THE RESOLVENT CONDITION AND UNIFORM POWER-BOUNDEDNESS

by EITAN TADMOR[34]

Let L be an operator with uniformly bounded powers:

$$\|L^k\| \leqslant M_P, \qquad k=1,2,\ldots. \tag{P}$$

Using the geometric expansion for the resolvent of such an operator, $(zI - L)^{-1}$, it follows that

$$\left\|(zI-L)^{-1}\right\| \leqq \frac{M_R}{|z|-1} \qquad \text{for all} \quad |z|>1, \tag{R}$$

with constant $M_R = M_P$.

In this talk we discuss the *inverse implication* of the above, namely, the power-boundedness of operators which satisfy the *resolvent condition* (R).

We begin with the finite-dimensional case, considering *families of matrices*. Thus, suppose L is given as a direct sum of finite-dimensional operators, their dimension being *uniformly* bounded, say $\leqslant N$. Then (R) $\Rightarrow$ (P) is in fact just one of the four implications contained in the Kreiss matrix theorem [6] which was subsequently treated by many authors, including [2–4], [7], [10–12], [14]. A simple derivation of this, which led to a power estimate sharper than the previous ones, was given at [15], asserting

$$\|L^k\| \leqslant \text{const}_R \cdot N, \qquad k=1,2,\ldots,$$

with the linear dependence on the dimension N being the best possible [8].

Turning to the infinite-dimensional case, we first note—using an argument due to Sz.-Nagy [13]—that *compact* operators satisfying (R) are necessarily similar to contractions:

$$\|TLT^{-1}\| \leqslant 1,$$

and hence are power-bounded with $M_p = \|T\| \cdot \|T^{-1}\|$. (The existence of such similarity in the finite-dimensional setup was proved in [6], [11].)

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Noncompact counterexamples of Foguel [1] and Halmos [5] indicate that the powers of general operators satisfying (R) may grow. How fast is the power growth permitted by the resolvent condition? An easy application of the Cauchy integral formula yields a *linear* upper bound:

$$\|L^n\| \leqslant e \cdot M_R \cdot n, \qquad n = 1,2,\dots .$$

In some cases this estimate can be improved on the basis of the following

LEMMA [16]. *Suppose L satisfies the resolvent condition* (R). *Let* d_n *denote the following minimax quantity*:

$$d_n = \min_{\zeta} \max_{\eta} |\zeta - \eta| \cdot \left\| \left(\left(1 + \frac{1}{n} \right) e^{i\eta} - L \right)^{-1} \right\| .$$

Then the following estimate holds:

$$\|L^n\| \leqslant \mathrm{const}_R \cdot d_n \log n, \qquad n = 2,3,\dots .$$

The last result yields a *logarithmic power growth* provided the spectrum of L is not "too dense" in the neighborhood of the unit circle. One such case is the dissipative case, where instead of (R) we have the stronger *dissipativity condition*

$$\|(zI - L)^{-1}\| \leqslant \frac{M_D}{|z-1|} \qquad \text{for all} \quad |z| > 1. \tag{D}$$

S. Friedland (private communication) has given an alternative proof of the logarithmic growth in this case. The same estimate applies if there is a *finite* number of simple poles on the unit circle. Finally, we give a counterexample satisfying (R), [9], with an *unbounded* number of such poles and with a logarithmic power growth.

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AN EFFICIENT PRECONDITIONING ALGORITHM AND ITS ANALYSIS

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Introduction

The purpose of this work is to suggest and analyze a new preconditioning for solving sparse linear systems, which is readily vectorized and very efficient for matrices arising from a finite-difference discretization of partial

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