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CONVERGENCE OF THE SPECTRAL VISCOSITY METHOD FOR NONLINEAR CONSERVATION LAWS

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1. INTRODUCTION. In recent years, spectral methods have become one of the standard tools for the approximate solution of nonlinear conservation laws, e.g., [3]. It is well known that the spectral methods enjoy high order of accuracy whenever the underlying solution is smooth. On the other hand, one of the main disadvantages of using spectral methods for nonlinear conservation laws, lies in the formation of Gibbs phenomena, once spontaneous shock discontinuities appear in the solution; the global nature of spectral methods then pollutes the unstable Gibbs oscillations overall the computational domain and prevent the convergence of spectral approximation in these cases. One of the standard techniques to mask the oscillatory behavior of spectral approximations is based on spectrally accurate post-processing of these approximations. Indeed, the convergence of such recovery techniques can be justified by linear arguments [1], [5]. However, for nonlinear problems we show by a series of prototype counterexamples, that spectral solutions with or without such post-processing techniques, do not converge to the correct 'physically' entropy solutions of the conservation laws. The main reason for this failure of convergence of spectral methods is explained by their lack of entropy dissipation.

A similar situation which involves unstable oscillations, is encountered with finite-difference approximations to nonlinear conservation laws. Here, the problem of oscillations is usually solved by the so called vanishing viscosity method. Namely, one adds artificial viscosity, such that on the one hand it retains the formal accuracy of the basic scheme, while on the other hand, it is sufficient to stabilize the Gibbs oscillations. The Spectral Viscosity Method (SVM) proposed in [6] attempts, in an analogous way, to stabilize the Gibbs oscillations and consequently to guarantee the convergence of spectral approximations, by augmenting them with proper viscous modifications.

2. THE SPECTRAL VISCOSITY METHOD. We consider one-dimensional system of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial (f(u))}{\partial x} = 0, \tag{2.1}$$

with prescribed initial data $u_0(x)$. We restrict our attention to the periodic initial-value problem (2.1) which avoids the separate question of boundary treatment. To solve this 2π-periodic problem by spectral methods, we use an N-trigonometric polynomial

$$u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx},$$

in order to approximate the spectral or pseudospectral Fourier projection $P_N u$ of the exact solution. Starting with $u_N(x,0) = P_N u_o(x)$, the classical Fourier method lets $u_N(x,t)$ evolves at later time according to the approximate model

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N(f(u_N(x,t)))] = 0. \tag{2.2}$$

We have already noted that the convergence to the entropy solution of (2.1), $u_N \underset{N\to\infty}{\longrightarrow} u$ may (and in fact, in some cases must) fail. Instead, we propose to replace (2.2) by appending to it a spectrally accurate vanishing viscosity modification which amounts to

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N(x,t))] = \varepsilon \frac{\partial}{\partial x}\left[Q_m(x,t) * \frac{\partial u_N}{\partial x}\right]. \tag{2.3}$$

Here, $Q_m(x,t)$ is a viscosity kernel of the form

$$Q_m(x,t) = \sum_{m\le|k|\le N} \hat{Q}_k(t) e^{ikx}. \tag{2.4}$$

This kind of spectral viscosity can be efficiently implemented in the Fourier rather than in the physical space, i.e.,

$$\varepsilon \frac{\partial}{\partial x}\left[Q_m(x,t) * \frac{\partial u_N}{\partial x}\right] \equiv -\varepsilon \sum_{m\le|k|\le N} k^2 \hat{Q}_k(t) \hat{u}_k(t) e^{ikx}. \tag{2.5}$$

It depends on two free parameters: the viscosity amplitude, $\varepsilon = \varepsilon(N)$, and the effective size of the inviscid spectrum, $m = m(N)$. In [6] it was shown that these two parameters can be chosen so that we have both, i.e., the spectral viscosity retains the spectral accuracy of the overall approximation as $m(N) \uparrow \infty$, and at the same time, it is sufficient to enforce the correct amount of entropy dissipation that is missing otherwise (with $\varepsilon = 0$).

Entropy dissipation is necessary for convergence; the lack of such dissipation in the classical Fourier methods is the main reason for its divergence. On the other hand, under appropriate assumptions, one can use compensated compactness arguments [8] to show that entropy dissipation induced by the SV method is sufficient for convergence.

3. CONVERGENCE. We shall discuss two model problems.

Example 3.1 The scalar case. We consider general nonlinear scalar conservation laws (2.1). The pseudospectral viscosity method collocated at the equidistant points $x_\nu = \frac{2\pi\nu}{N}$, takes the form

$$\frac{\partial}{\partial t} u_N(x_\nu,t) + \frac{\partial}{\partial x}[P_N f(u_N(x_\nu,t))] = -\frac{1}{N}\sum_{|k|=\sqrt{N}}^{N} \frac{(k-\sqrt{N})^2}{N-\sqrt{N}} k^2 \hat{u}_k(t) e^{ikx_\nu}. \tag{4.1}$$

It can be shown, [7], that the spectral viscosity on the right guarantees entropy dissipation and the L^∞-stability of the overall approximation; consequently convergence follows.

Example 3.2 The isenotropic gas dynamics equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x &= 0 \end{aligned} \tag{4.2}$$

for a polytropic gas $p = Const.\rho^\gamma$, $\gamma < 1$ are approximated in a similar fashion. Under appropriate L^∞-stability assumption (in particular, a uniform bound from the vacuum state), it can be shown [3], [7] that the spectral viscosity solution converges.

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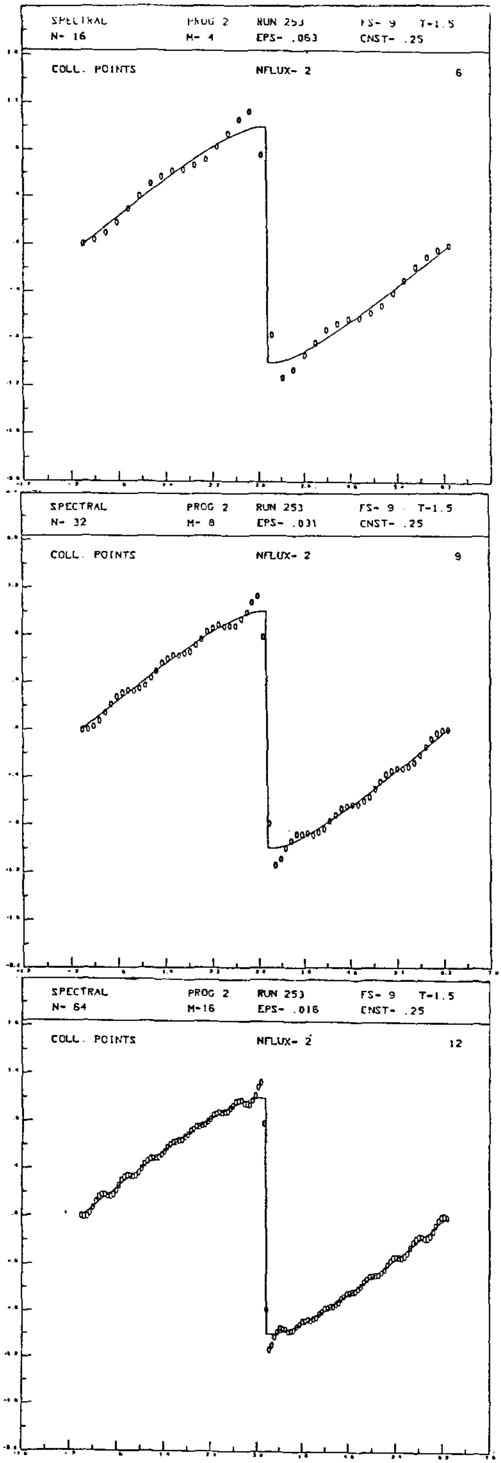
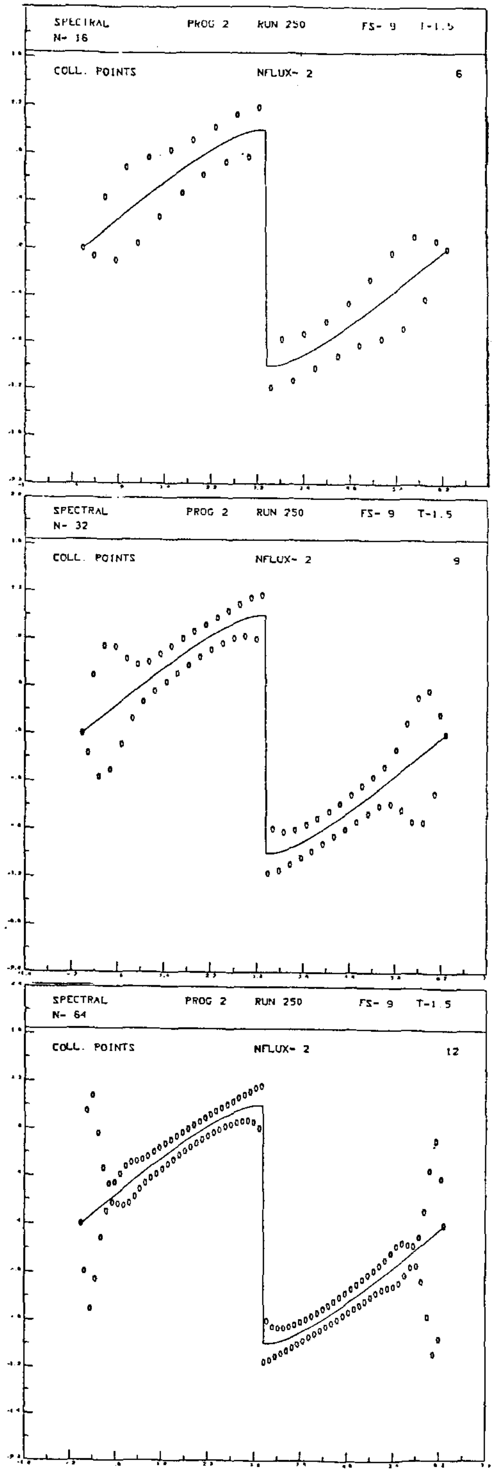
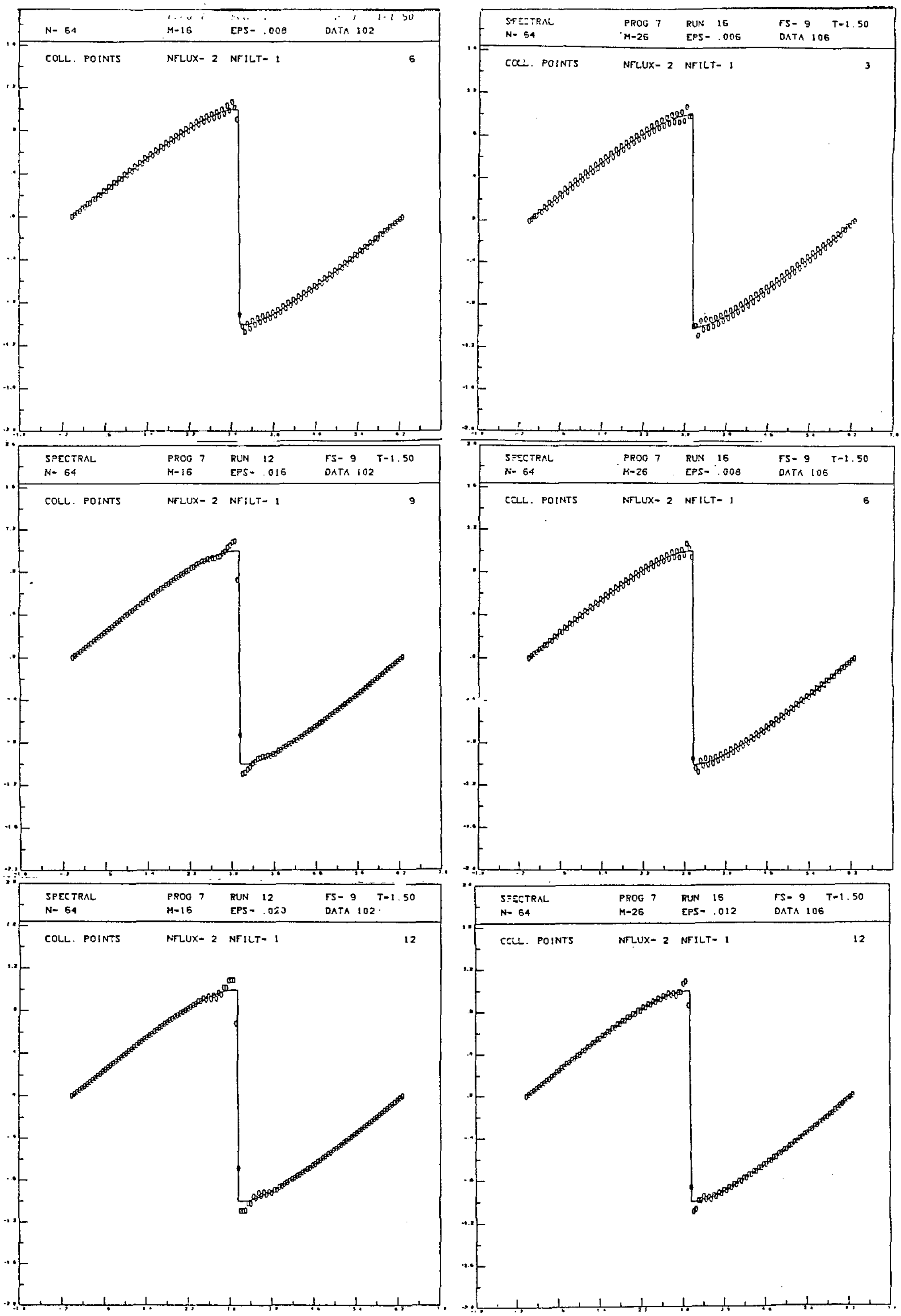
The Fourier method with spectral vanishing viscosity ... and without spectral viscosity.



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