *B as defined by* (3.15), *and $x_1, \ldots, x_m$ the vectors in* (3.12). *Then*

$$r(A^m) = r^m(A) \tag{3.16}$$

if and only if

$$(B^m x - x, x_j) = 0, \qquad 1 \leqslant j \leqslant m. \tag{3.17}$$

Proof By (3.14)

$$(B^m x, x) = \frac{e^{im\theta}}{r^m(A)}(A^m x, x) = \frac{|(A^m x, x)|}{r^m(A)} = \frac{r(A^m)}{r^m(A)}. \tag{3.18}$$

Therefore, Pearcy's Lemma implies that

$$\frac{1}{m}\left[\sum_{j=1}^{m} |x_j|^2 - \sum_{j=1}^{m} \omega_j(Bx_j, x_j)\right] = 1 - \frac{r(A^m)}{r^m(A)}. \tag{3.19}$$

Now, if $r(A^m) = r^m(A)$, then by (3.19)

$$\sum_{j=1}^{m} \omega_j(Bx_j, x_j) = \sum_{j=1}^{m} |x_j|^2. \tag{3.20}$$

Hence the left hand side of (3.20) is real and non-negative. Since for all x_j

$$|(Bx_j, x_j)| \leqslant r(B)|x_j|^2 = |x_j|^2, \tag{3.21}$$

we find that

$$\begin{aligned}\sum_{j=1}^{m} |x_j|^2 &= \sum_{j=1}^{m} \omega_j(Bx_j, x_j) \leqslant \sum_{j=1}^{m} |\omega_j(Bx_j, x_j)| \\ &= \sum_{j=1}^{m} |(Bx_j, x_j)| \leqslant \sum_{j=1}^{m} |x_j|^2. \end{aligned} \tag{3.22}$$

That is,

$$\sum_{j=1}^{m} \omega_j(Bx_j, x_j) = \sum_{j=1}^{m} |\omega_j(Bx_j, x_j)| = \sum_{j=1}^{m} |(Bx_j, x_j)| = \sum_{j=1}^{m} |x_j|^2. \tag{3.23}$$

From the left equality in (3.23) we have $\omega_j(Bx_j, x_j) \geqslant 0$; from the right equality and (3.21), $|(Bx_j, x_j)| = |x_j|^2$. Therefore

$$(\omega_j Bx_j, x_j) = |x_j|^2, \qquad 1 \leqslant j \leqslant m. \tag{3.24}$$

Now, substituting $\omega_j Bx_j$ from (3.13) into (3.24), we find that

$$|x_j|^2 = (x_j + B^m x - x, x_j) = |x_j|^2 + (B^m x - x, x_j), \quad 1 \leqslant j \leqslant m, \tag{3.25}$$

and (3.17) follows.

Conversely, if (3.17) holds, then by (3.13),

$$(\omega_j Bx_j, x_j) = (x_j + B^m x - x, x_j) = |x_j|^2 + (B^m x - x, x_j) = |x_j|^2, \quad 1 \leqslant j \leqslant m. \tag{3.26}$$

Hence (3.20) is satisfied, and by (3.19) $r(A^m) = r^m(A)$.

Lemma 1 enables us to prove the following theorem.

THEOREM 3 *Let A be an n-square matrix with minimal polynomial of degree p, and m an integer with $m \geqslant p$. Then A is spectral if and only if $r(A^m) = r^m(A)$.*

Proof By Theorem 2, the spectrality of A implies $r(A^m) = r^m(A)$. Next, suppose $r(A^m) = r^m(A)$. If $A = 0$, then A is obviously spectral.

Assume $A \neq 0$ and let $x = x(m)$, $B = B(m)$, and $x_1, \ldots, x_m$ be as in Lemma 1. By (3.12) and (3.17) we have

$$\sum_{k=0}^{m-1} \bar{\omega}_j^k (B^m x - x, B^k x) = \left(B^m x - x, \sum_{k=0}^{m-1} \omega_j^k B^k x\right) = (B^m x - x, x_j) = 0. \quad (3.27)$$

We conclude that the polynomial $P(z) = \sum_{k=0}^{m-1} (B^m x - x, B^k x) z^k$, which is of degree $m - 1$ at most, has m roots, $\bar{\omega}_1, \ldots, \bar{\omega}_m$. Hence all its coefficients vanish, i.e.,

$$(B^m x - x, B^k x) = 0, \qquad k = 0, \ldots, m - 1. \quad (3.28)$$

Clearly, the minimal polynomials of A and B are of the same degree, since $m \geqslant p$, there exists scalars α_j, $0 \leqslant j \leqslant m - 1$, such that

$$B^m = \sum_{j=0}^{m-1} \alpha_j B^j. \quad (3.29)$$

Therefore by (3.28) and (3.29)

$$(B^m x - x, B^m x) = \sum_{j=0}^{m-1} \bar{\alpha}_j (B^m x - x, B^j x) = 0. \quad (3.30)$$

By (3.30) and by (3.28) with $k = 0$, we obtain

$$(x, B^m x) = 1, \qquad (x, x) = (B^m x, B^m x) = 1, \quad (3.31)$$

i.e., the inner product of the unit vectors x and $B^m x$ is 1. This is true if and only if

$$B^m x = x. \quad (3.32)$$

Hence $\mu = 1$ is an eigenvalue of B^m, and we have $\rho(B^m) \geqslant 1$. Since $r(B) = 1$,

$$1 \leqslant \rho(B^m) = \rho^m(B) \leqslant r^m(B) = 1. \quad (3.33)$$

Consequently B is spectral and the spectrality of A follows.

Since the degree of the minimal polynomial of an n-square matrix does not exceed n, we may conclude the following.

COROLLARY 4 *An n-square matrix is spectral if and only if $r(A^n) = r^n(A)$.*

At this point it seems natural to ask whether in general an equality of the form $r(A^m) = r^m(A)$, for some $m < n$, implies spectrality. In general, the answer is negative even for the case $m = n - 1$, as can be seen from the example

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3.34)$$

Clearly $\rho(A) = 0$ and it can be verified that $r(A^2) = r^2(A) = \frac{1}{2}$.

Another result which follows immediately from Theorems 2 and 3 is given below.

COROLLARY 5 *If* $r(A^m) = r^m(A)$ *for some* m *with* $m \geqslant p$, *where* p *is the degree of the minimal polynomial of* A, *then* $r(A^k) = r^k(A)$ *for all* k.

An additional result can be derived from Corollaries 2 and 4.

COROLLARY 6 *For* $n = 2$, A *is normal if and only if* $r(A^2) = r^2(A)$.

We remark that Corollary 6 can be obtained directly by geometrical reasoning, using results of A. Brown [2].

Using Theorem 2 and Corollary 4 we prove our next theorem.

THEOREM 4 *Let* A, *with eigenvalues* $\mu_1, \ldots, \mu_n$, *be unitarily similar to a matrix of the form*

$$Q = \operatorname{diag}(\mu_1, \ldots, \mu_l) \oplus C, \qquad (C = C_{n-l \times n-l}), \tag{3.35}$$

so that at least one of the eigenvalues $\mu_1, \ldots, \mu_l$, *is on the spectral circle* $|z| = \rho(A)$. *Let* m *be an integer such that* $m \geqslant n - l$. *Then* A *is spectral if and only if*

$$r(A^m) = r^m(A). \tag{3.36}$$

Proof If A is spectral, then (3.36) holds by Theorem 2. Conversely, suppose A is unitarily similar to a matrix Q of the form (3.35), and that (3.36) holds. Since $Q = U^*AU$, we have $Q^m = U^*A^mU$, and by (1.12) and (3.36)

$$r(Q^m) = r^m(Q). \tag{3.37}$$

In addition

$$Q^m = \operatorname{diag}(\mu_1^m, \ldots, \mu_l^m) + C^m. \tag{3.38}$$

Thus by (3.37) and (1.10)

$$\max\{\rho^m(Q), r(C^m)\} = r(Q^m) = r^m(Q) = \max\{\rho^m(Q), r^m(C)\}. \tag{3.39}$$

If

$$r^m(Q) = r^m(C) > \rho^m(Q), \tag{3.40}$$

then by (3.39) we also have

$$r(Q^m) = r(C^m) > \rho^m(Q), \tag{3.41}$$

from which

$$r(C^m) = r^m(C). \tag{3.42}$$

Since C is $(n - l)$-square and $m \geqslant n - l$, it follows from Theorem 3 that C is spectral; hence $r(C) = \rho(C) \leqslant \rho(Q)$. This contradicts (3.40), so we must have

$$r^m(Q) = \rho^m(Q) \geqslant r^m(C). \tag{3.43}$$

This leads to

$$r(C^k) \leqslant r^k(C) \leqslant \rho^k(Q), \qquad k = 1, 2, 3, \ldots, \tag{3.44}$$

and for $k = n$ we obtain

$$r(Q^n) = \max\{\rho^n(Q), r(C^n)\} = \rho^n(Q) = \max\{\rho^n(Q), r^n(C)\} = r^n(Q). \quad (3.45)$$

By Corollary 4, Q is spectral. Hence A is spectral and the theorem follows.

Combining Theorems 1, 2 and 4, we may derive yet another final result.

COROLLARY 7 *Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$, ordered as in* (2.3), *and let m be an integer such that $m \geqslant n - s$. Then A is spectral if and only if A is unitarily similar to a triangular matrix $T = \Lambda \oplus B$ of the form* (2.4) *and* $r(A^m) = r^m(A)$.

The above results tend to show that the numerical radius can be useful in various applications. It is hoped that further research will actually bear this out.

References

[1] C. A. Berger and J. C. Stampfli, Mapping theorems for the numerical range, *Amer. J. Math.* **89** (1967), 1047–1055.
[2] A. Brown, On the Numerical Radius of a 2 × 2 Matrix, unpublished paper (1965).
[3] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand Co. Inc. (1967).
[4] T. Kato, Some mapping theorems for the numerical range, *Proc. Japan Acad.*, **41** (1965), 652–655.
[5] P. D. Lax and B. Wendroff, Difference schemes with high order of accuracy for solving hyperbolic equations, *Comm. Pure Appl. Math.* **17** (1964) 381–392.
[6] F. D. Murnaghan, On the field of values of a square matrix, *Proc. Nat. Acad. Sci.* **18** (1932) 246–248.
[7] W. V. Parker, Characteristic roots and field of values of a matrix, *Amer. Math. Soc. Bull.* **57** (1957) 103–108.
[8] C. Pearcy, An elementary proof of the power inequality for the numerical radius, *Mich. Math. J.* **13** (1966) 289–291.