

ESSENTIALLY NON-OSCILLATORY SPECTRAL VISCOSITY APPROXIMATIONS ¹

Eitan Tadmor
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv 69978 ISRAEL

Dedicated with appreciation to Heinz-Otto Kreiss

ABSTRACT

We study the approximate solution of nonlinear conservation laws by spectral methods. We show that spectral viscosity approximations of such equations are total-variation bounded. Moreover, they are upper-Lipschitz continuous, in agreement with Oleinik's E-entropy condition. It follows that the spectral viscosity approximations converge to the corresponding inviscid entropy solution and we prove convergence rate estimates.

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1. THE SPECTRAL APPROXIMATION

We are concerned here with spectral approximations of the scalar, genuinely nonlinear conservation law

$$(1.1a) \quad \frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u(x, t)) = 0, \quad f'' \geq \alpha > 0.$$

To single out a unique physically relevant weak solution, (1.1a) is augmented with the entropy condition

$$(1.1b) \quad \frac{\partial}{\partial t} \frac{u^2}{2} + \frac{\partial}{\partial x} F(u) \leq 0, \quad F(u) \equiv \int^u \xi f(\xi) d\xi.$$

Let $S_N u$ and $\psi_N u$ denote, respectively, the spectral-Fourier and the ψ -spectral-Fourier projections of $u(x, t)$, and let $P_N u$ stands for either one of these two projections. In order to solve the 2π -periodic initial-value problem (1.1a),(1.1b) by (pseudo-) spectral methods, we use an N -trigonometric polynomial

$$u_N(x, t) = \sum_{k=-N}^N \hat{u}_k(t) e^{ikx}$$

to approximate $P_N u$. Starting with $u_N(x, 0) = P_N u_0(x)$, the standard Fourier method, e.g., [Kr-Ol],[Go-Or],[CHQZ], lets $u_N(x, t)$ evolves according to the $(2N + 1)$ -dimensional approximate model

$$(1.2) \quad \frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} P_N f(u_N) = 0.$$

Although the spectral method (1.2) is spectrally accurate approximation of the conservation law (1.1), i.e., its local error does not exceed

$$(1.3) \quad \|(I - P_N)f(u_N)\|_{H^{-s}} \leq Const \cdot N^{-s} \|u_N\|_{L^2}, \quad \text{for any } s \geq 0,$$

the spectral solution, $u_N(x, t)$, need not approximate the corresponding entropy solution, $u(x, t)$. The following example shows what could go wrong.

Counterexample: [Ta1]. The spectral-Fourier approximation of the scalar equation (1.1a) reads

$$(1.4) \quad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} S_N f(u_N(x, t)) = 0.$$

Multiplying this by $u_N(x, t)$ and integrating over the 2π -period, we obtain that u_N -being orthogonal to $\frac{\partial}{\partial x}(I - S_N)f(u_N(x, t))$, satisfies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^{2\pi} u_N^2(x, t) dx &= - \int_0^{2\pi} u_N(x, t) \frac{\partial}{\partial x} f(u_N(x, t)) dx = \\ &= -F(u_N(x, t)) \Big|_{x=0}^{x=2\pi} = 0. \end{aligned}$$

Thus, the total quadratic entropy is conserved in time

$$(1.5) \quad \frac{1}{2} \int_0^{2\pi} u_N^2(x, t) dx = \frac{1}{2} \int_0^{2\pi} u_N^2(x, 0) dx,$$

which in turn yields the existence of a weak $L^2(x)$ -limit $\bar{u}(x, t) = w \lim_{N \rightarrow \infty} u_N(x, t)$. Yet, $\bar{u}(x, t)$ cannot be the entropy solution of a nonlinear equation (1.1a). Otherwise, $S_N f(u_N(x, t))$ and therefore $f(u_N(x, t))$ should tend in the weak distribution sense to $f(\bar{u}(x, t))$; consequently, since $f(u)$ is nonlinear, $\bar{u}(x, t) = s \lim_{N \rightarrow \infty} u_N(x, t)$, which by (1.5) should satisfy $\frac{1}{2} \int_0^{2\pi} \bar{u}^2(x, t) dx = \frac{1}{2} \int_0^{2\pi} \bar{u}^2(x, 0) dx$. But this is incompatible with the entropy condition (1.1b) if $\bar{u}(x, t)$ contains shock discontinuities. \square

The last example shows that the spectral method lacks *entropy dissipation*, which is inconsistent with the augmenting entropy condition (1.1b).

One of the main disadvantages of using spectral methods for nonlinear conservation laws lies in the formation of Gibbs phenomena, once spontaneous shock discontinuities appear in the solution.

The global nature of spectral methods then pollutes the unstable Gibbs oscillations overall the computational domain and the lack of entropy dissipation prevents the convergence of spectral approximations in these cases. In the next Section we discuss the convergence of the Spectral Viscosity method, which attempts to stabilize the Gibbs oscillations and consequently to guarantee the convergence of spectral methods, without sacrificing their overall spectral accuracy.

2. THE SPECTRAL VISCOSITY APPROXIMATION

We consider the Spectral Viscosity (SV) approximation

$$(2.1) \quad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} P_N f(u_N(x, t)) = \varepsilon_N \frac{\partial}{\partial x} Q_N * \frac{\partial}{\partial x} u_N(x, t),$$

subject to initial conditions

$$(2.2) \quad u_N(x, 0) = P_N u_0(x).$$

The left-hand side of (2.1) is the standard spectral approximation of (1.1). It is augmented, on the right-hand side of (2.1), by spectral viscosity with vanishing amplitude $\varepsilon_N \downarrow 0$. Here $Q_N(x, t) = \sum_{|k|=m_N}^N \hat{Q}_k(t) e^{ikx}$ is a viscosity kernel activated only on high-frequencies, say with wavenumbers $|k| \geq m_N$, which can be conveniently implemented in the Fourier space as

$$\varepsilon_N \frac{\partial}{\partial x} Q_N * \frac{\partial}{\partial x} u_N(x, t) \equiv -\varepsilon_N \sum_{|k|=m_N}^N k^2 \hat{Q}_k(t) \hat{u}_k(t) e^{ikx}.$$

We consider real symmetric viscosity kernels, $Q_N(x, t)$, with smoothly increasing Fourier coefficients, $\hat{Q}_k \equiv \hat{Q}_{|k|}(t)$, which satisfy

$$(2.3)_q \quad \hat{Q}_k(t) \geq 1 - \left(\frac{m_N}{|k|} \right)^q, \quad |k| \geq m_N, \quad \text{for some fixed } q \geq 2,$$

and we let the spectral viscosity parameters, (ε_N, m_N) , lie in the range

$$(2.4) \quad \varepsilon_N \sim \frac{1}{N^\theta}, \quad m_N < \text{Const} \frac{N^{\frac{\theta}{4}}}{\log N}, \quad \theta < 1.$$

Remark that this choice of spectral viscosity parameters is small enough to retain the formal spectral accuracy of the overall approximation, since

$$\|\varepsilon_N Q_N * \frac{\partial}{\partial x} u_N(\cdot, t)\|_{H^{-s}} \leq \text{Const} \cdot N^{-\frac{\theta}{4}s} \|u_N(\cdot, t)\|_{L^2}, \quad \text{for any } s \geq 0.$$

At the same time, it is sufficiently large to enforce the correct amount of entropy dissipation that is missing otherwise, when either $\varepsilon_N = 0$ or $m_N = N$. Indeed, it was shown in [Ta1],[Ma-Ta],[Ta2] that the SV approximation (2.1)-(2.4) has a bounded entropy production in the sense that

$$(2.5) \quad \varepsilon_N \left\| \frac{\partial}{\partial x} u_N(x, t) \right\|_{L_{loc}^2(x,t)}^2 \leq \text{Const}.$$

This together with an L^∞ -bound, e.g., [Ma-Ta],[Sc],[Ta2], imply the convergence of the SV approximation (2.1) - (2.4) by compensated compactness arguments.

In the next sections we shall deal with the convergence of the SV approximation by compactness arguments. Specifically, we show that the total-variation of the SV approximation is bounded uniformly w.r.t N . Moreover, the SV solutions are shown to be upper-Lipschitz continuous, in agreement with Oleinik's E-entropy condition. We conclude that the SV approximation converges to the entropy solution of (1.1) and we estimate the convergence rate.

3. A TOTAL-VARIATION BOUND

The presence of spectral viscosity on the right of (2.1) is responsible for a rapid decay of the Fourier coefficients located toward the end of the computed spectrum. This spectral decay result was proved in [Ma-Ta] for the special case of Burgers' equation where $f(u) = \frac{1}{2}u^2$, using an argument of [Kr]. The general case was analyzed in [Sc], where it was shown that the following spectral decay estimate,

$$(3.1) \quad \|(I - P_N)f(u_N(\cdot, t))\| \leq \text{Const}_s \cdot [N^{-s(1-\theta)} + N^{-r}e^{-Nt}],$$

holds, [Sc, Theorem 1]. Here, $s \geq 0$, is restricted only by the degree of smoothness of $f(\cdot)$, and $r \geq 0$ is determined by the smoothness of the initial conditions u_0 .

The last estimate shows that after a brief initial time interval (depending on how smooth u_0 is), the discretization error $(I - P_N)f(u_N)$ becomes spectrally small, independently whether or not the underlying entropy solution is smooth. We conclude that modulo the spectrally small error (3.1) which we ignore, the SV approximation is governed by the viscosity equation,

$$(3.2) \quad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} f(u_N(x, t)) = \varepsilon_N \frac{\partial}{\partial x} Q_N * \frac{\partial}{\partial x} u_N(x, t).$$

Therefore, it is enough to concentrate on total-variation and error estimates for (3.2), which we rewrite in the equivalent form

$$(3.3a) \quad \begin{aligned} \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} f(u_N(x, t)) &= \\ &= \varepsilon_N \frac{\partial^2}{\partial x^2} u_N(x, t) - \varepsilon_N \frac{\partial}{\partial x} R_N(x, t) * \frac{\partial}{\partial x} u_N(x, t), \end{aligned}$$

where

$$(3.3b) \quad R_N(x, t) = \sum_{k=-N}^N \hat{R}_k(t) e^{ikx}, \quad \hat{R}_k(t) \equiv \begin{cases} 1 & |k| < m_N, \\ 1 - \hat{Q}_k(t) & |k| \geq m_N. \end{cases}$$

Equation (3.3a) closely resembles the usual viscosity approximation of (1.1a), apart from the 'residual' kernel $R_N(x, t)$ on its RHS. The following lemma collects several a priori estimates taken from [Ta4], which show that $R_N(x, t)$ is 'sufficiently small'.

LEMMA 3.1. *Consider the SV kernel $Q_N(x, t)$ subject to the SV parameterization (2.3)_q–(2.4). Then $R_N(x, t) \equiv D_N(x) - Q_N(x, t)$ satisfies*

$$(3.4)_s \quad \begin{aligned} \left\| \frac{\partial^s}{\partial x^s} R_N(\cdot, t) \right\|_{L^1} + \left\| \frac{\partial^{s-1}}{\partial x^{s-1}} R_N(\cdot, t) \right\|_{L^\infty} &\leq \\ &\leq \text{Const} \cdot m_N^s \log m_N, \quad s \leq q. \end{aligned}$$

Equipped with Lemma 3.1 one can show that the SV approximation (2.1) has a total-variation bounded solution. Indeed, apart from a spectrally small truncation error (3.1) which is ignored, one may proceed with a total-variation estimate of (3.3a) in a standard fashion: we differentiate (3.3a) w.r.t. x , integrate against $\text{sgn}[\frac{\partial}{\partial x} u_N(x, t)]$, and in view of (3.4)₂ we obtain

$$\begin{aligned} \frac{d}{dt} \|u_N(\cdot, t)\|_{BV} &\leq \varepsilon_N \left\| \frac{\partial^2}{\partial x^2} R_N(\cdot, t) \right\|_{L^1} \|u_N(\cdot, t)\|_{BV} \leq \\ &\leq \text{Const}_N \cdot \|u_N(\cdot, t)\|_{BV}, \quad \text{Const}_N \sim \varepsilon_N m_N^2 \log m_N. \end{aligned}$$

taking into account (2.4), we find that $\text{Const}_N < \mathcal{O}\left(\frac{1}{\log N}\right)$, and we conclude

COROLLARY 3.2. *The SV approximation (2.1)–(2.4) is essentially non-oscillatory, in the sense that the increase of its initial total-variation is $o(1)$,*

$$(3.5) \quad \|u_N(\cdot, t)\|_{BV} \leq \left[1 + \mathcal{O}\left(\frac{1}{\log N}\right) \right] \|u_N(\cdot, 0)\|_{BV}.$$

4. CONVERGENCE RATE ESTIMATES

We say that $\{u_N(x, t)\}$ are *Lip⁺-stable*, if there exists a constant (independent of N), such that the following estimate is fulfilled²:

$$(4.1) \quad \|u_N(\cdot, t)\|_{Lip^+} \leq \text{Const}_T, \quad 0 \leq t \leq T.$$

Recall that the entropy solution of the nonlinear conservation law (1.1) with $f'' \geq \alpha > 0$ is identified by Oleinik's a priori estimate e.g., [La],[Ta3],

$$(4.2) \quad \|u(\cdot, t)\|_{Lip^+} \leq \frac{1}{\|u_0\|_{Lip^+}^{-1} + \alpha t}, \quad t \geq 0.$$

In particular, the entropy solution of (1.1) is *Lip⁺-stable*, as long as its initial conditions u_0 are *Lip⁺-bounded*.

We want to show that the SV approximation (2.1) is *Lip⁺-stable*.

First remark that the BV boundedness of the SV solution, (3.5), does not exclude the possibility of small high-frequencies oscillations. (By conservation, *Lip⁺-stability* implies BV-stability but not vice-versa). Such 'unphysical' oscillations violate the *Lip⁺-stability* condition (4.1). In order to prevent such unstable oscillations, we therefore need to slightly increase the amount of spectral viscosity. We do this without sacrificing formal spectral accuracy, requiring the spectral viscosity parameters to lie in the range (2.3)_q – (2.4) with $q \geq 3$.

As before, we ignore the spectrally small error (3.1) and turn to

²We let $\|\phi\|_{Lip}$, $\|\phi\|_{Lip^+}$ and $\|\phi\|_{Lip'}$ denote respectively, $\text{esssup}_{x \neq y} \left| \frac{\phi(x) - \phi(y)}{x - y} \right|$, $\text{esssup}_{x \neq y} \left[\frac{\phi(x) - \phi(y)}{x - y} \right]_+$ and $\text{sup}_{\psi} \frac{(\phi - \phi_0, \psi)}{\|\psi\|_{Lip}}$.

consider the viscosity equation (3.3a). A straightforward estimate of the latter, carried out in [Ta4] yields

$$(4.3a) \quad \frac{d}{dt} \|u_N(\cdot, t)\|_{Lip^+} + \alpha \|u_N(\cdot, t)\|_{Lip^+}^2 \leq c_N \|u_N(\cdot, t)\|_{Lip^+},$$

where according to (3.4)₃

$$(4.3b) \quad c_N \sim \varepsilon_N \left\| \frac{\partial^2}{\partial x^2} R_N(\cdot, t) \right\|_{L^\infty} \sim \varepsilon_N m_N^3 \log N < \frac{1}{(\log N)^2}.$$

Integration of (4.3) then yields the desired *Lip⁺-stability*, which we state as

LEMMA 4.1. *The SV solution of (2.1) – (2.2), (2.3)_q – (2.4) with $q \geq 3$ satisfies the a priori estimate*

$$(4.4) \quad \begin{aligned} \|u_N(\cdot, t)\|_{Lip^+} &\leq \frac{e^{c_N t}}{\|u_N(\cdot, 0)\|_{Lip^+}^{-1} + \frac{\alpha}{c_N}(e^{c_N t} - 1)} \leq \\ &\leq \frac{e^{c_N t}}{\|u_N(\cdot, 0)\|_{Lip^+}^{-1} + \alpha t}, \quad c_N \downarrow 0, \end{aligned}$$

in close agreement with (4.2). Thus the SV approximation (2.1)–(2.4) is *Lip⁺-stable*.

Next, we recall the main result of [Ne-Ta] which deals with the convergence rate of *Lip⁺-approximations*.

THEOREM 4.2. *Let $\{u_N(x, t), 0 \leq t \leq T\}$ be a family of *Lip⁺-stable approximate solutions of the conservation law (1.1), with *Lip⁺-bounded initial conditions. Assume that $\{u_N(x, t)\}$ are *Lip[']-consistent of order ε , i.e.,****

$$\|u_N(x, 0) - u_0(x)\|_{Lip'} + \left\| \frac{\partial}{\partial t} u_N(\cdot, t) + \frac{\partial}{\partial x} f(u_N(\cdot, t)) \right\|_{Lip'} \leq \text{Const}_T \cdot \varepsilon.$$

Then the following error estimates hold

$$\|u_N(\cdot, t) - u(\cdot, t)\|_{W^{-s,p}} \leq \text{Const}_T \cdot \varepsilon^{\frac{sp+1}{2p}}, \quad 1 \leq p < \infty, \quad s = 0, 1,$$

$$|u_N(x, t) - u(x, t)| \leq \text{Const}_T \cdot (1 + |u_x(\cdot, t)|_{loc}^{\frac{2}{3}}) \varepsilon^{\frac{1}{3}}.$$

Here $|u_x(\cdot, t)|_{loc} \equiv \|u_x(\cdot, t)\|_{L^\infty(x-\varepsilon^{\frac{1}{3}}, x+\varepsilon^{\frac{1}{3}})}$ measures the local smoothness of the entropy solution in the $\mathcal{O}(\varepsilon^{\frac{1}{3}})$ -neighborhood of x .

It remains to estimate the *Lip'*-size of the truncation error on the right of (3.2). By (3.4)₀, the viscosity kernel

$$Q_N(\cdot, t) \equiv D_N(\cdot) - R_N(\cdot, t),$$

has an L^1 -norm which does not exceed $\text{Const} \cdot \log N$. Hence, the SV approximation (2.1) is *Lip'*-consistent with the conservation law (1.1) of order $\varepsilon \sim \varepsilon_N \log N = N^{-\theta} \log N$, for

$$(4.5) \quad \begin{aligned} \|\varepsilon_N \frac{\partial}{\partial x} Q_N * \frac{\partial}{\partial x} u_N(\cdot, t)\|_{Lip'} &\leq \varepsilon_N \|Q_N(\cdot, t)\|_{L^1} \|u_N(\cdot, t)\|_{BV} \leq \\ &\leq \text{Const} \cdot N^{-\theta} \log N \|u_N(\cdot, 0)\|_{BV}. \end{aligned}$$

We may use now Theorem 4.2 to conclude

THEOREM 4.3. Consider the 2π -periodic conservation law (1.1) with smooth initial-data. Then the SV approximation (2.1)-(2.2), (2.3)_q- (2.4) with $q \geq 3$ converges to the entropy solution of (1.1), and the following error estimates hold for $0 < t \leq T$,

$$(4.6) \quad \|u_N(\cdot, t) - u(\cdot, t)\|_{W^{-s,p}} \leq \text{Const}_T \cdot (N^{-\theta} \log N)^{\frac{sp+1}{2p}},$$

$$(4.7) \quad |u_N(x, t) - u(x, t)| \leq \text{Const}_T \cdot (1 + |u_x(\cdot, t)|_{loc}^{\frac{2}{3}}) N^{-\frac{\theta}{3}} \log N,$$

Remarks.

1. According to (4.7), the pointwise convergence rate of the SV solution in smooth regions of the entropy solution is of order

$\sim N^{-\frac{1}{3}}$. Moreover, by post-processing the SV solution this convergence rate can be made arbitrarily close to N^{-1} , consult [Ta4], [Ne-Ta]. In fact, numerical experiments reported in [Ta2] show that by post-processing the SV solution, we recover the pointwise values in smooth regions of the entropy solution within spectral accuracy.

2. According to (4.6) with $(s, p) = (0, 1)$, the SV approximation has an L^1 -convergence rate of order $\sim N^{-\frac{1}{2}}$ in agreement with [Sc, §5]. This corresponds to the usual L^1 -convergence rate of order $\frac{1}{2}$ for monotone difference approximations, [Ku],[Sa].

3. The error estimates (4.6),(4.7) are not uniform in time. For arbitrary *Lip*⁺-bounded initial data u_0 an initial layer can be formed, after which the spectral viscosity becomes effective and guarantees the spectral decay of the discretization error indicated earlier. This can be avoided if we pre-process the initial data for the SV approximation by the de la Vallee Poussin's filter without sacrificing the formal spectral accuracy of the approximation, e.g., [Sc].

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