The Spectrally-Hyperviscous Navier-Stokes Equations

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1. The equations

The 3D spectrally-hyperviscous Navier-Stokes equations (SHNSE) are:

$$u_t + \nu A u + \mu A_\varphi u + (u \cdot \nabla)\, u + \nabla p = g, \quad (1)$$

$$\nabla \cdot u = 0. \quad (2)$$

- We assume (for now) that the domain Ω is a periodic box. Then A has eigenspaces $E_1, E_2, \cdots$.
- Let P_m project onto $E_1 \oplus \cdots \oplus E_m$, and $Q_m = I - P_m$; then $A_\varphi = Q_m A^\alpha$ or a smoothed-out version of this.
- Here $\alpha > 1$ and typically $\alpha = 2$; basically this is the hyperviscous version of Tadmor's *spectral vanishing viscosity* (SVV) method, and is in a sense an alternative version of spectral eddy viscosity.

2. Attractor results (Avrin, JDDE, 2008)

- We have $\dim_H \mathcal{A} \le \dim_F \mathcal{A} \le K(\nu/\mu)^c m^a \kappa_d^b$ where κ_d is the Kolmogorov wavenumber.
- $K(\alpha)$ depends also on the shape (but not the size) of Ω ; $(\nu/\mu)^c$ is of maneageable size, a is a small fractional power, and $b < 3$.
- As long as $m \le \kappa_d^3$ then $\dim_H \mathcal{A} \le \dim_F \mathcal{A} \le K m^a \kappa_d^b \le K \kappa_d^3$ for $m \le \kappa_d^3$; thus the degrees of freedom on $\mathcal{A}$ are within Landau-Lifschitz estimates even for huge m. For the more realistic choices $m \le \kappa_d$ we have that b is significantly lower, implying potential for degrees-of-freedom reduction in simulation.

3. Inertial manifold results (Avrin, JDDE, 2008)

- When $A_\varphi = Q_m A^\alpha$ an inertial manifold exists for large enough m whenever $\alpha \ge 3/2$.
- This result is obtained using a natural spectral gap property of the model. Note that regular hyperviscosity ($m = 0$) needs $\alpha \ge 5/2$.
- The results hold also (for larger still m) in the smoothed-out version of A_φ.

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4. Galerkin convergence (Avrin/Xiao, JDE, 2009)

- Let $w_N = u - u_N$ where (for some $N > m$) u_N is the Galerkin approximation to u. Let G be the Grashoff number; let $n \ge m$, then if $\lambda_{n+1}^{\alpha-5/4} > c_1(\nu/\mu)G$ we show for any interval $[0, t]$ that the convergence of $\sup_{0\le s\le t} \|Q_n w_N(s)\|^2_{H^{\beta/2}}$ depends linearly on $\|Q_n w_N(0)\|^2_{H^{\beta/2}}$, $\|Q_n(g - g_N)\|_2^2$, $\|A^{-(\alpha-\beta)/2} Q_N (u\cdot\nabla)u_2^2\|_2$, and $(\nu/\mu)^2 G^2 \sup_{0\le s\le t} \|P_n w_N(s)\|^2_{H^{\beta/2}}$.
- The convergence of $\sup_{0\le s\le t} \left\|A^{\beta/2} P_n w_N(s)\right\|_2^2$ to zero follows from a more standard Gronwall estimate, but with coefficients depending on only a fractional power of n.

5. Convergence to an inviscid limit

- We let $\nu \to 0$ in (1) (with $g = 0$) while keeping the term μA_φ fixed. The case $\nu = 0$ is similar to previous applications of SVV to the Euler equation. That μ can be chosen independently of ν is supported by our numerical experiments.
- Let $w_\nu = u - u_\nu$ where (for some $N > m$) u_ν solves (1) and u solves (1) with $\nu = 0$, respectively. If $\lambda_{n+1}^{\alpha-5/4} \ge c_2 \|u_0\|_2 \mu^{-1}$, the convergence of $\|Q_n w_\nu(t)\|_2^2$ depends linearly on $\|Q_n w_\nu(0)\|_2^2$, $\nu \|u_{\nu,0}\|_2^2$, and $\left[\|u_0\|_2^2 + \|u_{\nu,0}\|_2^2\right] \sup_{0\le s\le t} \|P_n w_\nu(s)\|^2_{H^{\beta/2}}$. The convergence of $\sup_{0\le s\le t} \|P_n w_\nu(s)\|_2^2$ is by Gronwall estimates but with coefficients whose dependence on n is like $n^{5/6}$.

6. Computational experiments (Xiao/Avrin/Deng)

Figures 1, 2 and 3 at right (extracted from a forthcoming paper) represent a summary of runs whose purpose is to find the optimal choices of the cutoff wavenumber $m = M$ and the spectral hyperviscosity coefficient μ, respectively, in simulations with high Reynolds numbers. In Fig. 1 the best results (dashed-dot line) in terms of agreement with the Kolmogorov power law hold for $M = 0.8 \times (N/2)$ where $N/2$ is the truncation order. In Fig. 2 the best results hold for μ at or on the order of 2×10^7, and as Fig. 3 illustrates, this value of μ seems especially optimal for larger values of $1/\nu$. Together Figs. 2 and 3 illustrate in the special case of total energy $T = 0.5$ the general observation (seen over the full course of the runs) that the optimal choice of μ is at or near $2 \times \sqrt{2T} \times 10^7$ for high Reynolds numbers, particularly for $Re{>}10^5$, and that otherwise this relation is independent of the choice of viscosity.

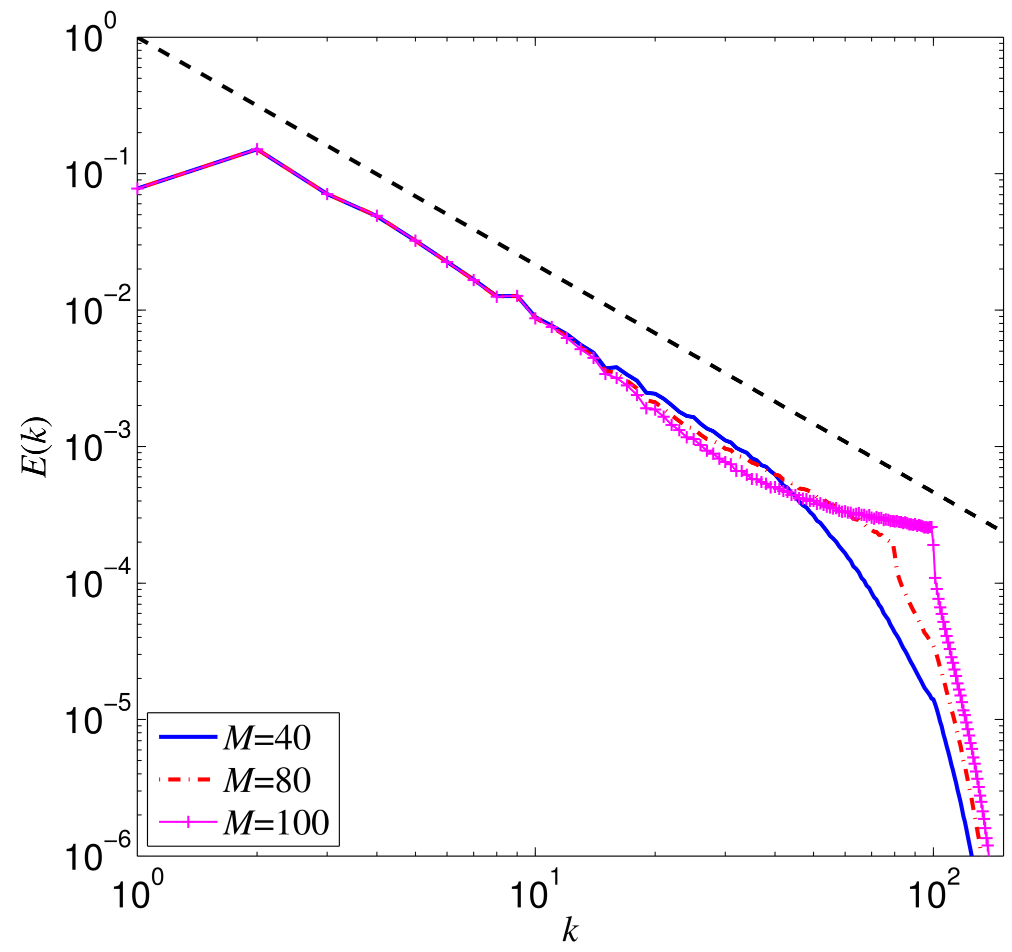

Fig. 1 The energy spectrum $E(k)$ versus the wave number k for SHNSE simulations of three different flows. The straight line represents a slope of $-5/3$. Here, Re= 5973270 and $k_d = 28155$.

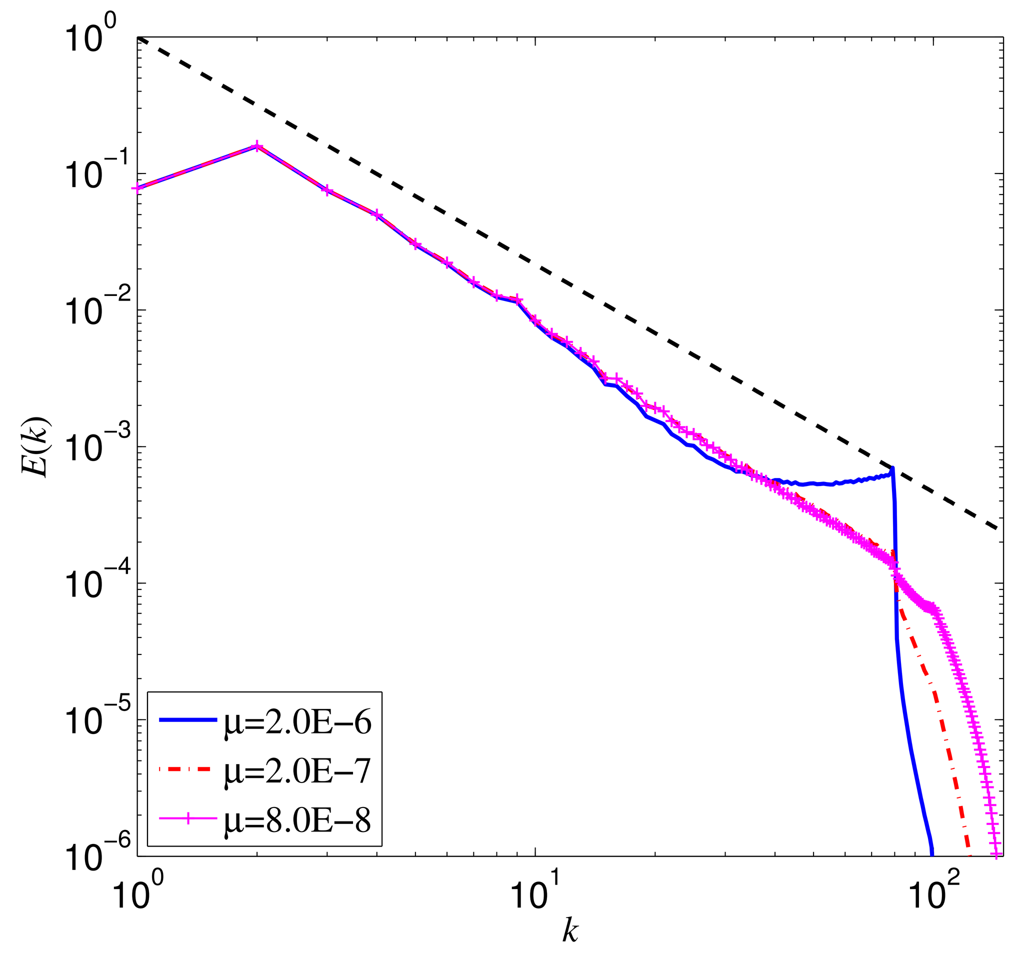

Fig. 2 The energy spectrum $E(k)$ versus the wave number k for SHNSE simulations using different values of the hyperviscosity coefficient μ. The straight line represents a slope of $-5/3$; $\nu = 1.73 \times 10^{-5}$.

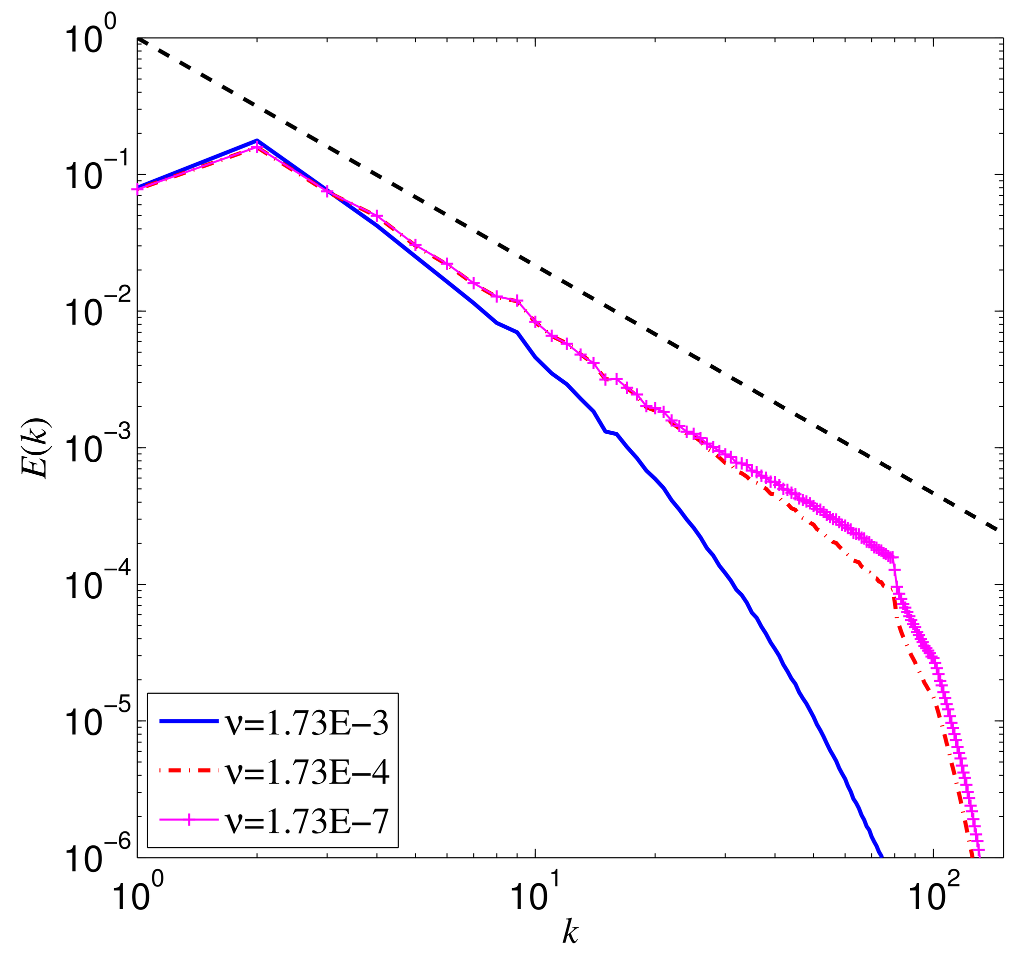

Fig. 3 The energy spectrum $E(k)$ versus the wave number k for SHNSE simulations using the same hyperviscosity coefficient μ for several different flows. The straight line represents a slope of $-5/3$.

References

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