The Equivalence of L_2-Stability, the Resolvent Condition, and Strict H-Stability

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ABSTRACT

The Kreiss matrix theorem asserts that a family of $N \times N$ matrices is L_2-stable if and only if either a resolvent condition (R) or a Hermitian norm condition (H) is satisfied. We give a direct, considerably shorter proof of the power-boundedness of an $N \times N$ matrix satisfying (R), sharpening former results by showing that power-boundedness depends, at most, linearly on the dimension N. We also show that L_2-stability is characterized by an H-condition employing a general H-numerical radius instead of the usual H-norm, thus generalizing a sufficient stability criterion, due to Lax and Wendroff.

1. INTRODUCTION

In studying the stability of difference approximations to pure initial-value systems, one encounters the problem of deciding the L_2-stability of a family of matrices, $\mathbf{F}$, i.e., the uniform power-boundedness of all matrices $A \in \mathbf{F}$.

There is a circle of ideas, first explored by H.-O. Kreiss [4], which gives the three necessary and sufficient conditions for a family $\mathbf{F}$ of $N \times N$ matrices to be L_2-stable; namely, the resolvent condition (R), the triangulization condition (S), and the strict H-stability condition (H). As given there ([4]; see also [14, Section 4.9]), the equivalence is proved by showing that each one of these conditions implies the next, and the circle is closed by finally showing that (H) implies L_2-stability.

A further study of the equivalence between L_2-stability and the resolvent condition, each of which was found to be valuable in its own right (e.g. [5, 12, 16]), is given in Section 2. Using a completely different approach than the one taken in Kreiss's matrix theorem [14, Section 4.9], we give a direct,

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considerably shorter proof of the equivalence between the two conditions, obtaining sharper results regarding their dependence on the typical dimension N. In Section 3 we show that L_2-stability is maintained if and only if strict H-stability holds, where a generalized H-numerical radius is employed instead of the usual H-norm used in Kreiss's matrix theorem, thus generalizing the Lax-Wendroff sufficient (but not necessary) stability criterion [8].

Before turning to discuss our results, we introduce some of the notation which will be used later on.

Let $\mathbb{C}$ denote the field of complex numbers, and let M_N denote the algebra of $N\times N$ complex matrices with identity matrix I. For a vector $u\in\mathbb{C}^N$ we denote its conjugate transpose by u^*, and by saying $H\leqslant J$, H and J being Hermitian matrices in M_N, we mean $u^*Hu\leqslant u^*Ju$ for all vectors u. Given N-dimensional vectors u, v, we denote their H-inner product and norm by

$$(u,v)_H=v^*Hu, \qquad |u|_H^2=(u,u)_H,$$

which reduce, in the special case $H=I$, to the standard notions of Euclidian inner product and norm. Similarily, for a matrix $A\in M_N$, we define its H-numerical radius and H-norm by

$$r_H(A)=\sup_{|u|_H=1}|(Au,u)_H|, \qquad \|A\|_H=\sup_{|u|_H=1}|Au|_H,$$

where in the special case $H=I$, we remain within the standard definitions of numerical radius and L_2-(spectral) norm. In particular we have

$$\|A\|=\|A\|_I=\sup_{|u|=|v|=1}|(Au,v)|, \tag{1.1}$$

as can be easily verified by taking $v=Au/|Au|$. Given a 2π-periodic real valued function $f(\theta)$, then $V[f]$ denotes its total variation over $[0,2\pi]$ and $\hat{f}_p\equiv(2\pi)^{-1}\int_0^{2\pi}f(\theta)e^{ip\theta}\,d\theta$ its pth (complex) Fourier coefficient, where the inequality [17, p. 48]

$$|\hat{f}_p|\leqslant\frac{V[f]}{\pi p}, \qquad p=1,2,\ldots, \tag{1.2}$$

holds.

2. THE RESOLVENT CONDITION

We start with the following definitions.

DEFINITION 2.1 (L_2-stability [14, p. 72]). A family of matrices, **F**, is said to be L_2-stable if there exists a constant (stability constant) $C_A > 0$ such that for all $A \in \mathbf{F}$ and all positive integers p,

$$\| A^p \| \leqslant C_A. \tag{A}$$

DEFINITION 2.2 (The resolvent condition). A family of matrices, **F**, is said to satisfy the resolvent condition if there exists a constant $C_R > 0$ such that for all $A \in \mathbf{F}$ and all complex numbers z with $|z| > 1$, the matrices $zI - A$ are nonsingular and the resolvent estimate

$$\|(zI-A)^{-1}\| \leqslant \frac{C_R}{|z|-1} \tag{R}$$

holds.

The proof that L_2-stability implies the resolvent condition is immediate. Indeed, for a power-bounded matrix A, we consider $(zI-A)^{-1}$, $|z|>1$ (whose existance is assured, since by the von Neumann condition the spectrum of A is contained inside the closed unit disc [14, Section 4.7]), and expand it by power series in A, obtaining [14, Section 4.9]

$$\|(zI-A)^{-1}\| = \left\| \sum_{p=0}^{\infty} A^p z^{-p-1} \right\| \leqslant C_A \sum_{p=0}^{\infty} |z|^{-p-1} = C_A(|z|-1)^{-1}.$$

Thus, the resolvent estimate (R) is satisfied with constant $C_R = C_A$. To prove the converse, however, a much more delicate analysis is required.

Here we note that the technique of verifying the resolvent condition in order to decide L_2-stability has been generally applied in numerical analysis to amplification matrices of *fixed* finite order, i.e., to the Fourier transforms of solution operators [5, 12, 16]. In other cases (e.g., approximations to mixed initial–boundary-value problems [6], time discretizations of spectral methods [3, Section 9]), the crucial question concerns the stability of the solution operators themselves, whose representation is made by a family of finite but *nonuniformly* bounded order. Therefore, of particular interest to us is the dependence of the stability constant C_{A} on the typical dimension N.

Assume that a given family $\mathbf{F}$ of $N \times N$ matrices satisfies the resolvent condition. Then by Kreiss's matrix theorem [14, Section 4.9] it follows, using the intermediate (S) and (H) steps, that $\mathbf{F}$ is L_2-stable with stability constant $C_{\mathrm{A}} \sim C_{\mathrm{R}}^{N^N}$. An extension of that theorem to families of unbounded finite order matrices was given in [2], where using the same intermediate steps it was shown that for each $A \in \mathbf{F}$ with minimal polynomial of degree s, $C_{\mathrm{A}} \sim C_{\mathrm{R}}^{s^s}$. Direct proofs of the implication of L_2-stability by the resolvent condition were given by Morton [12], Miller and Strang (who, in fact, directly proved the stronger strict H-stability) [11], and Miller [10]. The various proofs show the L_2-stability of $\mathbf{F}$ with stability constant $C_{\mathrm{A}} \sim 6^N(N+4)^{5N}$, $C_{\mathrm{A}} \sim N^N$, $C_{\mathrm{A}} \sim e^{9N^2}$, respectively. The proofs are rather involved ones and require sharp estimates on the distribution of the eigenvalues of the matrices $A \in \mathbf{F}$.

The following theorem, whose relatively simple proof is basically a modification of that of Laptev [7], sharpens the stability estimates obtained in all the above mentioned results.

THEOREM 2.1. *Let A be an N-dimensional matrix with minimal polynomial $m(z)$ of degree s, and assume it satisfies the resolvent condition* (R). *Then its powers are bounded by*

$$\| A^p \| \leq \frac{32 e s C_{\mathrm{R}}}{\pi}, \qquad p=0,1,\ldots. \tag{2.1}$$

Proof. Let p be a natural number, r be a real number with $r>1$, and u and v be N-dimensional unit vectors, and consider the (real) functions

$$\phi(\theta)=\mathrm{Re}\left[v^*(re^{i\theta}I-A)^{-1}u\right], \qquad \psi(\theta)=\mathrm{Im}\left[v^*(re^{i\theta}I-A)^{-1}u\right], \tag{2.2}$$

where using Schwartz's inequality and applying our resolvent estimate assumption (R), we have

$$|\phi(\theta)| \leq \frac{C_{\mathrm{R}}}{r-1}, \quad |\psi(\theta)| \leq \frac{C_{\mathrm{R}}}{r-1}, \qquad 0 \leq \theta \leq 2\pi. \tag{2.3}$$

Using the identity

$$(zI-A)^{-1} \equiv \frac{\mathbb{B}(z)}{m(z)}, \qquad \mathbb{B}(z)=\sum_{j=1}^{s}\frac{1}{j!}\frac{d^j[m(z)]}{dz^j}(A-zI)^{j-1}, \tag{2.4}$$

which can be verified by multiplying both sides by $(zI-A)m(z)$ and collecting terms, we find that each of the functions $\phi(\theta)$ and $\psi(\theta)$ is a rational trigonometric function with numerator of degree $2s-1$ and denominator of degree $2s$. Hence, their derivatives are vanishing at the $8s-1$ zeros of trigonometric polynomials of degree $4s-1$. Thus $\phi(\theta)$ and $\psi(\theta)$ have, at most, $8s$ different intervals of monotonicity, and therefore their total variations are bounded by

$$V[\phi] \leqslant 16s \max_{0\leqslant\theta\leqslant 2\pi}|\phi(\theta)|, \qquad V[\psi] \leqslant 16s \max_{0\leqslant\theta\leqslant 2\pi}|\psi(\theta)|. \tag{2.5}$$

Since by our assumption the spectrum of A is contained inside the closed unit disc, we may apply the Cauchy integral formula along the contour $|z|=r>1$ to obtain

$$(A^{p-1}u,v)=r^p\left(\hat{\phi}_p+i\hat{\psi}_p\right). \tag{2.6}$$

Combining (2.6), (1.2), (2.5), and (2.3) gives us

$$\begin{aligned}|(A^{p-1}u,v)| &\leqslant r^p\left(|\hat{\phi}_p|+|\hat{\psi}_p|\right) \leqslant \frac{r^p}{\pi p}(V[\phi]+V[\psi]) \\ &\leqslant \frac{16sr^p}{\pi p}\left(\max_{0\leqslant\theta\leqslant 2\pi}|\phi(\theta)|+\max_{0\leqslant\theta\leqslant 2\pi}|\psi(\theta)|\right) \leqslant \frac{32sr^pC_{\mathrm{R}}}{\pi p(r-1)},\end{aligned} \tag{2.7}$$

and by choosing $r=1+p^{-1}$, we finally conclude

$$\|A^{p-1}\| = \sup_{|u|=|v|=1}|(A^{p-1}u,v)| \leqslant \frac{32esC_{\mathrm{R}}}{\pi}. \qquad \blacksquare \tag{2.8}$$

Since the degree of the minimal polynomial of an $N\times N$ matrix does not exceed N, we may conclude

COROLLARY 2.1. *Let* **F** *be a family of* $N \times N$ *matrices satisfying the resolvent condition. Then* **F** *is* L_2*-stable with stability constant* $C_A \sim N$.

We close this section with some additional remarks.

(1) McCarthy and Schwartz [9] gave a counterexample of a family **F** satisfying the resolvent condition, where the dependence of the stability constant C_A on the dimension N is a logarithemic one. We conjecture that the linear dependence asserted in Corollary 2.1 is the best that can be obtained in the general case.

(2) The resolvent assumption made in Theorem 2.1 can be relaxed by requiring the resolvent estimate (R) to hold only in some *fixed* neighborhood of the unit circle $1<|z|\leqslant 1+\varepsilon$, rather than for all z outside the unit disc. Indeed, take r, $1<r\leqslant 1+\varepsilon$, and follow the proof line by line, but in this case choose for the estimate (2.7) $r=1+\varepsilon p^{-1}$, obtaining

$$\|A^{p-1}\|\leqslant \frac{32e^{\varepsilon}sC_R}{\pi\varepsilon}.$$

(3) Theorem 2.1 can be easily generalized for characterizing families **F** of $N\times N$ matrices which are weakly α-stable ($\alpha\geqslant 0$), namely [14, Section 5.2] those satisfying, for some constant $C_A>0$,

$$\|A^p\|\leqslant C_A p^{\alpha}, \qquad A\in \mathbf{F}, \quad p=0,1,\ldots. \tag{2.9}$$

Assume that **F** is α-stable, then it immediately follows that **F** satisfies a generalized resolvent condition of the form

$$\|(zI-A)^{-1}\|\leqslant \frac{C_R}{(|z|-1)^{\alpha+1}}, \qquad A\in\mathbf{F}, \quad |z|>1. \tag{2.10}$$

In fact, as in the special case $\alpha=0$ discussed above, (2.10) is valid with $C_R=C_A$. That the converse is also true was proved in [1]. An alternative, shorter and sharper, proof of the converse is obtained by repeating the proof of Theorem 2.1, replacing the term $r-1$ in the denominators of the estimates (2.3) (the only estimates whose derivation depends on our resolvent assumption) by $(r-1)^{\alpha+1}$, with a corresponding change for the estimate (2.7), and concluding that

$$\|A^{p-1}\|\leqslant C_A p^{\alpha}, \qquad C_A\equiv\frac{32esC_R}{\pi}.$$

Thus, **F** is α-stable, with a stability constant which grows at most linearily with the dimension N.

3. STRICT H-STABILITY CONDITION

DEFINITION 3.1 (Strict H-stability). A family of matrices, $\mathbf{F}$, is said to be strictly H-stable if there exists a constant $C_H>0$, and for each $A\in\mathbf{F}$ a positive definite Hermitian matrix H, such that

$$\begin{aligned} C_H^{-1} &\leqslant H \leqslant C_H I, \\ \|A\|_H &\leqslant 1. \end{aligned} \tag{H}$$

The strict H-stability condition, whose equivalence to L_2-stability of a family $\mathbf{F}$ of $N\times N$ matrices follows by Kreiss's matrix theorem ([14, Section 4.9]; see also [11]), has been used quite extensively in studying the stability properties of finite difference schemes [14, 15]. It therefore seems worthwhile to note another, somewhat milder, L_2-stability characterization of this type. For that purpose, let us first state

LEMMA 3.1 (Generalized Halmos inequality). *Given an N-dimensional matrix A, we have*

$$r_H(A^p)\leqslant r_H^p(A), \qquad p=0,1,\dots. \tag{3.1}$$

Proof. Let $H=T^*T$ for some nonsingular matrix $T\in M_N$. Applying the Halmos inequality [13], we obtain

$$r_H(A^p)=r(TA^pT^{-1})=r\big([TAT^{-1}]^p\big)\leqslant r^p(TAT^{-1})=r_H^p(A).$$

Alternatively, one may repeat Pearcy's proof [13], noting its independence of the inner product being employed; in particular, for the H-inner product, (3.1) follows. ∎

Lemma 3.1 enables us to prove

THEOREM 3.1. *A family $\mathbf{F}$ of $N\times N$ matrices is L_2-stable if and only if there exists a constant $C>0$, and for each $A\in\mathbf{F}$ a positive definite Hermitian matrix H, such that*

$$C^{-1}I\leqslant H\leqslant CI, \tag{3.2a}$$

$$r_H(A)\leqslant 1. \tag{3.2b}$$

Proof. For a positive definite Hermitian matrix $H=T^*T$, T a nonsingular matrix with $\|T^k\|=\|H^k\|^{1/2}$ for $k=\pm 1$, we have

$$r_H(A)=r(TAT^{-1})\leqslant\|TAT^{-1}\|=\|A\|_H \tag{3.3}$$

and

$$\begin{aligned}\|A\|&\leqslant\|T^{-1}\|\cdot\|T\|\cdot\|TAT^{-1}\|\leqslant\|T^{-1}\|\cdot\|T\|2r(TAT^{-1})\\&=2\|H^{-1}\|^{1/2}\cdot\|H\|^{1/2}r_H(A).\end{aligned} \tag{3.4}$$

Now the "only if" part of the theorem follows by Kreiss's matrix theorem together with (3.3). For the converse we apply (3.4) and (3.1), which, combined with our assumption (3.2), yield

$$\|A^p\|\leqslant 2Cr_H(A^p)\leqslant 2Cr_H^p(A)\leqslant 2C, \qquad p=0,1,2,\dots. \quad \blacksquare \tag{3.5}$$

For the special case $H=I$ Theorem 3.1 is reduced to the sufficient (but not necessary) stability criterion due to Lax and Wendroff [8], namely, $r(A)\leqslant 1$.

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