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Super Viscosity and Spectral Approximations of Nonlinear Conservation Laws

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Dedicated with friendship to Saul Abarbanel on his 60th birthday

1 Introduction

Let P_N stands for one of the standard spectral projections — Fourier, Chebyshev, Legendre .... It is well known that such spectral projections, $P_N u$, provide highly accurate approximations for sufficiently smooth u's. This superior accuracy is destroyed if u contains discontinuities. Indeed, $P_N u$ produces $\mathcal{O}(1)$ Gibbs' oscillations in the *local* neighborhoods of the discontinuities, and moreover, their *global* accuracy deteriorates to first-order.

We are interested in spectral approximations of nonlinear conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \tag{1.1}$$

subject to initial conditions, $u(x,0) = u_0$, and augmented with appropriate boundary conditions. The purpose of a spectral method is to compute an approximation to the *projection* of $u(\cdot,t)$ rather than $u(\cdot,t)$ itself. Consequently, since nonlinear conservation laws exhibit spontaneous shock discontinuities, the spectral approximation faces two difficulties:

- Stability. Numerical tests indicate that the convergence of spectral approximations to nonlinear conservation laws fails. In [T2]–[T4] we prove that this failure is related to the fact that spurious Gibbs oscillations pollute the entire computational domain, and that the lack of entropy dissipation then renders these spectral approximations unstable.

- Accuracy. The accuracy of the spectral computation is limited by the first order convergence rate of $P_N u(\cdot,t)$.

With this in mind we turn to discuss in §2 the Spectral Viscosity (SV) method introduced in [T2]. In this paper we restrict our attention to periodic problems. For a treatment of the nonperiodic case in terms of the Legendre SV method we refer to [MOT].

The purpose of the SV method is to stabilize the nonlinear spectral approximation without sacrificing its underlying spectral accuracy. This is achieved by augmenting the standard spectral approximation with high frequency regularization. In §3 we briefly review the convergence results of the periodic Fourier SV method, [T2]–[T5], [MT], [CDT], [S]. These convergence results employ high frequency regularization based on *second order* viscosity. In §4 we introduce spectral approximations based on "super-viscosity", i.e., high-frequency parabolic regularizations of order > 2. We prove the H^{-1}-stability of these spectral "super-viscosity" approximations, and together with L^∞-stability, convergence follows by compensated compactness arguments [Tr],[M].

We close this Introduction by referring to the numerical experiments reported in [T4],[MOT]. These numerical tests show that by *post-processing* the spectral (super)-viscosity approximation, the exact entropy solution is recovered within spectral accuracy. This post-processing is carried out as a highly accurate mollification and operated either in the physical space as in [GT],[AGT],[MOT], or in the dual Fourier space as in [KO],[MMO],[V]. It should be emphasized that the role of post-processing is essential in order to realize the highly accurate content of the SV solution.

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2 The Fourier spectral viscosity method

To solve the periodic conservation law (1.1) by a spectral method, one employs an N-degree trigonometric polynomial

$$u_N(x,t) = \sum_{|k|\le N} \widehat{u}_k(t)e^{ikx} ,$$

in order to approximate the Fourier projection of the exact entropy solution, $P_N u$.[1] Starting with $u_N(x,0) = P_N u_0(x)$, the classical spectral method lets $u_N(x,t)$ evolve according to the approximate model

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}\left[P_N\big(f(u_N)\big)\right] = 0 . \tag{2.1}$$

As we have already noted, the convergence of u_N towards the entropy solution of (1.1), $u_N \underset{N\to\infty}{\longrightarrow} u$, may fail, [T2]. Instead, we modify (2.1) by augmenting it with high frequency viscosity regularization which amounts to

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}\left[P_N f\big(u_N(x,t)\big)\right] = \varepsilon_N(-1)^{s+1}\frac{\partial^s}{\partial x^s}\left[Q_m(x,t) * \frac{\partial^s u_N}{\partial x^s}\right], \quad s \ge 1. \tag{2.2$_s$}$$

This kind of *spectral viscosity* can be efficiently implemented in Fourier space as

$$\varepsilon_N\frac{\partial^s}{\partial x^s}\left[Q_m(x,t) * \frac{\partial^s u_N}{\partial x^s}\right] := \varepsilon \sum_{m<|k|\le N} (ik)^{2s}\widehat{Q}_k(t)\widehat{u}_k(t)e^{ikx} . \tag{2.3}$$

It involves the following three ingredients:

- the viscosity amplitude, $\varepsilon = \varepsilon_N$,

$$\varepsilon \equiv \varepsilon_N \sim \frac{2\mathcal{C}_s}{N^{2s-1}}; \tag{2.4a}$$

  Here, $\mathcal{C}_s$ is a constant which may depend on the fixed order of super-viscosity, s. (A pessimistic upper bound of this constant will be specified below — consult [CDT, Theorem 2.1]).

[1] The spectral Fourier projection of $u(x)$ is given by $\sum_{|k|\le N}(u,e^{ikx})e^{ikx}$; the pseudospectral Fourier projection of $u(x)$ is given by $\sum_{|k|\le N} < u, e^{ikx} > e^{ikx}$, where $< u, e^{ikx} >:= \Delta x \sum_\nu u(x_\nu)e^{-ikx_\nu}$ is collocated at the $2N+1$ equidistant gridvalues $x_\nu = 2\pi\nu\Delta x$. $P_N u$ denotes either one of these two projections.

- the effective size of the *inviscid* spectrum, $m = m_N$,

$$m \equiv m_N \sim N^\theta, \qquad \theta < \frac{2s-1}{2s}; \tag{2.4b}$$

- the SV smoothing factors, $\widehat{Q}_k(t)$, which are activated only on high wavenumbers, $|k| > m_N$, satisfying

$$1 - \left(\frac{m}{|k|}\right)^{\frac{2s-1}{\theta}} \leq \widehat{Q}_k(t) \leq 1, \quad |k| > m_N. \tag{2.4c}$$

The SV method can be viewed as a compromise between the total-variation stable viscosity approximation – see (3.1) and $(4.1)_s$ below – which is restricted to first order accuracy (corresponding to $\theta = 0$), and the spectrally accurate yet unstable spectral method (2.1) (corresponding to $\theta = 1$). The additional SV on the right of $(2.2)_s$ is small enough to retain the *formal* spectral accuracy of the underlying spectral approximation, i.e., the following estimate holds

$$\|\varepsilon_N \frac{\partial^{s+p}}{\partial x^{s+p}}\left[Q_m(x,t) * \frac{\partial^s u_N}{\partial x^s}\right]\|_{L^2(x)} \leq \text{Const} \cdot N^{-\theta(q-p-1)} \|\frac{\partial^q u_N}{\partial x^q}\|_{L^2(x)}, \quad \forall q \geq p+1 > -\infty.$$

At the same time this SV is shown in §3 & 4 to be large enough so that it enforces a sufficient amount of entropy dissipation, and hence — by compensated compactness arguments — [Tr],[M], to prevent the unstable spurious Gibbs' oscillations.

3 Convergence of the Fourier SV method — 2nd order viscosity

The unique entropy solution of the scalar conservation law (1.1) is the one which is realized as the vanishing viscosity solution, $u = \lim_{\varepsilon \downarrow 0} u^\varepsilon$, where u^ε satisfies the standard viscosity equation

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\varepsilon(x,t)) = \varepsilon \frac{\partial^2}{\partial x^2} u^\varepsilon(x,t). \tag{3.1}$$

This section provides a brief review of the convergence results for the Fourier SV method $(2.2)_1$. The convergence analysis is based on the close resemblance of the Fourier SV method $(2.2)_1$ to the usual viscosity regularization (3.1). To quantify this similarity we rewrite $(2.2)_1$ in the equivalent form

$$\begin{aligned} \frac{\partial u_N}{\partial t} \quad &+ \quad \frac{\partial}{\partial x} f(u_N(x,t)) = \\ &= \quad \varepsilon_N \frac{\partial^2 u_N}{\partial x^2} - \varepsilon_N \frac{\partial}{\partial x}\left[R_N(x,t) * \frac{\partial u_N}{\partial x}\right] + \frac{\partial}{\partial x}(I - P_N) f(u_N), \end{aligned} \tag{3.2a}$$

where

$$R_N(x,t) := \sum_{k=-N}^{N} \hat{R}_k(t) e^{ikx}, \quad \hat{R}_k(t) \equiv \begin{cases} 1 & |k| < m_N, \\ 1 - \hat{Q}_k(t) & |k| \geq m_N. \end{cases} \tag{3.2b}$$

Observe that the SV approximation in (3.2a) contains two additional modifications to the standard viscosity approximation in (3.1).

{i} The second term on the right of (3.2a) measures the difference between the spectral viscosity, $\varepsilon_N \frac{\partial}{\partial x}\left[Q_m(x,t) * \frac{\partial u_N}{\partial x}\right]$, and the standard vanishing viscosity, $\varepsilon_N \frac{\partial^2 u_N}{\partial x^2}$. The following straightforward estimate shows this difference to be L^2-bounded, $\forall \theta < \frac{1}{2}$.

$$
\begin{aligned}
\|\varepsilon_N \frac{\partial}{\partial x}\left[R_N(\cdot,t) * \frac{\partial u_N}{\partial x}\right]\|_{L^2} &\leq \text{Const}\cdot\varepsilon_N\left[m_N^{1/\theta}\max_{|k|>m_N}|k|^{2-1/\theta} + m_N^2\right]\|u_N(\cdot,t)\|_{L^2} \\
&\leq \text{Const}\cdot N^{2\theta-1}\|u_N(\cdot,t)\|_{L^2} \leq \text{Const}\cdot\|u_N(\cdot,t)\|_{L^2}, \quad \theta\leq\tfrac{1}{2}.
\end{aligned}
$$

{ii} The spectral projection error contained in the third term on the right of (3.2a) does not exceed

$$\|(I-P_N)f(u_N(\cdot,t))\|_{L^2} \leq \text{Const}\frac{1}{N}\|\frac{\partial}{\partial x}u_N(\cdot,t)\|_{L^2}.$$

Equipped with the last two estimates one concludes the standard entropy dissipation bound, [T2], [MT], [T4], [CDT],

$$\|u_N(\cdot,t)\|_{L^2} + \sqrt{\varepsilon_N}\|\frac{\partial u_N}{\partial x}\|_{L^2_{\text{loc}}(x,t)} \leq \text{Const}, \qquad \varepsilon_N \sim \frac{1}{N}. \tag{3.3}$$

The inequality (3.3) is the usual statement of entropy stability familiar from the standard viscosity setup (3.1). For the L^∞-stability of the Fourier SV approximation consult e.g. [MT],[T3, §5] and [CDT, §4] for the one- and respectively, multi-dimensional problems. The convergence of the SV method then follows by compensated compactness arguments, [Tr],[M].

We note in passing that the the Fourier SV approximation $(2.2)_s$, (2.4) shares other familiar properties of the standard viscosity approximation (3.1), e.g., total variation boundedness, Oleinik's one-sided Lipschitz regularity (for $\theta<\frac{1}{3}$), L^1-convergence rate of order one-half, [S],[T4].

4 The Fourier SV method revisited – the super viscosity case

In this section we remove the restriction $\theta<\frac{1}{2}$ in (2.4b), which limits the portion of the inviscid spectrum. The key is to replace the standard second-order viscosity regularization (3.1) with the "super-viscosity" regularization

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x}f(u^\varepsilon(x,t)) = \varepsilon(-1)^{s+1}\frac{\partial^{2s}}{\partial x^{2s}}u^\varepsilon(x,t). \tag*{$(4.1)_s$}$$

The convergence analysis of the spectral "super-viscosity" method $(2.2)_s$ is linked to the behavior of the "super-viscosity" regularization $(4.1)_s$. To this end we rewrite $(2.2)_s$ in the equivalent form

$$
\begin{aligned}
\frac{\partial u_N}{\partial t} \quad &+ \quad \frac{\partial}{\partial x}f(u_N(x,t)) = \varepsilon_N(-1)^{s+1}\frac{\partial^{2s}u_N}{\partial x^{2s}} + \\
&+\varepsilon_N\frac{(-\partial)^s}{\partial x^s}\left[R_N(x,t)*\frac{\partial^s u_N}{\partial x^s}\right] + \frac{\partial}{\partial x}(I-P_N)f(u_N) = \\
&:= \quad \mathcal{I}_1(u_N)+\mathcal{I}_2(u_N)+\mathcal{I}_3(u_N).
\end{aligned} \tag{4.2}
$$

As before, we observe that the second and third terms on the right of (4.2), $\mathcal{I}_2(u_N)$ and $\mathcal{I}_3(u_N)$, are the two additional terms which distinguish the spectral "super-viscosity" approximation (4.2) from the super-viscosity regularization $(4.1)_s$. In the sequel we shall use the following upper-bounds on these two terms.

{i} The second term, $\mathcal{I}_2(u_N)$, measures the difference between the SV regularization in (4.2) and the "super-viscosity" in $(4.1)_s$. Using the SV parameterization in (2.4c), (2.4b) and (2.4a) (in this order), we find that this difference does not exceed

$$
\begin{aligned}
\|\varepsilon_N\frac{(-\partial)^s}{\partial x^s}\left[R_N(\cdot,t)*\frac{\partial^s u_N}{\partial x^s}\right]\|_{L^2} &\leq \varepsilon_N\left[m_N^{2s} + m_N^{\frac{2s-1}{\theta}}\max_{|k|>m_N}|k|^{2s-\frac{2s-1}{\theta}}\right]\|u_N(\cdot,t)\|_{L^2} \\
&\leq \text{Const}\cdot N^{2s\theta-2s+1}\|u_N(\cdot,t)\|_{L^2} \leq \\
&\leq \text{Const}\cdot\|u_N(\cdot,t)\|_{L^2}, \quad \forall\theta\leq\tfrac{2s-1}{2s}.
\end{aligned}
$$

Thus, the second term on the right of (4.2), $\mathcal{I}_2(u_N)$, is L^2-bounded :

$$\|\mathcal{I}_2(u_N)\|_{L^2(x)} \leq \text{Const}\|u_N(\cdot,t)\|_{L^2(x)}. \tag{4.3}$$

{ii} Regarding the third term, $\mathcal{I}_3(u_N)$, we shall make a frequent use of the spectral estimate which we quote from [CDT, §2.3][2],

$$\|\frac{\partial^p}{\partial x^p}(I-P_N)f(u_N(\cdot,t))\|_{L^2} \leq \mathcal{C}_q\frac{1}{N^{q-p}}\|\frac{\partial^q}{\partial x^q}u_N(\cdot,t)\|_{L^2}, \quad \forall q \geq p > -\infty,\ q > \frac{1}{2}. \tag{4.4}$$

(The restriction $q > \frac{1}{2}$ is required only for the *pseudospectral* Fourier projection, P_N, whose truncation estimate in provided in e.g., [T1, Lemma 2.2]). An upper bound on the constants $\mathcal{C}_s$ appearing on the right of (4.4) is given by [CDT, Theorem 7.1]

$$\mathcal{C}_s \sim \sum_{k=1}^{s}\|f(\cdot)\|_{C^k}\|u_N\|_{L^\infty}^{k-1};$$

this estimate may serve as a (pessimistic) bound for the same constant used in conjunction with the viscosity amplitude, ε_N, in (2.4a).

Next we turn to the behavior of the quadratic entropy of the SV solution, $U(u_N) = \frac{1}{2}u_N^2$. (A similar treatment applies to general convex entropy functions $U(u_N)$.) Multiplication of (4.2) by u_N implies

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}u_N^2 \quad &+ \quad \frac{\partial}{\partial x}\int^{u_N}\xi f'(\xi)d\xi = \\ &= \quad u_N\mathcal{I}_1(u_N) + u_N\mathcal{I}_2(u_N) + u_N\mathcal{I}_3(u_N) = \\ &:= \quad \mathcal{II}_1(u_N) + \mathcal{II}_2(u_N) + \mathcal{II}_3(u_N).\end{aligned} \tag{4.5}$$

The three expressions on the right (4.5) represent the quadratic entropy dissipation + production of the SV method. Successive "differentiation by parts" enable us to rewrite the first expression as

$$\begin{aligned}\mathcal{II}_1(u_N) \quad &\equiv \quad \varepsilon_N \sum_{\substack{p+q=2s-1 \\ 0\leq p<s}} (-1)^{s+p+1}\frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^q u_N}{\partial x^q}\right] - \varepsilon_N\left(\frac{\partial^s u_N}{\partial x^s}\right)^2 \\ &:= \quad \mathcal{II}_{11}(u_N) + \mathcal{II}_{12}(u_N).\end{aligned} \tag{4.6a}$$

Similarly, the second expression can be rewritten as

$$\begin{aligned}\mathcal{II}_2(u_N) \quad &\equiv \quad \varepsilon_N \sum_{p+q=s-1}(-1)^{s+p}\frac{\partial}{\partial x}\left(\frac{\partial^p u_N}{\partial x^p}\left[\frac{\partial^q R_N(x,t)}{\partial x^q} * \frac{\partial^s u_N}{\partial x^s}\right]\right) + \\ &\quad +\varepsilon_N\frac{\partial^s u_N}{\partial x^s}R_N(x,t) * \frac{\partial^s u_N}{\partial x^s} \\ &:= \quad \mathcal{II}_{21}(u_N) + \mathcal{II}_{22}(u_N).\end{aligned} \tag{4.6b}$$

[2] As usual we let $\partial_x^p w(x) := \sum_{k\neq 0}(ik)^p\hat{w}(k)e^{ikx}$. Note that if $\int w(x)dx = 0$ then $\partial_x^p w(x)$ with $p<0$ coincides with the $|p|$-th order primitive of $w(x)$.

Finally, we have for the third expression

$$
\begin{aligned}
\mathcal{II}_3(u_N) \quad &\equiv \quad \sum_{p=0}^{s-1}(-1)^p \frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^{-p}}{\partial x^{-p}}(I-P_N)f(u_N)\right]+ \\
&\quad\quad +(-1)^s \frac{\partial^s u_N}{\partial x^s}\frac{\partial^{-s+1}}{\partial x^{-s+1}}(I-P_N)f(u_N) \\
&:= \quad \mathcal{II}_{31}(u_N)+\mathcal{II}_{32}(u_N).
\end{aligned}
\tag{4.6c}
$$

We arrive at the following entropy estimate which plays an essential role in the convergence analysis of the SV method.

LEMMA 4.1 Entropy dissipation estimate. *There exists a constant,* Const $\sim \|u_N(\cdot,0)\|_{L^2}$, *(but otherwise is independent of* N*), such that the following estimate holds*

$$
\|u_N(\cdot,t)\|_{L^2}+\sqrt{\varepsilon_N}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2_{loc}(x,t)} \leq \text{Const}, \qquad \varepsilon_N = \frac{2\mathcal{C}_s}{N^{2s-1}}. \tag{4.7}
$$

REMARK. Observe that the entropy dissipation estimate in (4.7) is considerably *weaker* in the "super-viscosity" case where $s>1$, than in the standard viscosity regularization, $s=1$ quoted in (3.3).

PROOF. Spatial integration of (4.5) yields

$$
\frac{1}{2}\frac{d}{dt}\|u_N(\cdot,t)\|^2_{L^2}+\varepsilon_N\|\frac{\partial^s}{\partial x^s}u_N(\cdot,t)\|^2_{L^2} = (u_N,\mathcal{I}_2(u_N))_{L^2(x)}+(u_N,\mathcal{I}_3(u_N))_{L^2(x)}.
$$

According to (4.3), the first expression on the right of the last inequality does not exceed

$$
|(u_N,\mathcal{I}_2(u_N))_{L^2}| \leq \text{Const}\cdot\|u_N(\cdot,t)\|^2_{L^2}. \tag{4.8}
$$

According to (4.6c), the second expression on the right$= (-1)^s\frac{\partial^s u_N}{\partial x^s}\frac{\partial^{-s+1}}{\partial x^{-s+1}}(I-P_N)f(u_N)$, and by (4.4) it does not exceed

$$
|(u_N,\mathcal{I}_3(u_N))_{L^2}| \leq \|\frac{\partial^s u_N}{\partial x^s}\|_{L^2}\cdot\frac{\mathcal{C}_s}{N^{2s-1}}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2} \leq \frac{1}{2}\varepsilon_N\|\frac{\partial^s}{\partial x^s}u_N(\cdot,t)\|^2_{L^2}. \tag{4.9}
$$

(In fact, in the spectral case, the second expression vanishes by orthogonality). The result follows from Gronwall's inequality. ■

Equipped with Lemma 4.1 we now turn to the main result of this section, stating

THEOREM 4.2. Convergence. *Consider the Fourier "super-viscosity" approximation* $(2.2)_s$–(2.4), *subject to* L^∞*-initial data,* $u_N(\cdot,0)$. *Then uniformly bounded* u_N *converges to the unique entropy solution of the convex conservation law* (1.1).

PROOF. We proceed in three steps.

Step 1. L^∞-stability. The L^∞-stability for spectral viscosity of 2nd order, $s=1$, follows by L^p-iterations along the lines of [MT] and [CDT], (we omit the details). The issue of an L^∞ bound for spectral viscosity of 'super' order $s>1$ remains an open question. The intricate part of this question could be traced to the fact that already the underlying super-viscosity regularization $(4.1)_s$, *lacks* monotonicity for $s>1$: instead, it exhibits additional oscillations which are added to the spectral Gibbs' oscillations (Both types of oscillations are post-processed without sacrificing neither stability nor spectral accuracy).

Step 2. H^{-1}-stability. We want to show that both — the local error on the right hand-side of (4.2), $\sum_{1\le j\le 3}\mathcal{I}_j(u_N)$, and the *quadratic* entropy dissipation + production on the right of (4.5), $\sum_{1\le j\le 3}\mathcal{II}_j(u_N)$, belong to a compact subset of $H^{-1}_{loc}(x,t)$.

To this end we first prepare the following. Bernstein's inequality gives us $\forall p < s \le q$

(4.10)

$$\begin{aligned}
\|\varepsilon_N\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^q u_N}{\partial x^q}\right]\|_{L^2_{loc}(x,t)} &\le \text{Const}\cdot\varepsilon_N\|\frac{\partial^p u_N}{\partial x^p}\|_{L^\infty}\cdot\|\frac{\partial^q u_N}{\partial x^q}\|_{L^2_{loc}(x,t)} \le \\
\ldots\text{by Bernstein inequality}\ldots &\le \text{Const}\cdot\varepsilon_N\cdot N^p\|u_N\|_{L^\infty}\cdot N^{q-s}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2_{loc}(x,t)} \le \\
\ldots\text{by Lemma 4.1}\ldots &\le \text{Const}\cdot\sqrt{\varepsilon_N}\cdot N^{p+q-s}\|u_N\|_{L^\infty}\sim\sqrt{2\mathcal{C}_s}\cdot N^{p+q-2s+\frac{1}{2}}\cdot\|u_N\|_{L^\infty}.
\end{aligned}$$

Consider now the first two expressions, $\mathcal{I}_1(u_N)$ and $\mathcal{II}_1(u_N)$. The inequality (4.10) with $(p,q) = (0, 2s-1)$ implies that $\mathcal{I}_1(u_N)$ tends to zero in $H^{-1}_{loc}(x,t)$, for

$$\|\mathcal{I}_1(u_N)\|_{H^{-1}_{loc}(x,t)} \le \text{Const}\cdot\sqrt{2\mathcal{C}_s/N}\cdot\|u_N\|_{L^\infty}\to 0. \tag{4.11}$$

We turn now to the expression $\mathcal{II}_1(u_N)$ in (4.6a): its first half tends to zero in $H^{-1}_{loc}(x,t)$, for by (4.10) we have $\forall p+q = 2s-1$,

$$\begin{aligned}
\|\mathcal{II}_{11}(u_N) &\equiv \varepsilon_N\cdot\sum_{\substack{p+q=2s-1\\ 0\le p<s}}(-1)^{s+p}\frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^q u_N}{\partial x^q}\right]\|_{H^{-1}_{loc}(x,t)} \le \\
&\le \text{Const}\cdot\sqrt{2\mathcal{C}_s/N}\cdot\sum_{\substack{p+q=2s-1\\ 0\le p<s}}\|u_N\|_{L^\infty} \le \\
&\le \text{Const}\cdot s\sqrt{2\mathcal{C}_s/N}\cdot\|u_N\|_{L^\infty}\to 0;
\end{aligned} \tag{4.12}$$

the second half of $\mathcal{II}_1$ in (4.6a), $-\varepsilon_N\left(\frac{\partial^s u_N}{\partial x^s}\right)^2$, is bounded in $L^1_{loc}(x,t)$, consult Lemma 4.1, and hence by Murat's Lemma [M], belongs to a compact subset of $H^{-1}_{loc}(x,t)$. We conclude

$$\mathcal{II}_{12}(u_N)\underset{H^{-1}_{loc}(x,t)}{\longrightarrow}\le 0. \tag{4.13}$$

We continue with the next pair of expressions, $\mathcal{I}_2(u_N)$ and $\mathcal{II}_2(u_N)$. According to (4.3), $\mathcal{I}_2(u_N)$ — and therefore also $\mathcal{II}_2(u_N) = u_N\mathcal{I}_2(u_N)$ — are L^2-bounded, and hence belong to a compact subset of $H^{-1}_{loc}(x,t)$; in fact, by repeating our previous arguments which led to (4.3) one finds that

$$\|\mathcal{I}_2(u_N)\|_{H^{-1}(x,t)} \le \text{Const}\cdot\varepsilon_N m_N^{s-1}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2(x,t)} \le \text{Const}\cdot\sqrt{\varepsilon_N}m_N^{s-1}\sim\sqrt{2\mathcal{C}_s}\cdot N^{-\frac{2s-1}{2s}}\to 0. \tag{4.14}$$

A similar treatment shows that the first half of $\mathcal{II}_2(u_N)$ in (4.6b) tends to zero in $H^{-1}_{loc}(x,t)$, for

$$\begin{aligned}
\|\mathcal{II}_{21}(u_N) &\equiv \varepsilon_N\sum_{p+q=s-1}(-1)^{s+p}\frac{\partial}{\partial x}\left(\frac{\partial^p u_N}{\partial x^p}\left[\frac{\partial^q R_N(x,t)}{\partial x^q}*\frac{\partial^s u_N}{\partial x^s}\right]\right)\|_{H^{-1}_{loc}(x,t)} \le \\
&\le \varepsilon_N\cdot\sum_{p+q=s-1}N^p\|u_N\|_{L^\infty}\cdot m_N^q\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2_{loc}(x,t)} \le \\
&\le \text{Const}\cdot\sqrt{\varepsilon_N}\sum_{p+q=s-1}N^{p+q}\|u_N\|_{L^\infty}\le s\sqrt{2\mathcal{C}_s/N}\cdot\|u_N\|_{L^\infty}\to 0.
\end{aligned} \tag{4.15}$$

The second half of $\mathcal{II}_2(u_N)$ is L^1-bounded, for

$$\|\mathcal{II}_{22}(u_N) \equiv \varepsilon_N \frac{\partial^s u_N}{\partial x^s} R_N(x,t) * \frac{\partial^s u_N}{\partial x^s}\|_{L^1} \leq \text{Const} \cdot \varepsilon_N \|\frac{\partial^s u_N}{\partial x^s}\|^2_{L^2_{loc}(x,t)} \leq \text{Const}. \tag{4.16}$$

Finally we treat the third pair of expressions, $\mathcal{I}_3(u_N)$ and $\mathcal{II}_3(u_N)$. The spectral decay estimate (4.4) with $(p,q) = (0,s)$, together with Lemma 4.1 imply that $\mathcal{I}_3(u_N)$ tends to zero in $H^{-1}_{loc}(x,t)$; indeed

$$\|\mathcal{I}_3(u_N) \equiv \frac{\partial}{\partial x}(I - P_N)f(u_N)\|_{H^{-1}_{loc}(x,t)} \leq \frac{\mathcal{C}_s}{N^s}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2} \sim \sqrt{2\mathcal{C}_s/N} \to 0. \tag{4.17}$$

A similar argument applies to the expression $\mathcal{II}_3(u_N)$ given in (4.6c). Sobolev inequality – consult (4.10), followed by the spectral decay estimate (4.4) imply that the first half of $\mathcal{II}_3(u_N)$ does not exceed

$$\begin{aligned}
\|\mathcal{II}_{31}(u_N) \quad &\equiv \quad \sum_{p=0}^{s-1}(-1)^p \frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^{-p}}{\partial x^{-p}}(I-P_N)f(u_N)\right]\|_{H^{-1}_{loc}(x,t)} \leq \\
&\leq \quad \sum_{p=0}^{s-1}\|\frac{\partial^p u_N}{\partial x^p}\|_{L^\infty} \cdot \|\frac{\partial^{-p}}{\partial x^{-p}}(I-P_N)f(u_N)\|_{L^2_{loc}(x,t)} \leq \\
&\leq \quad \text{Const} \cdot \sum_{p=0}^{s-1} N^p \|u_N\|_{L^\infty}\frac{\mathcal{C}_s}{N^{s+p}}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2_{loc}(x,t)} \leq \\
&\sim \quad \text{Const} \cdot s\sqrt{2\mathcal{C}_s/N}\|u_N\|_{L^\infty} \to 0.
\end{aligned} \tag{4.18}$$

According to Lemma 4.1, the second half of $\mathcal{II}_3(u_N)$ is L^1-bounded, for

$$\|\mathcal{II}_{32}(u_N) \equiv \frac{\partial^s u_N}{\partial x^s}\frac{\partial^{-s+1}}{\partial x^{-s+1}}(I-P_N)f(u_N)\|_{L^1} \leq \|\frac{\partial^s u_N}{\partial x^s}\|_{L^2}\frac{\mathcal{C}_s}{N^{2s-1}}\|\frac{\partial^s u_N}{\partial x^s}\|_{L^2} \leq \text{Const}, \tag{4.19}$$

and hence by Murat's Lemma [M], belongs to a compact subset of $H^{-1}_{loc}(x,t)$. We conclude that the entropy dissipation of the Fourier spectral "super-viscosity" method, for both linear and quadratic entropies, belongs to a compact subset of $H^{-1}_{loc}(x,t)$.

Step 3. Convergence. It follows that the SV solution u_N converges strongly (in $L^p_{loc}, \forall p < \infty$) to a weak solution of (1.1). In fact, except for the L^1-bounded terms $\mathcal{II}_{22}(u_N)$ and $\mathcal{II}_{32}(u_N)$, we have shown that all the other expressions which contribute to the entropy dissipation tend either to zero or to a negative measure. Using the strong convergence of u_N it follows that $\mathcal{II}_{22}(u_N)$ and $\mathcal{II}_{32}(u_N)$ also tend to zero, consult [MT]. Hence the convergence to the unique entropy solution. ■

REMARKS.

1. Low pass filter [G]. We note that the spectral "super-viscosity" in $(2.2)_s$ allows for an increasing order of parabolicity, $s \sim N^\mu$, $\mu < 1/2$ (at least for bounded $\mathcal{C}_s$'s). This enables us to rewrite the spectral "super-viscosity" method in the form

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N)] = -N\sum_{|k|\leq N}\sigma(\frac{k}{N})\hat{u}_k(t)e^{ikx},$$

where $\sigma(\xi)$ is a symmetric low pass filter satisfying

$$\sigma(\xi)\begin{cases} \leq |\xi|^{2s}, & |\xi| \leq 1, \\ \geq |\xi|^{2s} - \frac{1}{N}, & |\xi| > 0. \end{cases}$$

In particular, for $s \sim N^{\mu}$, one is led to a low pass filter which is C^{∞}-tailored at the origin, consult [V].

2. Super viscosity regularization. Theorem 4.2 implies the convergence of the regularized "super-viscosity" approximation u^{ε} in $(4.1)_s$, to the entropy solution of the convex conservation law (1.1). Unlike the regular viscosity case, the solution operator associated with $(4.1)_{s>1}$ is *not* monotone — here there are "spurious" oscillations, on top of the Gibbs' oscillations due to the Fourier projection. What we have shown is that the oscillations of either type do not cause instability. Moreover, these oscillations contain, in some weak sense, highly accurate information on the exact entropy solution; this could be revealed by post-processing the spectral (super)-viscosity approximation, e.g. [MOT].

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