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Convergence results for a class of spectrally hyperviscous models of 3-D turbulent flow

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ABSTRACT

We consider the spectrally hyperviscous Navier–Stokes equations (SHNSE) which add hyperviscosity to the NSE but only to the higher frequencies past a cutoff wavenumber m_0. In Guermond and Prudhomme (2003) [18], subsequence convergence of SHNSE Galerkin solutions to dissipative solutions of the NSE was achieved in a specific spectral-vanishing-viscosity setting. Our goal is to obtain similar results in a more general setting and to obtain convergence to the stronger class of Leray solutions. In particular we obtain subsequence convergence of SHNSE strong solutions to Leray solutions of the NSE by fixing the hyperviscosity coefficient μ while the spectral hyperviscosity cutoff m_0 goes to infinity. This formulation presents new technical challenges, and we discuss how its motivation can be derived from computational experiments, e.g. those in Borue and Orszag (1996, 1998) [3,4]. We also obtain weak subsequence convergence to Leray weak solutions under the general assumption that the hyperviscous coefficient μ goes to zero with no constraints imposed on the spectral cutoff. In both of our main results the Aubin Compactness Theorem provides the underlying framework for the convergence to Leray solutions.

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1. Introduction

We obtain subsequence convergence results for solutions of the 3-D spectrally hyperviscous Navier–Stokes equations (SHNSE):

$$u_t + \nu A u + \mu A_\varphi u + (u \cdot \nabla) u + \nabla p = g, \tag{1.1a}$$

$$\nabla \cdot u = 0. \tag{1.1b}$$

Here $A = -\triangle$, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, $g = (g_1, g_2, g_3)$ is the external force, and p is the pressure. We have that $u_i = u(x, t)$, $g_i = g_i(x, t)$, $i = 1, 2, 3$, and $p = p(x, t)$ where $x \in \Omega$, a domain in $\mathbb{R}^3$. We assume that Ω is a periodic box; then "modding out" the constant vectors as in standard practice, A has eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ with corresponding eigenspaces $E_1, E_2, \ldots$ Let P_m be the projection onto $E_1 \oplus \cdots \oplus E_m$, let $Q_m = I - P_m$ and let P_{E_j} be the projection onto each E_j. The general class of operators A_φ considered in [1] satisfy $A_\varphi = \sum_{j=1}^\infty a(\lambda_j) P_{E_j}$ such that (for a constant $\alpha > 1$) $A_\varphi \geq A_m \equiv Q_m A^\alpha$ in the sense of quadratic forms, i.e., $(A_\varphi v, v) \geq (A_m v, v)$.

Specializing to what would arise in typical computations we identify here an *applicable distinguished class* (ADC) in which $A_\varphi = \sum_{j=m_0+1}^m d_j \lambda_j^\alpha P_{E_j} + Q_m A^\alpha = \sum_{j=m_0+1}^m d_j P_{E_j} A^\alpha + Q_m A^\alpha$ where $\{d_j\}_{j=m_0+1}^m$ is such that $0 < d_j \leq 1$ and $d_j \uparrow 1$. Included in the ADC is the special case $m_0 = m$ and $d_{m_0+1} = d_{m+1} = 1$ in which case $a(\lambda)$ is a Heaviside function and $A_\varphi = A_m$.

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The ADC derives much of its motivation from the basic idea in large-eddy-simulation (LES) of adding the divergence of a subgrid-scale (SGS) stress tensor to the Navier–Stokes equations (NSE) in order to model the effect of the subgrid scales on the resolvable scales. The most common LES approach models the SGS stress tensor using a scalar eddy viscosity. A number of standard implementations of this approach have well-known limitations, however (see e.g. [3,4,22]), and in response the concept of spectral-eddy viscosity (SEV) was introduced in [22]; therein it is demonstrated that for eddy viscosity to accurately model the effects of the subgrid scales, the eddy viscosity coefficient ν_{ev} must depend spectrally on j. As developed in [7,8] a working fit to the predictions of [22] has the nondimensional form $d_j = \nu_{ev}(j) = \nu_0(j/N) = K_0^{-3/2}[0.441 + 15.2 \exp(-3.03N/j)]$ for $j \leq N$ where $K_0 = 2.1$ denotes the Kolmogorov constant. It was suggested in [8] that the introduction of a hyperviscous term $\nu_h A^\alpha u$ could approximate the spectral growth of $\nu_{ev}(j)$; see also [6] where arguments employing vector–calculus techniques and certain physical assumptions are used to show that a hyperviscous term of order $\alpha = 2$ can approximate the SGS tensor. The resulting technique has been widely employed; see the references in e.g. [6] and in particular the computational experiments in [3,4] which use the hyperviscosity term $\nu^2 A^2$ (and higher powers of it) in robust and effective application to high-Reynolds number regimes. Moreover, the computational results using hyperviscous terms are observed in [4] to be virtually identical to those using the full SGS stress tensor directly.

Since for high Reynolds numbers the hyperviscosity coefficient $\mu = \nu^2$ is small, it therefore makes sense that in some cases the hyperviscous term may be negligible when acting on the low-frequency components of u. For higher frequencies the combined term $\nu^2 \lambda_j$ would then become more significant due to the growth of λ_j with j. Thus we can envision a cutoff m_0 such that $\nu^2 \lambda_j$ is significant for $j > m_0$ and insignificant for $j \leq m_0$. In this way and for $m = m_0$ we can motivate the operator $A_\varphi = A_m \equiv Q_m A^\alpha$ on physical grounds as a specific SGS modeling technique, and more subtle gradations of negligible vs. nonnegligible terms can be modeled by more general operators in the ADC. Such cases will be the focus of Theorem 2 below.

This idea of having a spectral cutoff $m_0 > 0$ in the ADC versions of the operators A_φ is related to the spectral-vanishing-viscosity (SVV) method, introduced in the study of gas dynamics in [32] to obtain stable and accurate approximation to the correct entropy solution of conservation laws. In SVV the distribution of the coefficients d_j is similar to the SEV case but spectral viscosity is added only to the high wavenumber modes. SVV was first applied to the incompressible 3-D NSE in [19] for Reynolds numbers up to Re = 395, and was used in obtaining improved accuracy in long-term computations of 2-D flows past a circular cylinder in [31] with Re = 100 and Re = 500. Higher Reynolds numbers have since been treated effectively in 3-D using SVV, e.g. in [20] with Re = 1250 in a study of a triangular duct, in [29,28] in computing the turbulent wake of a cylinder with Re = 3900, and in [27] with Re = 768 000 in a treatment of the Ahmed car-body problem. In these works SVV "can be thought of as using hyperviscous dissipation that will affect only the high Fourier modes" [19], and in fact the SVV terms qualitatively resemble spectrally-applied hyperviscous terms in truncation due to the generally exponential growth of the coefficients d_j.

In fact spectral hyperviscosity, or superviscosity, has already been used in the modeling of gas dynamics in [33]; H^1-stability was established, and convergence proven under the same L^∞-stability assumption used in [31]. Applying spectral hyperviscosity directly to the 3-D NSE in computations was suggested in [19], discussed experimentally in [6], and advocated in [18,17]. With the coefficients μ and m_0 chosen in (1.1) to depend on certain negative and positive powers, respectively, of the truncation order N according to a specific version of SVV methodology, it was shown in [18] that a subsequence of the resulting Galerkin solutions u_N converges to a weak dissipative solution of the NSE using the definition of dissipative solution advanced in [5,12,30]. The choices of μ and m_0 in this case are designed to tune the system (1.1) so as to "introduce the least possible dissipation while ensuring that the limit solution is dissipative" and the results represent the first of theoretical type for (1.1) with $0 < m_0 \leq m$.

In [1] global regularity for the general class of A_φ described above was established for $\alpha \geq 5/4$, generalizing the classical results in [26] which held for the case $m_0 = m = 0$. Estimates on the dimension of the attractor for (1.1) were obtained in [1] by adapting elements of the "CFT" framework [9,11,34,35] and the generalized Lieb–Thirring inequalities [34–36]. Additionally in [1] the machinery developed in [15,16] was adapted to establish the existence of an inertial manifold of dimension m_0 for A_φ in the ADC case. In [2] the strong convergence of Galerkin solutions was obtained and continuous dependence on data was established with estimates optimized for the high frequencies.

In this paper we similarly show as in [18] that the system (1.1) approximates the NSE through subsequence convergence, but here the target solutions will be Leray solutions and we will consider a wider class of operators A_φ. First introduced and explored among the classical results in [23,25,24], we recall that u is a Leray solution of (1.1) if u solves an appropriate weak version of (1.1) (see (3.12) below) and also satisfies the energy inequality in a particular weak form (see (3.16) below). Let P be the Leray projection onto the solenoidal vectors; then setting as in standard practice $H = PL^2(\Omega)$, $V = PH^1(\Omega)$, and $V' = V = PH^{-1}(\Omega)$, the following is the first of our main results.

Theorem 1. *Let $\{u_k\}_{k=1}^\infty$ be the corresponding strong solutions of (1.1) with $u_k(x, 0) = u_0(x)$ and with $\mu = \mu_k$ such that $\mu_k \downarrow 0$ as $k \to \infty$. Assume that $\alpha \geq 5/4$, and let the corresponding $A_\varphi = A_{\varphi,k}$ be in the applicable distinguished class. Then on each interval $[0, T]$ there exists a subsequence, also denoted $\{u_k\}_{k=1}^\infty$, and a divergence-free vector $u \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$ such that $u_k \to u$ strongly in $L^2([0, T]; H)$, $u_k \to u$ weakly in $L^2([0, T]; V)$, $\frac{d}{dt}u_k \to \frac{d}{dt}u$ weakly in $L^r([0, T]; PH^{-\alpha})$, and u is a Leray weak solution of the NSE. Here $r = 4/(5 - 2\alpha)$ if $5/4 \leq \alpha < 3/2$ and $r = 2$ if $\alpha \geq 3/2$.*

We note that the assumption $\mu_k \downarrow 0$ as $k \to \infty$ is the only restriction we place on the μ_k and there is no restriction placed on m_0 (in particular m_0 can be constant or arbitrarily depend on k). With A_φ as in Theorem 1 we have that $\mu_k \|A_\varphi u\|_2 \leq$

$\mu_k \|A^\alpha u\|_2 \sim \mu_k \|u\|_{H^{2\alpha}(\Omega)}$ since $0 < d_j \leq 1$, so we also have the same sense of the hyperviscosity term A_φ being "spectrally small" as in [18, Proposition 3.1]. In fact since Theorem 1 can be readily adapted to include Galerkin approximations to the solutions u_k, it therefore strengthens the results in [18] by demonstrating that any scheme of dissipation represented by the μ_k will, as long as $\mu_k \downarrow 0$ as $k \to \infty$, guarantee subsequence convergence to a Leray solution. We also note that Leray solutions represent a significantly stronger subclass of the dissipative solutions.

We note also that $\frac{d}{dt} u_k \to \frac{d}{dt} u$ in a different and weaker Banach space than in the classical results of Leray (see also [10] and the references contained therein), but in a stronger Banach space than in [18] if $\alpha \geq 3/2$. We also note that a result similar to Theorem 1 was established for the NS-α model in [13], along with global regularity and attractor results. In particular subsequence convergence to a weak solution in the sense of (3.12) below was established as $\alpha \downarrow 0$, and $\frac{d}{dt} u_k \to \frac{d}{dt} u$ in again a different and weaker Banach space than in the classical results of Leray.

We now examine the case in which μ and the d_j are fixed (e.g. $\mu = \nu^2$ as in [4,5]), but $m_0 \to \infty$; such a convergence case has not been considered previously from the theoretical point of view. Replacing m_0 by k for a corresponding $m = m_k \geq k$ where $k \to \infty$, and denoting again the resulting A_φ by $A_{\varphi,k}$, the following is our second main result.

Theorem 2. *Let $A_{\varphi,k}$ be as above, let $\{u_k\}_{k=1}^\infty$ be the corresponding strong solutions of (1.1) such that $k = m_0$ and $k \to \infty$, and assume that $\alpha > 5/4$. Then on each interval $[0, T]$ there exists a subsequence, also denoted $\{u_k\}_{k=1}^\infty$, and a divergence-free vector $u \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$ such that $u_k \to u$ strongly in $L^2([0, T]; H)$, $u_k \to u$ weakly in $L^2([0, T]; V)$, $\frac{d}{dt} u_k \to \frac{d}{dt} u$ weakly in $L^{4/3}([0, T]; V')$ and u is an appropriate weak solution of (1.1). If we further assume that $\alpha > 3/2$ then u is a Leray weak solution of (1.1).*

By an appropriate weak solution of (1.1) we mean that (3.12) is satisfied below. To obtain this we assume from the outset the same condition on α as in [18]. To further obtain that u is a Leray weak solution, i.e. that the energy inequality (3.16) below is satisfied, we need the further assumption $\alpha > 3/2$, but this restriction leaves out no cases of practical interest that we are aware of.

In Theorem 2 we again have a sense in which adding the spectrally hyperviscous term produces a perturbation of the NSE which is suitably spectrally small. This will be evident in the Proof of Theorem 2 and will be discussed in detail in the conclusion; in particular, if P_k projects onto the first k eigenspaces of A and $Q_k = I - P_k$, then we will show that a subsequence of $P_k u_k$ converges weakly to u and that $Q_k u_k$ vanishes so that in some sense $P_k u_k$ is approximately a Galerkin solution. Also in Theorem 2 we have in this case that $\frac{d}{dt} u_k \to \frac{d}{dt} u$ in exactly the same Banach space as in the classical result of Leray. In both Theorems 1 and 2 the Aubin Compactness Theorem plays a central role in establishing convergence to Leray solutions. After some preliminary observations and calculations in Section 2, we will prove Theorem 1 in Section 3 and Theorem 2 in Section 4.

2. Preliminaries

We express the Sobolev inequalities on Ω in terms of the operator $A = -\triangle$:

$$\|v\|_q \leq M_1 \|A^\theta v\|_p \tag{2.1}$$

where $q \leq 3p/(3 - 2\theta p)$ and $M_1 = M_1(\theta, p, q, \Omega)$. For the semigroup $\exp(-tA)$ we have the decay estimate

$$\|e^{-tA} v\|_2 \leq \|v\|_2 e^{-\lambda_1 t}. \tag{2.2}$$

Like the standard NSE, (1.1) satisfies an energy inequality, which we derive as follows: taking the inner product of both sides of (1.1) with u we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \nu \|A^{1/2} u\|_2^2 + \mu \|A_\varphi^{1/2} u\|_2^2 = (g, u) \tag{2.3}$$

where we note that since $\operatorname{div} u = 0$ we have that $(\nabla p, u) = 0$ and $((u \cdot \nabla) u, u) = -((\operatorname{div} u) u, u) = 0$.

Now

$$(g, u) = (A^{-1/2} g, A^{1/2} u) \leq \frac{\nu}{2} \|A^{1/2} u\|_2^2 + \frac{1}{2\nu} \|A^{-1/2} g\|_2^2; \tag{2.4}$$

combining (2.4) with (2.3) and multiplying by 2 we have our basic energy inequality

$$\frac{d}{dt} \|u\|_2^2 + \nu \|A^{1/2} u\|_2^2 + 2\mu \|A_\varphi^{1/2} u\|_2^2 \leq \frac{1}{\nu \lambda_1} \|g\|_2^2 \tag{2.5}$$

where we note that by Poincaré's inequality $\|A^{-1/2} g\|_2 \leq \lambda_1^{-1/2} \|g\|_2$; note that (2.5) reduces to the standard NSE energy inequality when $\mu = 0$. Integrating both sides of (2.5) we have that

$$\|u\|_2^2 + \nu \int_0^T \|A^{1/2} u\|_2^2 \, ds + 2\mu \int_0^T \|A_\varphi^{1/2} u\|_2^2 \, ds \leq \|u_0\|_2^2 + \frac{1}{\nu \lambda_1} \int_0^T \|g\|_2^2 \, ds. \tag{2.6}$$

Discarding the term $u \left\| A_\varphi^{1/2} u \right\|_2^2$ in (2.5) and again using Poincaré's inequality we obtain

$$\frac{d}{dt} \|u\|_2^2 + \nu \lambda_1 \|u\|_2^2 \leq \frac{1}{\nu \lambda_1} \|g\|_2^2 \tag{2.7}$$

so that, setting

$$L_g = \sup_{t \geq 0} \|g\|_2 \tag{2.8}$$

we have that

$$\frac{d}{dt} \|u\|_2^2 + \nu \lambda_1 \|u\|_2^2 \leq \frac{L_g^2}{\nu \lambda_1}. \tag{2.9}$$

Solving the differential inequality (2.9) we have that for $u_0 = u(x, 0)$

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu \lambda_1 t} + \int_0^t \left(\frac{L_g^2}{\nu \lambda_1} \right) e^{-\nu \lambda_1 (t-s)} ds \tag{2.10}$$

or, since $L_g^2 / (\nu \lambda_1)$ is a constant,

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu \lambda_1 t} + \left(\frac{L_g}{\nu \lambda_1} \right)^2. \tag{2.11}$$

Thus, we have the a priori estimate

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \left(\frac{L_g}{\nu \lambda_1} \right)^2 \equiv U_g^2. \tag{2.12}$$

This concludes our preliminary observations and in the next section we will prove Theorem 1.

3. Proof of Theorem 1

For $[t_0, t] \subset [0, T]$ we take the $L^2([t_0, t]; H)$ inner product of both sides of (1.1) with a smooth (divergence-free) test function $v = v(x, t)$ and using self-adjointness we obtain the variational formulation

$$(u_k(t), v) + \int_{t_0}^t \nu \left(A^{1/2} u_k, A^{1/2} v \right) + \mu_k \left(A_\varphi^{1/2} u_k, A_\varphi^{1/2} v \right) + ((u_k \cdot \nabla) u_k, v) ds = (u_k(t_0), v) + \int_{t_0}^t (g, v) \, ds \tag{3.1}$$

where $(\cdot, \cdot)$ denotes the inner product in H. Note that since the $a(\lambda_j)$ are positive real numbers for $j \geq m_0 + 1$ in the definition of A_φ, we can define $A_\varphi^{1/2}$ by $A_\varphi^{1/2} = \sum_{j=m_0}^\infty a(\lambda_j)^{1/2} P_{E_j}$. We first establish suitable convergence of subsequences of $\{u_k\}_{k=1}^\infty$. We first assume that $\alpha \geq 3/2$; applying $A^{-\alpha/2}$ and the Leray projection P to both sides of (1.1), using (2.8) and (2.12), taking L^2-norms, and noting that $\alpha/2 \geq 3/4$, we have for an appropriate M_1 from (2.1) that

$$\begin{aligned}
\left\| A^{-\alpha/2}(u_k)_t \right\|_2 &\leq \nu \left\| A^{1-\alpha/2} u_k \right\|_2 + \mu_k \left\| A^{-\alpha/2} A_\varphi^{1/2} u_k \right\|_2 + \left\| P A^{-\alpha/2} (u_k \cdot \nabla u_k) \right\|_2 + \left\| A^{-\alpha/2} P g \right\|_2 \\
&\leq \frac{\nu \lambda_1^{1/2}}{\lambda_1^{\alpha/2}} \left\| A^{1/2} u_k \right\|_2 + \frac{\mu_k}{\lambda_1^{\alpha/2}} \left\| A_\varphi^{1/2} u_k \right\|_2 + \left\| A^{-(\alpha/2 - 3/4)} A^{-3/4} (u_k \cdot \nabla u_k) \right\|_2 + \lambda_1^{-\alpha/2} \|g\|_2 \\
&\leq \frac{\nu \lambda_1^{1/2}}{\lambda_1^{\alpha/2}} \|\nabla u_k\|_2 + \frac{\mu_k}{\lambda_1^{\alpha/2}} \left\| A_\varphi^{1/2} u_k \right\|_2 + \frac{1}{\lambda_1^{\alpha/2 - 3/4}} M_1 \|u_k\|_2 \|\nabla u_k\|_2 + \lambda_1^{-\alpha/2} \|g\|_2 \\
&\leq \nu \lambda_1^{(1-\alpha)/2} \|\nabla u_k\|_2 + \mu_k \lambda_1^{-\alpha/2} \left\| A_\varphi^{1/2} u_k \right\|_2 + \lambda_1^{-(\alpha/2 - 3/4)} M_1 U_g \|\nabla u_k\|_2 + \lambda_1^{-\alpha/2} L_g \tag{3.2}
\end{aligned}$$

where we have used Poincaré's inequality in the second line of (3.2). Squaring both sides of (3.2), there is a constant C_0 such that

$$\left\| A^{-\alpha/2}(u_k)_t \right\|_2^2 \leq C_0 \left[\nu^2 \lambda_1^{1-\alpha} \|\nabla u_k\|_2^2 + \mu_k^2 \lambda_1^{-\alpha} \left\| A_\varphi^{1/2} u_k \right\|_2^2 + \lambda_1^{-(\alpha-3/2)} M_1^2 U_g^2 \|\nabla u_k\|_2^2 + \lambda_1^{-\alpha} L_g^2 \right]. \tag{3.3}$$

From (3.3) and from $(\mu_k)^2 \leq \mu_k$ (without loss of generality we can assume that $\mu_k \leq 1$) we have that there is a constant $C_1 = C_1(\nu, \lambda_1, \alpha, M_1, U_g)$ such that

$$\left\| A^{-\alpha/2}(u_k)_t \right\|_2^2 \leq C_1 \left[\nu \left\| A^{1/2} u \right\|_2^2 + 2\mu_k \left\| A_\varphi^{1/2} u \right\|_2^2 \right] + \lambda_1^{-\alpha} L_g^2. \tag{3.4}$$

Integrating both sides of (3.4) and using (2.6), (2.8) we have that

$$\int_0^T \left\| A^{-\alpha/2}(u_k)_t \right\|_2^2 dt \le C_1 \left[\|u_0\|_2^2 + \frac{1}{\nu\lambda_1} L_g^2 T \right] + \lambda_1^{-\alpha} L_g^2 T. \tag{3.5}$$

From (3.5) we thus have that $\frac{d}{dt} u_k$ is uniformly bounded in $L^2([0,T]; PH^{-\alpha})$. If $5/4 \le \alpha < 3/2$, then the third term in the second line of (3.2) can be estimated by $\left\| A^{-\alpha/2}(u_k \cdot \nabla u_k) \right\|_2 \le M' \|u_k\|_2^{\alpha-1/2} \|\nabla u_k\|_2^{5/2-\alpha} \le M' U_g^{\alpha-1/2} \|\nabla u_k\|_2^{5/2-\alpha}$ for an appropriate constant M' (see e.g. [10, Chapter 6]); substituting the right-hand side of this for the third term in the last line of (3.2), it then follows from (2.6) that each term in the last line of (3.2) and hence $(u_k)_t$ is uniformly bounded in $L^r([0,T]; PH^{-\alpha})$ for $r = 4/(5-2\alpha)$. For what follows next we need the Aubin Compactness Theorem; the following is the version stated in [14, p. 224]:

Theorem 3 (*Aubin Compactness Theorem*). *Let X_0, X, X_1 be three Banach spaces and suppose that X_0 and X_1 are reflexive, $X_0 \subset X$ with compact injection and $X \subset X_1$ with continuous injection. Let $T > 0$ and $p_0, p_1 > 1$. Consider the space*

$$\mathcal{Y} = \left\{ v \in L^{p_0}([0,T]; X_0); \frac{d}{dt} v \in L^{p_1}([0,T]; X_1) \right\}$$

endowed with the norm $\|v\|_{\mathcal{Y}} = \|v\|_{L^{p_0}([0,T];X_0)} + \|\frac{d}{dt} v\|_{L^{p_1}([0,T];X_1)}$. Then the injection of $\mathcal{Y}$ into $L^{p_0}([0,T]; X)$ is compact.

We set $X_0 = V, X = H, X_1 = PH^{-\alpha}(\Omega)$, and set $p_0 = 2, p_1 = r$, where as in Theorem 1 $r = 4/(5-2\alpha)$ if $5/4 \le \alpha < 3/2$ and $r = 2$ if $\alpha \ge 3/2$. Note that X_1 has the norm $\|v\|_{PH^{-\alpha}(\Omega)} = \|A^{-\alpha/2} v\|_2$. Standard Sobolev theory shows that the hypotheses of Theorem 3 are met. From (2.6) and (3.5) we have that the sequence $\{u_k\}_{k=1}^\infty$ is contained in closed balls in X_0, X, and X_1, from which we conclude from Theorem 3 that there is a subsequence (also denoted $\{u_k\}_{k=1}^\infty$) contained in $L^2([0,T]; V)$ and a $u \in L^2([0,T]; V)$ such that $\|u_k - u\|_{L^2([0,T];H)} \to 0$, $u_k \to u$ weakly in $L^2([0,T]; V)$, and $(u_k)_t \to u_t$ weakly in $L^r([0,T]; PH^{-\alpha}(\Omega))$ as $k \to \infty$. From the weak convergence in $L^2([0,T]; V)$ it follows that

$$\nu \int_{t_0}^t \left(A^{1/2} u_k, A^{1/2} v \right) ds \to \nu \int_{t_0}^t \left(A^{1/2} u, A^{1/2} v \right) ds \tag{3.6}$$

as $k \to \infty$, while using that $|d_j| \le 1$ we have for each t that

$$\begin{aligned} \left| \mu_k \left(A_\varphi^{1/2} u_k(t), A_\varphi^{1/2} v(t) \right) \right| &= \mu_k \left| \left(u_k(t), A_\varphi v(t) \right) \right| \\ &\le \mu_k \|u_k(t)\|_2 \|A_\varphi v(t)\|_2 \\ &\le \mu_k U_g \|A^\alpha v(t)\|_2 \downarrow 0 \end{aligned} \tag{3.7}$$

and thus by the Dominated Convergence Theorem we have that

$$\int_{t_0}^t \mu_k \left(A_\varphi^{1/2} u_k, A_\varphi^{1/2} v \right) ds = \int_{t_0}^t \mu_k \left(u_k, A_\varphi v \right) ds \to 0 \tag{3.8}$$

as $k \to \infty$. Note that (3.7) is independent of the choice of m_0 and in particular of any potential dependence of A_φ on k. Since $u_k \to u$ weakly in $L^2([0,T]; PH^1(\Omega))$ we may also assume that $u_k(t) \to u(t), u_k(t_0) \to u(t_0)$ weakly in $PH^1(\Omega)$ for all $t, t_0 \in [0,T] \setminus E$ where E has Lebesgue measure zero, so that

$$(u_k(t_0), v) \to (u(t_0), v), (u_k(t), v) \to (u(t), v) \tag{3.9}$$

as $k \to \infty$ for all $t, t_0 \in [0,T] \setminus E$. For the nonlinear term we can use the classical Galerkin-approximation proof of weak Leray solutions of the NSE or directly note that for the appropriate tensor product $\otimes$ we have that

$$\begin{aligned} |((u_k \cdot \nabla)u_k, v) - ((u \cdot \nabla)u, v)| &\le |(((u_k - u) \cdot \nabla)u_k, v)| + |((u \cdot \nabla)(u_k - u), v)| \\ &\le \|((u_k - u) \cdot \nabla)u_k\|_1 \|v\|_\infty + |(\nabla(u_k - u), u \otimes v)| \\ &\le \|u_k - u\|_2 \|\nabla u_k\|_2 \|v\|_\infty + |(\nabla(u_k - u), u \otimes v)| \end{aligned} \tag{3.10}$$

where in the third line we use the estimate $|(((u_k - u) \cdot \nabla)u_k, v)| \le \|((u_k - u) \cdot \nabla)u_k\|_1 \|v\|_\infty$ and observe that $u \otimes v$ is in $L^2(\Omega)$ since $\|u \otimes v\|_2 \le \|u\|_2 \|v\|_\infty$. From this it follows by (2.6), the Dominated Convergence Theorem, from $u_k \to u$ weakly in $L^2([0,T]; PH^1(\Omega))$, and from $u_k \to u$ strongly in $L^2([0,T]; PL^2(\Omega))$ that

$$\int_{t_0}^t (g, v) + (u_k, (u_k \cdot \nabla)v) ds \to \int_{t_0}^t (g, v) + (u, (u \cdot \nabla)v) ds. \tag{3.11}$$

Combining (3.6), (3.8), (3.9), and (3.11) with (3.1) we have that

$$(u(t), v) + \nu \int_{t_0}^t \left(A^{1/2} u, A^{1/2} v \right) + ((u \cdot \nabla) u, v) ds = (u(t_0), v) + \int_{t_0}^t (g, v) \, ds \tag{3.12}$$

and thus u is a suitable weak solution of the NSE.

Meanwhile from (2.3) we have, neglecting the term involving $A_\varphi^{1/2} u_k$, that

$$\frac{1}{2} \|u_k(t)\|_2^2 + \nu \int_{t_0}^t \left\| A^{1/2} u_k \right\|_2^2 ds \le \frac{1}{2} \|u_k(t_0)\|_2^2 + \int_{t_0}^t (g, u_k) \, ds. \tag{3.13}$$

For $t_0 \in [0, T] \setminus E$ we have that

$$\frac{1}{2} \|u_k(t_0)\|_2^2 + \int_{t_0}^t (g, u_k) \, ds \to \frac{1}{2} \|u(t_0)\|_2^2 + \int_{t_0}^t (g, u) \, ds \tag{3.14}$$

and we have that

$$\limsup_{k \to \infty} \left[\frac{1}{2} \|u_k(t)\|_2^2 + \nu \int_{t_0}^t \left\| A^{1/2} u_k \right\|_2^2 ds \right] \ge \limsup_{k \to \infty} \frac{1}{2} \|u_k(t)\|_2^2 + \nu \liminf_{k \to \infty} \int_{t_0}^t \left\| A^{1/2} u_k \right\|_2^2 ds$$

$$\ge \frac{1}{2} \|u(t)\|_2^2 + \nu \int_{t_0}^t \left\| A^{1/2} u \right\|_2^2 ds \tag{3.15}$$

where we have used Fatou's lemma in the last line of (3.15). Combining (3.14) with (3.15) we have that

$$\frac{1}{2} \|u(t)\|_2^2 + \nu \int_{t_0}^t \left\| A^{1/2} u \right\|_2^2 ds \le \frac{1}{2} \|u(t_0)\|_2^2 + \int_{t_0}^t (g, u) \, ds \tag{3.16}$$

and hence u satisfies the standard energy inequality. Combining this with (3.12) we have that u is a Leray weak solution of the NSE. This completes the Proof of Theorem 1.

4. Proof of Theorem 2

Let P_k be the projection onto the first $k = m_0$ eigenspaces of A and let $Q_k = I - P_k$. With $A_\varphi = A_{\varphi,k}$ and with the decomposition $u_k = P_k u_k + Q_k u_k$ the variational formulation (3.1) becomes

$$(P_k u_k(t), v) + \int_{t_0}^t \nu \left(A^{1/2} P_k u_k, A^{1/2} v \right) + \mu \left(A_\varphi^{1/2} P_k u_k, A_\varphi^{1/2} v \right) + ((P_k u_k \cdot \nabla) P_k u_k, v) ds$$

$$= (P_k u_k(t_0), v) + \int_{t_0}^t (g, v) \, ds - \mathcal{Q}_k(u, v) \tag{4.1}$$

where

$$\mathcal{Q}_k(u, v) = (Q_k u_k(t), v) + \int_{t_0}^t \nu \left(A^{1/2} Q_k u_k, A^{1/2} v \right) + \mu \left(A_\varphi^{1/2} Q_k u_k, A_\varphi^{1/2} v \right) - (Q_k u_k(t_0), v) \, ds + \mathcal{Q} \mathcal{N}_k(u, v) \tag{4.2}$$

with

$$\mathcal{Q} \mathcal{N}_k(u, v) = \int_{t_0}^t ((Q_k u_k \cdot \nabla) P_k u_k, v) + ((P_k u_k \cdot \nabla) Q_k u_k, v) + ((Q_k u_k \cdot \nabla) Q_k u_k, v) ds. \tag{4.3}$$

We first show that $\mathcal{Q}_k(u, v) \to 0$ as $k \to \infty$. For this it will be useful to show for large enough β that $\|A^{\beta/2} Q_k u_k(t)\|_2 \to 0$ uniformly in t as $k \to \infty$. In fact we will show this provided that $\alpha > 5/4$ and $0 \le \beta < 2\alpha - 5/2$. Taking the H-inner product of both sides of (1.1) with $A^\beta Q_k u_k = Q_k A^\beta u_k$, and noting that $((u_k \cdot \nabla) u_k, Q_k A^\beta u_k v) = (Q_k (u_k \cdot \nabla) P_k u_k, A^\beta Q_k u_k v) + (Q_k (u_k \cdot \nabla) Q_k u_k, A^\beta Q_k u_k)$ where we have used that $Q_k^2 = Q_k$, we have that

$$\frac{1}{2} \frac{d}{dt} \|A^{\beta/2} Q_k u_k\|_2^2 + \nu \left\| A^{(1+\beta)/2} Q_k u_k \right\|_2^2 + \mu \left\| A^{\beta/2} A_\varphi^{1/2} Q_k u_k \right\|_2^2$$

$$\le |(Q_k (u_k \cdot \nabla) P_k u_k, A^\beta Q_k u_k)| + |(Q_k (u_k \cdot \nabla) P_k u_k, A^\beta Q_k u_k)| + (Q_k g, A^\beta Q_k u_k)$$

$$\le |(A^{-(\alpha-\beta)/2} Q_k (u_k \cdot \nabla) P_k u_k, A^{(\alpha+\beta)/2} Q_k u_k)| + |(A^{-(\alpha-\beta)/2} Q_k (u_k \cdot \nabla) Q_k u_k, A^{(\alpha+\beta)/2} Q_k u_k)|$$

$$+ |(A^{-(1-\beta)/2} Q_k g, A^{(1+\beta)/2} Q_k u_k)|. \tag{4.4}$$

Using that $A_\varphi^{1/2}Q_ku_k \geq d_{k+1}A^{\alpha/2}Q_ku_k$ in the sense of quadratic forms, setting $d = \mu d_{k+1}$, and using Young's inequality we have that

$$\frac{1}{2}\frac{d}{dt}\|A^{\beta/2}Q_ku_k\|_2^2 + \nu\left\|A^{(1+\beta)/2}Q_ku_k\right\|_2^2 + d\left\|A^{(\alpha+\beta)/2}Q_ku_k\right\|_2^2$$
$$\leq \frac{1}{d}\|A^{-(\alpha-\beta)/2}Q_k(u_k\cdot\nabla)P_ku_k\|_2^2 + \frac{d}{4}\left\|A^{(\alpha+\beta)/2}Q_ku_k\right\|_2^2 + \frac{1}{d}\|A^{-(\alpha-\beta)/2}Q_k(u_k\cdot\nabla)Q_ku_k\|_2^2$$
$$+ \frac{d}{4}\left\|A^{(\alpha+\beta)/2}Q_ku_k\right\|_2^2 + \frac{1}{2\nu}\left\|A^{-(1-\beta)/2}Q_kg\right\|_2^2 + \frac{\nu}{2}\left\|A^{(1+\beta)/2}Q_ku_k\right\|_2^2. \tag{4.5}$$

After combining the terms in (4.5), multiplying by 2, using Poincaré on the term involving g, neglecting the term $\nu\left\|A^{(1+\beta)/2}Q_ku_k\right\|_2^2$, and using (2.8), we have that

$$\frac{d}{dt}\|A^{\beta/2}Q_ku_k\|_2^2 + d\left\|A^{(\alpha+\beta)/2}Q_ku_k\right\|_2^2 \leq \frac{2}{d}\|A^{-(\alpha-\beta)/2}Q_k(u_k\cdot\nabla)P_ku_k\|_2^2$$
$$+ \frac{2}{d}\|A^{-(\alpha-\beta)/2}Q_k(u_k\cdot\nabla)Q_ku_k\|_2^2 + \frac{1}{\nu\lambda_{k+1}^{1+\beta}}\|Q_kg\|_2^2$$
$$\leq \frac{2}{d}\|A^{-(\alpha-\beta)/2}\|_2^2\|(u_k\cdot\nabla)P_ku_k\|_2^2$$
$$+ \frac{2}{d}\|A^{-(\alpha-\beta)/2}Q_k\|_2^2\|(u_k\cdot\nabla)Q_ku_k\|_2^2 + \frac{1}{\nu\lambda_{k+1}^{1+\beta}}\|g\|_2^2$$
$$\leq \frac{2}{d\lambda_{k+1}^{\alpha-\beta}}\|u_k\|_2^2\|\nabla P_ku_k\|_\infty^2 + \frac{2}{d\lambda_{k+1}^{\alpha-\beta}}\|u_k\|_2^2\|\nabla Q_ku_k\|_\infty^2 + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2. \tag{4.6}$$

Using (2.1) with the appropriate constant M_2, we have from (2.12), (4.6), and Poincaré's inequality that

$$\frac{d}{dt}\|A^{\beta/2}Q_ku_k\|_2^2 + d\left\|A^{(\alpha+\beta)/2}Q_ku_k\right\|_2^2 \leq \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}U_g^2\|A^{5/4}P_ku_k\|_2^2 + \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}U_g^2\|A^{5/4}Q_ku_k\|_2^2 + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2$$
$$\leq \frac{2M_2}{d\lambda_{k+1}^{\alpha-\beta}}\lambda_k^{5/2}U_g^2\|P_ku_k\|_2^2 + \frac{2M_2}{d\lambda_{k+1}^{2\alpha-5/2}}U_g^2\|A^{(\alpha+\beta)/2}Q_ku_k\|_2^2 + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2$$
$$\leq \frac{2M_2U_g^4}{d\lambda_{k+1}^{\alpha-\beta-5/2}} + \frac{2M_2U_g^2}{d\lambda_{k+1}^{2\alpha-5/2}}\|A^{(\alpha+\beta)/2}Q_ku_k\|_2^2 + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2. \tag{4.7}$$

Next we choose k large enough so that

$$\lambda_{k+1}^{2\alpha-5/2} \geq 4M_2U_g^2d^{-2} \tag{4.8a}$$

which guarantees that

$$\frac{2M_2U_g^2}{d\lambda_{k+1}^{2\alpha-5/2}} \leq \frac{d}{2}. \tag{4.8b}$$

Using the condition (4.8) in (4.7) and combining terms, we obtain that

$$\frac{d}{dt}\|A^{\beta/2}Q_ku_k\|_2^2 + (d/2)\left\|A^{(\alpha+\beta)/2}Q_ku_k\right\|_2^2 \leq \frac{2M_2U_g^4}{d\lambda_{k+1}^{\alpha-\beta-5/2}} + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2 \tag{4.9}$$

from which after applying Poincaré's inequality again we have that

$$\frac{d}{dt}\|A^{\beta/2}Q_ku_k\|_2^2 + (d/2)\lambda_{k+1}^\alpha\left\|A^{\beta/2}Q_ku_k\right\|_2^2 \leq \frac{2M_2U_g^4}{d\lambda_{k+1}^{\alpha-\beta-5/2}} + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2. \tag{4.10}$$

Integrating the inequality (4.10) we have that

$$\|A^{\beta/2}Q_ku_k(t)\|_2^2 \leq \|A^{\beta/2}Q_ku_0\|_2^2e^{-(d/2)\lambda_{k+1}^\alpha t} + \int_0^t\left[\frac{2M_2U_g^4}{d\lambda_{k+1}^{\alpha-\beta-5/2}} + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2\right]e^{-(d/2)\lambda_{k+1}^\alpha(t-s)}ds$$
$$\leq \|A^{\beta/2}Q_ku_0\|_2^2 + \left[\frac{2M_2U_g^4}{d\lambda_{k+1}^{\alpha-\beta-5/2}} + \frac{1}{\nu\lambda_{k+1}^{1-\beta}}L_g^2\right]\int_0^t e^{-(d/2)\lambda_{k+1}^\alpha(t-s)}ds$$

$$\leq \|A^{\beta/2}Q_k u_0\|_2^2 + \left[\frac{2M_2 U_g^4}{d\lambda_{k+1}^{\alpha-\beta-5/2}} + \frac{1}{\nu\lambda_{k+1}^{1-\beta}} L_g^2 \right] \int_0^\infty e^{-(d/2)\lambda_{k+1}^\alpha s} ds$$

$$\leq \|A^{\beta/2}Q_k u_0\|_2^2 + \frac{4M_2 U_g^4}{d^2\lambda_{k+1}^{2\alpha-\beta-5/2}} + \frac{2}{d^2\lambda_{k+1}^{\alpha-\beta+1}} L_g^2. \tag{4.11}$$

Thus we see from (4.11) and $\|A^{\beta/2}Q_k u_0\|_2 = \|Q_k A^{\beta/2} u_0\|_2$ that $\|A^{\beta/2}Q_k u_k(t)\|_2 \to 0$ as $k \to \infty$ uniformly in t. Then in particular $\|Q_k u_k(t)\|_2 \to 0$ in the case $\beta = 0$ and for $\beta > 0$ by Poincaré's inequality for any nonnegative γ with $\gamma < \beta$

$$\|A^{\gamma/2}Q_k u_k(t)\|_2^2 \leq \frac{1}{\lambda_{k+1}^{(\beta-\gamma)}} \|A^{\beta/2}Q_k u_k(t)\|_2^2$$

$$\leq \frac{1}{\lambda_{k+1}^{(\beta-\gamma)}} \left[\|A^{\beta/2}Q_k u_0\|_2^2 + \frac{4M_2 U_g^4}{d^2\lambda_{k+1}^{2\alpha-\beta-5/2}} + \frac{2}{d^2\lambda_{k+1}^{\alpha-\beta+1}} L_g^2 \right] \tag{4.12}$$

so that also $\|A^{\gamma/2}Q_k u_k(t)\|_2 \to 0$ as $k \to \infty$ uniformly in t.

To apply this fact to (4.1), we first estimate the nonlinear terms in $\mathcal{QN}_k(u, v)$; as a starting point for this we note that for a given sufficiently smooth $a(x, t)$ the operator $a \cdot \nabla$ is skew-adjoint, so in particular we have in a calculation similar to (3.10) that

$$|((P_k u_k \cdot \nabla)Q_k u_k, v)| = -|(Q_k u_k, (P_k u_k \cdot \nabla)v)|$$

$$\leq \|Q_k u_k\|_2 \|P_k u_k\|_2 \, \|\nabla v\|_\infty$$

$$\leq \|Q_k u_k\|_2 \|u_k\|_2 \, \|\nabla v\|_\infty$$

$$\leq \|Q_k u_k\|_2 U_g \, \|\nabla v\|_\infty \tag{4.13}$$

and in turn the term in the last line of (4.13) is bounded by $\|u_k\|_2 U_g \|\|\nabla v\|_\infty \leq U_g^2 \|\|\nabla v\|_\infty$. Similarly the other terms in the integrand of (4.3) are bounded by $\|Q_k u_k\|_2 U_g \|\|\nabla v\|_\infty \leq U_g^2 \|\|\nabla v\|_\infty$, so by the Dominated Convergence Theorem $\mathcal{QN}_k(u, v) \to 0$ as $k \to \infty$. For the other integral in $\mathcal{Q}_k(u, v)$, we have for example that $|(Q_k u_k(t), v)| \leq \|Q_k u_k\|_2 \, \|v\|_2$ and that $\left| \left(A_\varphi^{1/2} Q_k u_k, A_\varphi^{1/2} v \right) \right| = \left| \left(Q_k u_k, A_\varphi v \right) \right| \leq \|Q_k u_k\|_2 \, \|A_\varphi v\|_2$; the other terms in the first integral on the right-hand side can be estimated similarly, so this integral also goes to zero as $k \to \infty$ by the Dominated Convergence Theorem. Hence $\mathcal{Q}_k(u, v) \to 0$ as $k \to \infty$, and thus in essence by (4.1) $P_k u_k$ solves the weak form of the NSE except for a small perturbing term that goes to zero.

We next show that $P_k u_k$ has a convergent subsequence in a suitable topology. Applying the Leray projection P and P_k to both sides of (1.1) and noting that $P_k A_\varphi u = 0$ since $k = m_0 < m_{0+1}$, we have that $P_k u_k$ satisfies the equation

$$(P_k u_k)_t = \nu A (P_k u_k) + P_k P (u_k \cdot \nabla) u_k + P_k f \tag{4.14}$$

where $f = Pg$. Applying $A^{-1/2}$ to both sides of (4.14) as in [11, Chapter 6] and taking L^2-norms, we have that

$$\left\| A^{-1/2} (P_k u_k)_t \right\|_2 \leq \nu \left\| A^{1/2} P_k u_k \right\|_2 + \left\| P_k P A^{-1/2} (u_k \cdot \nabla u_k) \right\|_2 + \left\| A^{-1} P_k f \right\|_2$$

$$\leq \nu \left\| A^{1/2} u_k \right\|_2 + \left\| A^{-1/2} (u_k \cdot \nabla u_k) \right\|_2 + \lambda_1^{-1} \|P_k f\|_2$$

$$\leq \nu \left\| A^{1/2} u_k \right\|_2 + M' \|u_k\|_2^{1/2} \|\nabla u_k\|_2^{3/2} + \lambda_1^{-1} \|f\|_2 L_g \tag{4.15}$$

where in the last line of (4.15) we have again used the classical Sobolev inequality (as in the development after (3.5)) for an appropriate constant M'. By (2.6) each term on the right-hand side of (4.15) is in $L^{4/3}([0, T])$ and hence $\frac{d}{dt} P_k u_k \in L^{4/3}([0, T]; V')$. Thus we can again apply the Aubin Compactness Theorem (Theorem 3), with $X_0 = V, X = H$, and $X_1 = V'$, and with $p_0 = 2, p_1 = r$, from which we conclude that there is a subsequence (also denoted $\{P_k u_k\}_{k=1}^\infty$) contained in $L^2([0, T]; H)$ and a $u \in L^2([0, T]; V)$ such that $\|P_k u - u\|_{L^2([0,T];H)} \to 0, P_k u_k \to u$ weakly in $L^2([0, T]; V)$, and $(P_k u_k)_t \to u_t$ weakly in $L^2([0, T]; V')$ as $k \to \infty$. We have that (3.6)–(3.11) follow as in Section 3 above with $P_k u_k$ replacing u_k; together with $\mathcal{Q}_k(u, v) \to 0$ as $k \to \infty$ we obtain that u satisfies (3.12). Note that so far we only need that $\alpha > 5/4$, since we only have used (4.11) with $\beta = 0$.

Meanwhile, taking the H-inner product of both sides of (4.14) with $P_k u_k$ and integrating we have that

$$\frac{1}{2} \|P_k u_k(t)\|_2^2 + \nu \int_{t_0}^t \left\| A^{1/2} P_k u_k \right\|_2^2 ds = \frac{1}{2} \|P_k u_k(t_0)\|_2^2 + \int_{t_0}^t (g, P_k u_k) ds - \int_{t_0}^t (P (u_k \cdot \nabla) Q_k u_k, P_k u_k) ds \tag{4.16}$$

since $P_k^2 = P_k$ and since $(P (u_k \cdot \nabla) P_k u_k, P_k u_k) = 0$; also note that we can replace f by g since $(I - P)g$ is orthogonal to $P_k u_k$. Using that $PP_k u_k = P_k u_k$, using that $\|P_k v\|_2 \leq \|v\|_2$, using Poincaré's inequality, using that $A^{-1/2}\text{div}$ is a bounded operator

on $L^2(\Omega)$ with $\|A^{-1/2}\mathrm{div}\|_2 \leq 1$, using (2.1) for an appropriate constant M_3, and using (4.11) with $\beta = 1/2$, we have that

$$
\begin{aligned}
|(P(u_k \cdot \nabla) Q_k u_k, P_k u_k)| &= |((u_k \cdot \nabla) Q_k u_k, P_k u_k)| = \left|\left(A^{-1/2}(u_k \cdot \nabla) Q_k u_k, A^{1/2} P_k u_k\right)\right| \\
&\leq \|A^{-1/2}\mathrm{div}(u_k \otimes Q_k u_k)\|_2 \|P_k A^{1/2} u_k\|_2 \\
&\leq \|A^{-1/2}\mathrm{div}\|_2 \|u_k\|_6 \|Q_k u_k\|_3 \|A^{1/2} u_k\|_2 \\
&\leq M_3 \|A^{1/2} u_k\|_2 \|A^{1/4} Q_k u_k\|_2 \|A^{1/2} u_k\|_2 \\
&\leq M_3 \left[\|A^{1/4} Q_k u_k\|_2^2 + \frac{4M_2 U_g^4}{d^2 \lambda_{k+1}^{2\alpha-3}} + \frac{2}{d^2 \lambda_{k+1}^{\alpha+1/2}} L_g^2\right]^{1/2} \|\nabla u_k\|_2^2.
\end{aligned}
\tag{4.17}
$$

If $\alpha - 3/2 > 0$ as in the second assumption on α in Theorem 2, then the term in brackets in (4.17) goes to zero for each t as $k \to \infty$, so by (2.6), (4.17), and the Dominated Convergence Theorem the third term on the right-hand side of (4.16) goes to zero as $k \to \infty$. For a set E of Lebesgue measure zero and for $t_0 \in [0, T] \setminus E$ we thus have that

$$
\frac{1}{2}\|P_k u_k(t_0)\|_2^2 + \int_{t_0}^t (g, P_k u_k)\, ds - \int_{t_0}^t (P(u_k \cdot \nabla) Q_k u_k, P_k u_k)\, ds \to \frac{1}{2}\|u(t_0)\|_2^2 + \int_{t_0}^t (g, u)\, ds.
\tag{4.18}
$$

Since (3.15) holds as before with $P_k u_k$ replacing u_k, we thus have that (3.16) as well as (3.12) hold for the u produced here in the development following (4.15). Hence u is a Leray weak solution of the NSE and this completes the Proof of Theorem 2.

5. Conclusion

That the applicable distinguished class operators A_φ in Theorem 2 represent a "small" hyperviscosity perturbation of the NSE is reflected both in terms of the estimate (4.11) and the mechanics of showing that (3.12) and (3.16) are satisfied. In particular the perturbation term $\mathcal{Q}_k(u, v) \to 0$ as $k \to \infty$ and similarly the term involving $Q_k u_k$ in (4.18) goes to zero as well. Taken together, the convergence behavior of these perturbative terms shows that in a tangible sense $P_k u_k$ for large k is close to and behaves like a Galerkin solution for the NSE.

We observe also that (4.11) gives an estimate that streamlines the proof of global regularity for (1.1); in fact $\|A^{1/2} u(t)\|_2^2 = \|A^{1/2} P_k u(t)\|_2^2 + \|A^{1/2} Q_k u(t)\|_2^2 \leq \lambda_k \|P_k u(t)\|_2^2 + \|A^{1/2} Q_k u(t)\|_2^2$ and the latter term is bounded by (4.11). We also note that if k is larger than the Kolmogorov wavenumber κ we have an explicit estimate in (4.11) which reflects the Kolmogorov theory [21] that wavenumbers in the dissipative range are insignificant dynamically; this is shown here in a concrete theoretical sense where (4.11) is used to demonstrate that $\mathcal{Q}_k(u, v) \to 0$ as $k \to \infty$.

As we have noted Theorem 2 is designed to be applicable in the case that μ is fixed and in particular in the case $\mu = \nu^2$ as motivated by [4,5]. In this case we have argued that we can envision a cutoff m_0 such that $\nu^2 \lambda_j$ is significant for $j \geq m_0$ and insignificant for $j \leq m_0$; of future interest within the framework of this approach would be further theoretical or empirical investigation toward determining or estimating the optimal spectral cutoff m_0.

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