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Finite-difference approximation of a one-dimensional Hamilton–Jacobi/elliptic system arising in superconductivity

A. J. BRIGGS, J. R. CLAISSE AND C. M. ELLIOTT
Centre for Mathematical Analysis and Its Applications, University of Sussex, Falmer, Brighton, BN1 9QH, UK

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Finite-difference approximations to an elliptic–hyperbolic system arising in vortex density models for type II superconductors are studied. The problem can be formulated as a non-local Hamilton–Jacobi equation on a bounded domain with zero Neumann boundary conditions. Monotone schemes are defined and shown to be stable. An L^∞ error bound is proved for the approximations of the unique viscosity solution.

Keywords: Hamilton–Jacobi equation; elliptic–hyperbolic system; vortex density; evolution; superconductivity; finite difference schemes; viscosity solutions.

1. Introduction

We study finite-difference approximations of the following system:

$$\left.\begin{aligned} u_t - f(q-u)|u_x| &= 0 && x \in \Omega && t > 0 \\ u_x &= 0 && x \in \partial\Omega && t > 0 \\ u(x,0) &= u_0(x) && x \in \overline{\Omega} && \\ -\mu q_{xx} + q &= u && x \in \Omega && t \geqslant 0 \\ q_x &= (q_\infty)_x && x \in \partial\Omega && t \geqslant 0 \end{aligned}\right\} \tag{1.1}$$

where $\Omega = (0, L)$, $\mu > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and monotone non-decreasing. The function $q_\infty : \mathbb{R} \to \mathbb{R}$ is a given, twice continuously differentiable, function satisfying

$$-\mu(q_\infty)_{xx} + q_\infty = 0. \tag{1.2}$$

This is a one-dimensional form of the elliptic hyperbolic system

$$\left.\begin{aligned} u_t - f(q-u)|\nabla u| &= 0 && x \in \Omega && t > 0 \\ \nabla u \cdot \underline{\nu} &= 0 && x \in \partial\Omega && t > 0 \\ u(x,0) &= u_0(x) && x \in \overline{\Omega} && \\ -\mu \Delta q + \chi_\Omega q &= \chi_\Omega u && x \in \mathbb{R}^2 && t \geqslant 0 \\ \nabla(q - q_\infty) &\in L^2(\mathbb{R}^2) && && t \geqslant 0 \end{aligned}\right\} \tag{1.3}$$

where q_∞ is a given function, harmonic outside some large ball, and $\Omega \subset \mathbb{R}^2$ is a bounded domain and $\underline{\nu}$ is the unit outward pointing normal to $\partial\Omega$.

Here $D = \Omega \times \mathbb{R}$ is an infinitely long cylinder of type II superconducting material subject to an applied magnetic field $\underset{\sim}{H}_\infty = (\underline{H}_\infty, 0)$ transverse to the axis of the cylinder. Defining the operator $\nabla^\perp := (-\partial_2, \partial_1)$, the functions u and q are scalar potentials for the vortex density $\underline{\omega} = \nabla^\perp u$ and the magnetic field $\underline{H} = \nabla^\perp q$. The function f models the pinning of vorticity. The elliptic equation is the London equation, and the first-order equation is the law of motion for the vortex density. The system (1.3) is a two-dimensional reduction of the vortex density model derived by Chapman (1995) arising as an average over a model for the motion of very many individual line vortices. The model for the motion of line vortices can be derived from an asymptotic limit of the Ginzburg–Landau equations. In engineering applications there are many millions of vortices making it appropriate to use a vortex density model. Existence, uniqueness and long-time behaviour of the two-dimensional system in the case of f being the identity was considered by Elliott *et al.* (1998). Also included in this work was a mean curvature term. The system (1.1) corresponds to D being an infinite slab. An alternative two-dimensional reduction of the three-dimensional mean-field model is that of an infinitely long cylinder subject to an applied field parallel to the axis; this has been studied in Chapman *et al.* (1996); Elliott & Styles (2000, 2001); Schatzle & Styles (1999) and Styles (1997). See also Chapman (2000) for a review of mathematical models for flux penetration and vortex density motion in type II superconductors and Barnes *et al.* (1999) for an engineering application.

For convenience of the mathematical and numerical analysis we may rewrite the system (1.1) using the solution $\mathcal{K}$ of the elliptic equation. We set $q = \mathcal{K}u + q_\infty$, where $\mathcal{K}u := \hat{q}$ is defined by

$$\begin{aligned} -\mu\hat{q}_{xx} + \hat{q} &= u \quad x \in \Omega \quad t \geqslant 0 \\ \hat{q}_x &= 0 \quad x \in \partial\Omega \quad t \geqslant 0. \end{aligned} \tag{1.4}$$

Thus formally (1.1) may be considered as a non-local Hamilton–Jacobi equation set in a bounded interval $\Omega := (0, L) \subset \mathbb{R}$, with a homogeneous Neumann boundary condition, of the following form

$$\left.\begin{aligned} u_t + H(\mathcal{F}(u), u_x) &= 0 \qquad x \in \Omega \qquad t > 0 \\ u_x &= 0 \qquad x \in \partial\Omega \qquad t > 0 \\ u(x, 0) &= u_0(x) \qquad x \in \overline{\Omega} \end{aligned}\right\}. \tag{1.5}$$

Here $u_0 : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $\mathcal{L}_0$, and $\|u_0\|_{L^\infty(\Omega)} =: \mathcal{R}$.
The Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ is of the following form:

$$H(a, \alpha) = -f(a)|\alpha|. \tag{1.6}$$

The function $f : \mathbb{R} \to \mathbb{R}$ will have the properties

$$f(0) = 0, \tag{1.7}$$

$$\mathcal{L}_f[a - b]^- \leqslant f(a) - f(b) \leqslant \mathcal{L}_f[a - b]^+, \quad \forall a, b \in \mathbb{R}, \tag{1.8}$$

that is to say, f is Lipschitz continuous with Lipschitz constant $\mathcal{L}_f$, and monotone non-decreasing.

From this it follows that H is locally Lipschitz continuous, and monotone non-increasing with respect to its first argument, so we may write

$$\mathcal{L}_f(B[b-a]^- - A|\alpha-\beta|) \leqslant H(a,\alpha) - H(b,\beta) \leqslant \mathcal{L}_f(B[b-a]^+ + A|\alpha-\beta|), \tag{1.9}$$

where $A = \max\{|a|, |b|\}$ and $B = \max\{|\alpha|, |\beta|\}$.

The operator $\mathcal{F}$ will be used only for notational convenience since we are solely concerned with $\mathcal{F}$ having the form

$$\mathcal{F}(v) := \mathcal{K}v - v + q_\infty. \tag{1.10}$$

Here $q_\infty \in W^{1,\infty}(\Omega)$ is a given function with

$$\|q_\infty\|_{L^\infty(\Omega)} =: Q_0 \text{ and } \|(q_\infty)_x\|_{L^\infty(\Omega)} =: Q_1, \tag{1.11}$$

and $\mathcal{K} : L^2(\Omega) \to H^1(\Omega)$ is a bounded linear operator with the following properties: for all $v \in C(\bar{\Omega})$,

(1)

$$\inf_{x\in\overline{\Omega}} v(x) \leqslant \mathcal{K}v \leqslant \sup_{x\in\overline{\Omega}} v(x), \tag{1.12}$$

(2) $\exists c_\mathcal{K} \in \mathbb{R}^+$ such that

$$\|(\mathcal{K}v)_x\|_{L^\infty(\Omega)} \leqslant c_\mathcal{K}\|v\|_{L^\infty(\Omega)} \tag{1.13}$$

(3)

$$\mathcal{K}(v+r) = \mathcal{K}v + r, \quad \forall r \in \mathbb{R}. \tag{1.14}$$

That $\mathcal{K}$ defined by (1.4) has these properties which can easily be shown by standard arguments.

The theory of viscosity solutions of Hamilton–Jacobi equations was developed, for example, in Crandall *et al.* (1984) and Crandall & Lions (1983). See Crandall *et al.* (1992) and the books of Bardi & Capuzzo-Dolcetta (1997); Barles (1994) for reviews. Viscosity solutions with Neumann boundary condition are covered in Barles & Lions (1991) and Giga & Sato (1993). Numerical approximations of the Cauchy problem based on monotone finite-difference schemes were studied in Crandall & Lions (1984). See also Souganidis (1985) for an extension to more general Hamiltonians. The work by Perthame & Sanders (1988) considered a stationary problem on a bounded domain with Neumann boundary conditions.

Section 2 starts by giving a definition of viscosity solution for (1.5). We then prove the existence and uniqueness of a Lipschitz-continuous function satisfying this notion of solution. In Section 3 we give a definition of a numerical solution for (1.5) as well as proving a number of stability/monotonicity results that combine to show that the numerical solution has properties that are directly analogous to those of the viscosity solution stated in Theorem 2.1. The principal result of this paper is Theorem 4.1 in which we prove an L^∞

error bound of order $\sqrt{\Delta t}$ for monotone finite-difference schemes as defined in Section 3. Section 5 contains four different schemes which are all shown to satisfy the properties necessary for Theorem 4.1 to apply. Then in Section 6 we display a variety of numerical computations that compare the different schemes and the effect that altering the function f has upon the numerical solution. The experimental observed rates of convergence agree with our error bound.

Our problem (1.1) is a coupled system and, in general, the theory of viscosity solutions does not apply to systems because an essential requirement in the uniqueness theory is that a comparison result holds. However, the structure of (1.1) is such that a comparison principal does hold. This was exploited in the theory of Elliott *et al.* (1998). It is convenient to rewrite (1.1) using the solution operator, $\mathcal{K}$, of the London equation (1.4) as a non-local Hamilton–Jacobi equation (1.5). By using the properties of $\mathcal{K}$ we show that (1.5) enjoys a comparison principle. The numerical schemes we propose and analyse for (1.1) are derived so that there is a natural discrete analogue of $\mathcal{K}$ and so that the discretization enjoys a discrete comparison principle. This enables various stability results to be proved which are analogues of estimates which hold for the continuous problem. The error bound is proved by adapting the argument used in proving uniqueness for the continuous problem. The structural conditions we impose on H are fully exploited in the proof of comparison and in the numerical analysis. The methodology follows that for the Cauchy problem for classical Hamilton–Jacobi equations but the argument requires modification in order to deal with the particular non-local structure.

Work is in progress to extend these results to the two-dimensional case, to time-dependent applied fields and to the nucleation of vorticity on the boundary of Ω: see Claisse (2000) for details.

2. The continuous problem

In what follows we adopt the notation $\Omega_T := \Omega \times (0, T]$ and $\partial\Omega_T := \partial\Omega \times (0, T]$. Define the notation $g_{\nu_\xi}(\xi, \tau)$ to mean the directional derivative of a function $g \in C^1(\overline{\Omega_T})$ at the point (ξ, τ) in the direction ν_ξ, where ν_ξ is the outward pointing normal at $\xi \in \partial\Omega$.
We define our notion of solution in the following way.

A function $u \in C(\overline{\Omega} \times [0, T])$ is said to be a sub-solution of (1.5) if the following definition holds.

DEFINITION 2.1

$$\begin{cases} \text{For every } \phi \in C^1(\overline{\Omega}_T), \text{ if } (\xi,\tau) \in \overline{\Omega}_T \text{ is a maximum point of } u - \phi \text{ then,} \\ \qquad \phi_t(\xi,\tau) + H(\mathcal{F}(u)(\xi,\tau), \phi_x(\xi,\tau)) \leqslant 0 \qquad (\xi,\tau) \in \Omega_T \\ \min\{\phi_t(\xi,\tau) + H(\mathcal{F}(u)(\xi,\tau), \phi_x(\xi,\tau)), \phi_{\nu_\xi}(\xi,\tau)\} \leqslant 0 \qquad (\xi,\tau) \in \partial\Omega_T. \end{cases}$$

A function $u \in C(\overline{\Omega} \times [0, T])$ is said to be a super-solution of (1.5) if the following definition holds.

DEFINITION 2.2

$$\begin{cases} \text{For every } \phi \in C^1(\overline{\Omega}_T), \text{ if } (\xi,\tau) \in \overline{\Omega}_T \text{ is a minimum point of } u - \phi \text{ then,} \\ \qquad \phi_t(\xi,\tau) + H(\mathcal{F}(u)(\xi,\tau), \phi_x(\xi,\tau)) \geqslant 0 \qquad (\xi,\tau) \in \Omega_T \\ \max\{\phi_t(\xi,\tau) + H(\mathcal{F}(u)(\xi,\tau), \phi_x(\xi,\tau)), \phi_{\nu_\xi}(\xi,\tau)\} \geqslant 0 \qquad (\xi,\tau) \in \partial\Omega_T. \end{cases}$$

If u is both a sub-solution and a super-solution of (1.5), then u is said to be a viscosity solution.

The consequences of this definition of solution in relation to (1.5) are summed up in Theorem 2.1 which is proved over the course of Sections 2.1 and 2.2. Observe that the constants in these estimates are independent of time.

THEOREM 2.1 There exists a unique viscosity solution u of (1.5), furthermore

(1) $\|u(t) - v(t)\|_{L^\infty(\Omega)} \leqslant \|u_0 - v_0\|_{L^\infty(\Omega)}$ for all $t > 0$;
(2) $\|u(t)\|_{L^\infty(\Omega)} \leqslant \mathcal{R}$ for all $t > 0$;
(3) u is Lipschitz continuous in space for all $t \geqslant 0$, with Lipschitz constant $\mathcal{L}_u := \max\{\mathcal{L}_0, c_{\mathcal{K}}\mathcal{R} + Q_1\}$, i.e. $|u(x,t) - u(y,t)| \leqslant \mathcal{L}_u|x - y|$ for all $x, y \in \overline{\Omega}$;
(4) u is Lipschitz continuous in time for all $x \in \overline{\Omega}$, with Lipschitz constant $\tilde{\mathcal{L}}_u := \mathcal{L}_f\mathcal{L}_0(2\mathcal{R} + Q_0)$, i.e. $|u(x,t) - u(x,s)| \leqslant \tilde{\mathcal{L}}_u|t - s|$ for all $t, s \geqslant 0$.

2.1 *Existence of viscosity solutions*

Here we will deal with the existence of viscosity solutions; the following section will demonstrate uniqueness. The approach will be to consider problems approximating (1.5), then show that a limit function exists and that it satisfies our definition of solution.
So, consider the parabolic regularization of (1.5) below:

$$\left.\begin{aligned} u_t^\epsilon + H_\epsilon(\mathcal{F}(u^\epsilon), u_x^\epsilon) &= \epsilon u_{xx}^\epsilon && (x,t) \in \Omega_T \\ u_x^\epsilon &= 0 && (x,t) \in \partial\Omega_T \\ u^\epsilon(x,0) &= u_0^\epsilon(x) && x \in \overline{\Omega} \end{aligned}\right\} \tag{2.1}$$

where $H_\epsilon(a,\alpha) := -f_\epsilon(a)|\alpha|_\epsilon$ and $\epsilon > 0$.

The function $f_\epsilon : \mathbb{R} \to \mathbb{R}$ is such that $f_\epsilon \in C^1(\mathbb{R})$ and $f_\epsilon \to f$ uniformly as $\epsilon \to 0$; also f_ϵ has similar properties to the properties of the function f given at (1.7) and (1.8), in that it is Lipschitz continuous with Lipschitz constant $\mathcal{L}_f$, and $f_\epsilon(0) = 0$; however, we will demand in addition that f_ϵ be strictly increasing.

The function $|\cdot|_\epsilon : \mathbb{R} \to \mathbb{R}^+$ is defined by

$$|\alpha|_\epsilon := \sqrt{\alpha^2 + \epsilon^2} - \epsilon \qquad \forall \alpha \in \mathbb{R}, \tag{2.2}$$

so $|\cdot|_\epsilon$ is smooth, Lipschitz continuous with Lipschitz constant 1, and $|\cdot|_\epsilon \to |\cdot|$ uniformly as $\epsilon \to 0$. Note that $|0|_\epsilon = 0$ and $|\alpha|_\epsilon \leqslant |\alpha|$ for all $\alpha \in \mathbb{R}$.

With f_ϵ and $|\cdot|_\epsilon$ so defined we can write for H_ϵ a statement analogous to (1.9),

$$\mathcal{L}_f(B[b-a]^- - A|\alpha - \beta|) \leqslant H_\epsilon(a,\alpha) - H_\epsilon(b,\beta) \leqslant \mathcal{L}_f(B[b-a]^+ + A|\alpha - \beta|), \tag{2.3}$$

where $A = \max\{|a|, |b|\}$ and $B = \max\{|\alpha|, |\beta|\}$.

The initial datum $u_0^\epsilon \in C^\infty(\bar{\Omega})$ is chosen such that

$$\|u_0^\epsilon\|_{L^\infty(\Omega)} \leqslant \mathcal{R} \text{ and } \|(u_0^\epsilon)_x\|_{L^\infty(\Omega)} \leqslant \mathcal{L}_0;$$

we also require that there exists $C_0 \in \mathbb{R}^+$ such that

$$\|(u_0^\epsilon)_{xx}\|_{L^\infty(\Omega)} \leqslant \frac{C_0}{\sqrt{\epsilon}}.$$

Standard results (see, for example, Ladyzhenskaja *et al.*, 1968) give the existence of a unique u^ϵ solving (2.1) with sufficient regularity to perform the calculations below which yield estimates uniform in ϵ.

For convenience we use u to denote u^ϵ for $t > 0$.

(1) Let u and v be solutions of (2.1) with initial data u_0^ϵ and v_0^ϵ respectively. Define the constant $k := \|u_0^\epsilon - v_0^\epsilon\|_{L^\infty(\Omega)}$, and let $w := u - v - k$ and $w_0^\epsilon := u_0^\epsilon - v_0^\epsilon - k$ so that w solves

$$\left.\begin{aligned} w_t + H_\epsilon(\mathcal{F}(u), u_x) - H_\epsilon(\mathcal{F}(v), v_x) &= \epsilon w_{xx} \qquad && (x,t) \in \Omega_T \\ w_x &= 0 && (x,t) \in \partial\Omega_T \\ w(x,0) &= w_0^\epsilon(x) && x \in \overline{\Omega}. \end{aligned}\right\} \tag{2.4}$$

Note that

$$\begin{aligned} H_\epsilon(\mathcal{F}(v), v_x) - H_\epsilon(\mathcal{F}(u), u_x) &\leqslant \mathcal{L}_f(B[\mathcal{F}(u) - \mathcal{F}(v)]^+ + A|u_x - v_x|) \\ &\leqslant C([\mathcal{K}w - w]^+ + |w_x|) \end{aligned} \tag{2.5}$$

where A and B are as defined at (2.3), and $C := \mathcal{L}_f \max\{A, B\}$.
Also, since $\mathcal{K}(w^+ - w) \geqslant 0$ by (1.12), we have $\mathcal{K}[w]^+ \geqslant \mathcal{K}w$, and hence

$$(\mathcal{K}w - w)w^+ \leqslant [\mathcal{K}w - w]^+ w^+ \leqslant \mathcal{K}[w]^+ w^+. \tag{2.6}$$

Multiplying the first equation of (2.4) by w^+ and then integrating over Ω we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (w^+)^2 + \epsilon\int_\Omega (w^+)_x^2 &= \int_\Omega (H_\epsilon(\mathcal{F}(v), v_x) - H_\epsilon(\mathcal{F}(u), u_x))w^+ \\ &\leqslant C\int_\Omega ([\mathcal{K}w - w]^+ + |w_x|)w^+ \text{ by (2.5)} \\ &\leqslant C\int_\Omega (\mathcal{K}(w^+) + |(w^+)_x|)w^+ \text{ by (2.6)} \\ &\leqslant \frac{C_\epsilon}{2}\int_\Omega (w^+)^2 + \epsilon\int_\Omega (w^+)_x^2. \end{aligned}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w^+(t)\|_{L^2(\Omega)}^2 \leqslant C_\epsilon\|w^+(t)\|_{L^2(\Omega)}^2 \quad \text{for all } t \in [0, T].$$

Gronwall's inequality shows that $w(t) \leqslant 0$ for all $t \in [0, T]$ and thus

$$\sup_{(x,t)\in\overline{\Omega}\times[0,T]} \{u(x,t) - v(x,t)\} \leqslant \|u_0^\epsilon - v_0^\epsilon\|_{L^\infty(\Omega)}.$$

Repeating this argument with $w := k - (u - v)$ and $w_0^\epsilon := k - (u_0^\epsilon - v_0^\epsilon)$ provides the necessary lower bound and hence

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \leqslant \|u_0^\epsilon - v_0^\epsilon\|_{L^\infty(\Omega)} \text{ for all } t \in [0, T].$$

(2) $\|u(t)\|_{L^\infty(\Omega)} \leqslant \mathcal{R}$ for all $t \in [0, T]$ follows immediately from (1) by putting $v_0^\epsilon \equiv 0$.

(3) Define $v := u_x$ where u solves (2.1), and differentiate (2.1) with respect to x to obtain

$$\left.\begin{aligned} v_t + \partial_1 H_\epsilon(\mathcal{F}(u), v)(\mathcal{F}(u))_x & \\ + \partial_2 H_\epsilon(\mathcal{F}(u), v)v_x = \epsilon v_{xx} & \qquad (x,t) \in \Omega_T \\ v = 0 & \qquad (x,t) \in \partial\Omega_T \\ v(x,0) = (u_0^\epsilon)_x(x) & \qquad x \in \overline{\Omega}. \end{aligned}\right\} \tag{2.7}$$

Let $v(\hat{x},\hat{t}) = \sup_{(x,t)\in\overline{\Omega}\times[0,T]} v(x,t)$ and suppose that $v(\hat{x},\hat{t}) > \mathcal{L}_0$, thus $\hat{t} \in (0, T]$; also $v = 0$ on $\partial\Omega$, so $\hat{x} \notin \partial\Omega$. At such a maximum point of v we have

$$v_t(\hat{x},\hat{t}) \geqslant 0, \quad v_x(\hat{x},\hat{t}) = 0, \quad v_{xx}(\hat{x},\hat{t}) \leqslant 0.$$

Substituting this into (2.7) yields

$$0 \geqslant \partial_1 H_\epsilon(\mathcal{F}(u), v)((\mathcal{K}u)_x - v + (q_\infty)_x). \tag{2.8}$$

But

$$\partial_1 H_\epsilon(\mathcal{F}(u), v) = -f_\epsilon'(\mathcal{F}(u))|v|_\epsilon < 0$$

since f_ϵ is strictly monotone increasing and $v(\hat{x},\hat{t}) \neq 0$. This implies that

$$(\mathcal{K}u)_x - v + (q_\infty)_x \geqslant 0,$$

which when rearranged yields

$$v \leqslant (\mathcal{K}u)_x + (q_\infty)_x \leqslant c_\mathcal{K}\mathcal{R} + Q_1. \tag{2.9}$$

Considering this bound in light of the assumption that $v(\hat{x},\hat{t}) > \mathcal{L}_0$ we obtain

$$\sup_{x\in\overline{\Omega}} v \leqslant \max\{\mathcal{L}_0, c_\mathcal{K}\mathcal{R} + Q_1\} = \mathcal{L}_u.$$

Similarly, by considering a minimum of v, we find

$$\inf_{x\in\overline{\Omega}} v \geqslant -\mathcal{L}_u,$$

hence

$$\|u_x(t)\|_{L^\infty(\Omega)} \leqslant \mathcal{L}_u \text{ for all } t \in [0,T].$$

(4) Define $v := u_t$ where u solves (2.1), and differentiate (2.1) with respect to t to obtain

$$\left.\begin{aligned} v_t + \partial_1 H_\epsilon(\mathcal{F}(u), u_x)(\mathcal{K}v - v) & \\ + \partial_2 H_\epsilon(\mathcal{F}(u), u_x)v_x = \epsilon v_{xx} & \qquad (x,t) \in \Omega_T \\ v_x = 0 & \qquad (x,t) \in \partial\Omega_T \\ v(x,0) = u_t(x,0) & \qquad x \in \overline{\Omega}. \end{aligned}\right\} \tag{2.10}$$

Define the constant $k := \|u_t(\cdot,0)\|_{L^\infty(\Omega)}$, and let $w := u_t - k$ so that w solves

$$\left.\begin{aligned} w_t + \partial_1 H_\epsilon(\mathcal{F}(u), u_x)(\mathcal{K}w - w) & \\ + \partial_2 H_\epsilon(\mathcal{F}(u), u_x) w_x &= \epsilon w_{xx} && (x,t) \in \Omega_T \\ w_x &= 0 && (x,t) \in \partial\Omega_T \\ w(x,0) &= u_t(x,0) - k && x \in \overline{\Omega}. \end{aligned}\right\} \tag{2.11}$$

Define the constant C by

$$\max\left\{ \sup_{|u|\leqslant\mathcal{R},|u_x|\leqslant\mathcal{L}_u} -\partial_1 H_\epsilon(\mathcal{F}(u), u_x), \sup_{|u|\leqslant\mathcal{R},|u_x|\leqslant\mathcal{L}_u} |\partial_2 H_\epsilon(\mathcal{F}(u), u_x)| \right\} \leqslant \mathcal{L}_f \max\{\mathcal{L}_u, 2\mathcal{R} + Q_0\} =: C. \tag{2.12}$$

Like in part (1), we multiply the first equation of (2.11) by w^+, integrate over Ω and by (2.12) and (2.6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w^+(t)\|^2_{L^2(\Omega)} \leqslant C_\epsilon \|w^+(t)\|^2_{L^2(\Omega)} \quad \text{for all } t \in (0,T].$$

Appealing to Gronwall's inequality, as in the proof of (2.1.1), we obtain

$$\|u_t\|_{L^\infty(0,T,L^\infty(\Omega))} \leqslant \|u_t(\cdot,0)\|_{L^\infty(\Omega)}.$$

Finally we look for a bound for $\|u_t(\cdot,0)\|_{L^\infty(\Omega)}$. By the regularity of u we have the following:

$$\begin{aligned} u_t(\cdot,0) &= \lim_{t\to 0}\{\epsilon u_{xx}(\cdot,t) - H_\epsilon(\mathcal{F}(u)(\cdot,t), u_x(\cdot,t))\} \\ &= \epsilon(u_0^\epsilon)_{xx}(\cdot) - H_\epsilon(\mathcal{F}(u_0^\epsilon)(\cdot), (u_0^\epsilon)_x(\cdot)). \end{aligned}$$

Thus, making use of the appropriate bounds on u_0^ϵ and its derivatives, we obtain

$$\|u_t(\cdot,0)\|_{L^\infty(\Omega)} \leqslant C_0\sqrt{\epsilon} + \mathcal{L}_f(2\mathcal{R} + Q_0)\mathcal{L}_0 := \tilde{\mathcal{L}}_u^\epsilon. \tag{2.13}$$

That $u^\epsilon, u_x^\epsilon, u_t^\epsilon \in L^\infty(0,T;L^\infty(\Omega))$, and are bounded independently of $\epsilon \leqslant \epsilon_0$, implies that there exists $u \in W^{1,\infty}(\Omega_T)$ being the limit of a subsequence of u^ϵ, which satisfies properties 1–4 of Theorem 2.1. It remains to show that u is a viscosity solution of (1.5). For the details of this argument, see Briggs *et al.* (1999).

2.2 *Uniqueness of viscosity solutions*

Suppose that u and v are both viscosity solutions of (1.5), with $u \neq v$, and further suppose that u is Lipschitz continuous in space with Lipschitz constant $\mathcal{L}_u$ where existence is proved in Section 2.1. Let $\mathcal{R}$ be a fixed positive constant such that

$$\max\{\|u(\cdot,t)\|_{L^\infty(\Omega)}, \|v(\cdot,t)\|_{L^\infty(\Omega)}\} \leqslant \mathcal{R}.$$

Then

$$\max\{\|\mathcal{F}(u(\cdot,t))\|_{L^\infty(\Omega)}, \|\mathcal{F}(v(\cdot,t))\|_{L^\infty(\Omega)}\} \leqslant 2\mathcal{R} + Q_0.$$

Let $\tilde{A} = 2\mathcal{R} + Q_0$, $\tilde{B} = \mathcal{L}_u + 1$ and $\mathcal{L}_H = \mathcal{L}_f \max\{\tilde{A}, \tilde{B}\}$. In order to prove uniqueness we initially work on a reduced time interval $[0, T]$ for $T = \frac{1}{2\mathcal{L}_H}$.

We may suppose that $\sup_{(x,t)\in\overline{\Omega}_T} |u(x,t) - v(x,t)| = \sup_{(x,t)\in\overline{\Omega}_T}\{u(x,t) - v(x,t)\}$; if this were not the case then one would use a similar argument with u and v interchanged in the test function Φ defined at (2.16).

Plainly there exist $(x_0, t_0) \in \overline{\Omega}_T$, such that

$$u(x_0, t_0) - v(x_0, t_0) = \sup_{(x,t)\in\bar{\Omega}_T}\{u(x,t) - v(x,t)\} =: \sigma > 0.$$

Let $\beta, \alpha > 0$ and $\beta \leqslant \frac{1}{2}$. Let $\gamma \in \left(0, \frac{1}{2T}\right]$ be fixed.

Define functions $d : \overline{\Omega} \to \mathbb{R}$ and $\Gamma : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ by

$$d(x) := \frac{x(x-L)}{L}, \quad \Gamma(x,y) := d(x) + d(y). \tag{2.14}$$

Note that $d(\xi) = 0$ and $d'(\xi) \cdot \nu_\xi = 1$ where $\xi \in \partial\Omega$ and $-\frac{L}{4} \leqslant d(x) < 0$ when $x \in \Omega$.

Now define the functions $\Psi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, and $\Phi : \overline{\Omega} \times [0, T] \times \overline{\Omega} \times [0, T] \to \mathbb{R}$, by

$$\Psi(x,y) := \frac{|x-y|^2}{\alpha} + \beta\Gamma(x,y), \tag{2.15}$$

$$\Phi(x,t,y,s) := u(x,t) - v(y,s) - \frac{\gamma}{2}\sigma(t+s) - \frac{(t-s)^2}{\alpha} - \Psi(x,y). \tag{2.16}$$

The use of Γ in (2.15) is to deal with the Neumann boundary condition. The remaining part of Ψ differs from that of Crandall & Lions (1983) because we are able to use the boundedness of the domain Ω. See also Giga & Sato (1993).

Note that for $\xi, \eta \in \partial\Omega$, using the properties of d,

$$\Psi_x(\xi, y) \cdot \nu_\xi = \left(\frac{2(\xi - y)}{\alpha} + \beta d'(\xi)\right) \cdot \nu_\xi \geqslant \beta,$$

$$-\Psi_y(x, \eta) \cdot \nu_\eta = -\left(-\frac{2(x-\eta)}{\alpha} + \beta d'(\eta)\right) \cdot \nu_\eta \leqslant -\beta.$$

Let Φ be maximized by $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$; such a point must exist since Φ is a continuous function on a compact set. Considering Φ evaluated at (x_0, t_0, x_0, t_0) and at $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$, we find that

$$\begin{aligned}\Phi(\hat{x}, \hat{t}, \hat{y}, \hat{s}) &= u(\hat{x}, \hat{t}) - v(\hat{y}, \hat{s}) - \frac{\gamma}{2}\sigma(\hat{t} + \hat{s}) - \frac{1}{\alpha}(\hat{t} - \hat{s})^2 - \Psi(\hat{x}, \hat{y}) \\ &\geqslant \Phi(x_0, t_0, x_0, t_0) \\ &\geqslant \sigma - \gamma\sigma T \end{aligned} \tag{2.17}$$

and rearranging yields the following inequality:

$$\frac{1}{\alpha}|\hat{x} - \hat{y}|^2 + \frac{1}{\alpha}(\hat{t} - \hat{s})^2 \leqslant 2\mathcal{R} - \sigma + \gamma\sigma T + \frac{\beta L}{2}. \tag{2.18}$$

In the light of (2.18) observe the following implications.

(a) Let $\{\alpha_l\}_{l\in\mathbb{N}}$ be a sequence such that $\alpha_l > 0$ for all $l \geqslant 1$ and $\alpha_l \to 0$ as $l \to \infty$, and let

$(\hat{x}_l, \hat{t}_l, \hat{y}_l, \hat{s}_l)$ be a point maximising Φ for $\alpha = \alpha_l$. The inequality (2.18) implies that there exists a subsequence of l (for convenience also denoted by l) such that

$$\frac{|\hat{x}_l - \hat{y}_l|^2}{\alpha_l} \to \xi \text{ and } \frac{(\hat{t}_l - \hat{s}_l)^2}{\alpha_l} \to \tau \text{ as } l \to \infty \tag{2.19}$$

where $\xi, \tau \in \mathbb{R}^+$.
This in turn implies that

$$\hat{x}_l, \hat{y}_l \to \hat{x}_0 \in \overline{\Omega} \text{ and } \hat{t}_l, \hat{s}_l \to \hat{t}_0 \in [0, T] \text{ as } l \to \infty. \tag{2.20}$$

Plainly

$$\Phi(x, t, y, s) \;\; (\text{for } \alpha = \alpha_l) \leqslant \Phi(\hat{x}_l, \hat{t}_l, \hat{y}_l, \hat{s}_l) \tag{2.21}$$

for all $(x, t, y, s) \in \overline{\Omega} \times [0, T] \times \overline{\Omega} \times [0, T]$. Letting $(x, t, y, s) = (\hat{x}_0, \hat{t}_0, \hat{x}_0, \hat{t}_0)$ in (2.21) yields the following inequality:

$$\begin{aligned} u(\hat{x}_0, \hat{t}_0) - v(\hat{x}_0, \hat{t}_0) - \gamma\sigma\hat{t}_0 - 2\beta d(\hat{x}_0) &\leqslant u(\hat{x}_l, \hat{t}_l) - v(\hat{y}_l, \hat{s}_l) - \frac{\gamma}{2}\sigma(\hat{t}_l + \hat{s}_l) \\ &\quad - \frac{1}{\alpha_l}(\hat{t}_l - \hat{s}_l)^2 - \frac{1}{\alpha_l}|\hat{x}_l - \hat{y}_l|^2 - \beta\Gamma(\hat{x}_l, \hat{y}_l). \end{aligned}$$

Letting $l \to \infty$ gives $\xi + \tau \leqslant 0$, which implies that $\xi = 0$ and $\tau = 0$. It follows that we may write (2.19) as

$$\frac{|\hat{x} - \hat{y}|^2}{\alpha} \to 0 \text{ and } \frac{(\hat{t} - \hat{s})^2}{\alpha} \to 0 \text{ as } \alpha \to 0. \tag{2.22}$$

(b) $\hat{t}_0 > 0$. First note that by (2.17) we have the following lower bound for $\Phi(\hat{x}_l, \hat{t}_l, \hat{y}_l, \hat{s}_l)$:

$$\Phi(\hat{x}_l, \hat{t}_l, \hat{y}_l, \hat{s}_l) \geqslant \sigma - \gamma\sigma T. \tag{2.23}$$

By the continuity of Φ,

$$\Phi(\hat{x}_0, \hat{t}_0, \hat{x}_0, \hat{t}_0) = \lim_{l\to\infty} \Phi(\hat{x}_l, \hat{t}_l, \hat{y}_l, \hat{s}_l) \geqslant \sigma - \gamma\sigma T \tag{2.24}$$

so

$$u(\hat{x}_0, \hat{t}_0) - v(\hat{x}_0, \hat{t}_0) \geqslant \sigma - \gamma\sigma T + \gamma\sigma\hat{t}_0 + 2\beta d(\hat{x}_0). \tag{2.25}$$

Suppose now that $\hat{t}_0 = 0$ then we have that

$$\begin{aligned} 0 &\geqslant \sigma - \gamma\sigma T + 2\beta d(\hat{x}_0) \\ &\geqslant \frac{\sigma}{2} + 2\beta d(\hat{x}_0) \text{ by our choice of } \gamma \leqslant \tfrac{1}{2T} \\ &> 0 \quad \text{for } \beta \text{ sufficiently small,} \end{aligned} \tag{2.26}$$

a contradiction, hence $\hat{t}_0 > 0$.
Having established the results (a), (b) we continue with our proof.
First observe that since at $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ Φ is minimized, the function

$$(x, t) \to u(x, t) - \frac{\gamma}{2}\sigma t - \Psi(x, \hat{y}) - \frac{(t - \hat{s})^2}{\alpha}$$

has a maximum at $(x, t) = (\hat{x}, \hat{t})$. Therefore, setting $\phi : \overline{\Omega}_T \to \mathbb{R}$ as

$$\phi(x, t) := \frac{\gamma}{2}\sigma t + \Psi(x, \hat{y}) + \frac{(t - \hat{s})^2}{\alpha},$$

we see that definition (2.1) must be satisfied, i.e.

$$\phi_t(\hat{x}, \hat{t}) + H(\mathcal{F}(u)(\hat{x}, \hat{t}), \phi_x(\hat{x}, \hat{t})) \leqslant 0 \quad (\hat{x}, \hat{t}) \in \Omega_T, \tag{2.27}$$

$$\min\{\phi_t(\hat{x}, \hat{t}) + H(\mathcal{F}(u)(\hat{x}, \hat{t}), \phi_x(\hat{x}, \hat{t})), \phi_{\nu_{\hat{x}}}(\hat{x}, \hat{t})\} \leqslant 0 \quad (\hat{x}, \hat{t}) \in \partial\Omega_T. \tag{2.28}$$

However, by construction, $\phi_{\nu_{\hat{x}}}(\hat{x}, \hat{t}) = \nu_{\hat{x}} \cdot \Psi_x(\hat{x}, \hat{y}) \geqslant \beta > 0$ so we need only consider (2.27). This is the point where we have exploited the use of Γ in (2.15).

Substituting in the derivatives of ϕ yields

$$\frac{\gamma}{2}\sigma + \frac{2(\hat{t} - \hat{s})}{\alpha} + H(\mathcal{F}(u)(\hat{x}, \hat{t}), \Psi_x(\hat{x}, \hat{y})) \leqslant 0. \tag{2.29}$$

Similarly, the function

$$(y, s) \to v(y, s) + \frac{\gamma}{2}\sigma s + \Psi(\hat{x}, y) + \frac{(\hat{t} - s)^2}{\alpha}$$

has a minimum at $(y, s) = (\hat{y}, \hat{s})$. Making use of definition (2.2) and properties of Ψ we obtain the following inequality:

$$-\frac{\gamma}{2}\sigma + \frac{2(\hat{t} - \hat{s})}{\alpha} + H(\mathcal{F}(v)(\hat{y}, \hat{s}), -\Psi_y(\hat{x}, \hat{y})) \geqslant 0. \tag{2.30}$$

Combining (2.29) and (2.30), and inserting the derivatives of Ψ gives

$$\begin{aligned}
\gamma\sigma &\leqslant H(\mathcal{F}(v)(\hat{y}, \hat{s}), -\Psi_y(\hat{x}, \hat{y})) - H(\mathcal{F}(u)(\hat{x}, \hat{t}), \Psi_x(\hat{x}, \hat{y}))\\
&\leqslant \mathcal{L}_f(\max\{|\Psi_x(\hat{x}, \hat{y})|, |\Psi_y(\hat{x}, \hat{y})|, \}[\mathcal{F}(u)(\hat{x}, \hat{t}) - \mathcal{F}(v)(\hat{y}, \hat{s})]^+ \cdot\\
&\quad + \max\{|\mathcal{F}(u)(\hat{x}, \hat{t})|, |\mathcal{F}(v)(\hat{y}, \hat{s})|\}|\Psi_x(\hat{x}, \hat{y}) + \Psi_y(\hat{x}, \hat{y})|)\\
&\leqslant \mathcal{L}_H((\mathcal{F}(u)(\hat{x}, \hat{t}) - \mathcal{F}(v)(\hat{y}, \hat{s}))^+ + 2\beta),
\end{aligned} \tag{2.31}$$

where we have used (1.9) which is justified since

$$\begin{aligned}
&\max\{|\mathcal{F}(u)(\hat{x}, \hat{t})|, |\mathcal{F}(v)(\hat{y}, \hat{s})|\} \leqslant \tilde{A},\\
&\quad \max\{|\Psi_x(\hat{x}, \hat{y})|, |\Psi_y(\hat{x}, \hat{y})|\} \leqslant \tilde{B}.
\end{aligned} \tag{2.32}$$

This is the point where we have exploited the structure (1.6)–(1.9). The former of these these inequalities is a consequence of (1.12) and the latter is obtained as follows. Note first that the function $x \to u(x, \hat{t}) - \Psi(x, \hat{y})$ has a maximum at $x = \hat{x}$. Hence, if $\hat{x} \in \Omega$,

$$\begin{aligned}(u(x, \hat{t}) - \Psi(x, \hat{y})) - (u(\hat{x}, \hat{t}) - \Psi(\hat{x}, \hat{y}))) &\leqslant 0 \\ \Rightarrow \Psi(\hat{x}, \hat{y}) - \Psi(x, \hat{y}) &\leqslant \mathcal{L}_u |\hat{x} - x| \\ \Rightarrow |\Psi_x(\hat{x}, \hat{y})| &\leqslant \mathcal{L}_u. \end{aligned} \tag{2.33}$$

If $\hat{x} \in \partial\Omega$ then we have the one-sided bounds, $\Psi_x(0, \hat{y}) \geqslant -\mathcal{L}_u$ and $\Psi_x(L, \hat{y}) \leqslant \mathcal{L}_u$. However, by the construction of Ψ, $\Psi_x(0, \hat{y}) \leqslant -\beta$ and $\Psi_x(L, \hat{y}) \geqslant \beta$; thus, taking $\beta < \mathcal{L}_u$, we have that (2.33) is for all $\hat{x} \in \overline{\Omega}$.

Since v is not necessarily Lipschitz continuous we cannot use an equivalent argument for $|\Psi_y(\hat{x}, \hat{y})|$. However,

$$\begin{aligned}|\Psi_y(\hat{x}, \hat{y})| &= |-\Psi_x(\hat{x}, \hat{y}) + \beta(d'(\hat{x}) + d'(\hat{y}))| \\ &\leqslant \mathcal{L}_u + |\beta(d'(\hat{x}) + d'(\hat{y}))| \\ &\leqslant \mathcal{L}_u + 2\beta \end{aligned} \tag{2.34}$$

which yields (2.32) since $\beta \leqslant \frac{1}{2}$.

Since (2.31) is true for all $\alpha > 0$ it is certainly true for all α_l. Letting $l \to \infty$ we obtain

$$\gamma\sigma \leqslant \mathcal{L}_H[\mathcal{F}(u)(\hat{x}_0, \hat{t}_0) - \mathcal{F}(v)(\hat{x}_0, \hat{t}_0)]^+ + 2\mathcal{L}_H\beta. \tag{2.35}$$

We now use the properties of the solution operator $\mathcal{K}$. By the definition of $\mathcal{F}$, the property (1.12) of $\mathcal{K}$ and (2.25) we obtain the following bound:

$$\begin{aligned}\mathcal{F}(u)(\hat{x}_0, \hat{t}_0) - \mathcal{F}(v)(\hat{x}_0, \hat{t}_0) &= \mathcal{K}(u - v)(\hat{x}_0, \hat{t}_0) - (u - v)(\hat{x}_0, \hat{t}_0) \\ &\leqslant \sigma - \left(\sigma - \gamma\sigma T - \frac{\beta L}{2}\right) \\ &= \gamma\sigma T + \frac{\beta L}{2} \end{aligned} \tag{2.36}$$

and hence (2.35) becomes

$$\gamma\sigma \leqslant \mathcal{L}_H\left(\gamma\sigma T + \beta\left(2 + \frac{L}{2}\right)\right). \tag{2.37}$$

Rearranging and noting that $\mathcal{L}_H T = \frac{1}{2}$ yields

$$\sigma \leqslant \beta\frac{2\mathcal{L}_H}{\gamma}\left(2 + \frac{L}{2}\right) = C\beta. \tag{2.38}$$

Now, as β can be made as small as we like, this implies that $\sigma = 0$; thus $u(x, t) = v(x, t)$ for all $(x, t) \in \Omega_T$. Repeated use of this result implies uniqueness of solutions to (1.5). (Uniqueness is proved on successive time intervals $I_k := [T_k, T_{k+1}]$ where $T_k := \frac{k}{2\mathcal{L}_H}$ and for each of these problems the initial conditions at time T_k are defined to be the unique value of the solution for previous time interval evaluated at time T_k.)

3. Numerical schemes

3.1 *Notation*

We introduce the following notation for the finite difference grid.
Mesh sizes are defined by

$$h := \tfrac{L}{M} \text{ and } \Delta t := \tfrac{T}{N} \text{ where } M, N \in \mathbb{N}, \text{ with } \lambda := \tfrac{\Delta t}{h}. \tag{3.1}$$

Grid points are denoted by

$$x_j := jh \text{ for } j = 0, 1, \ldots, M \text{ and } t_k := k\Delta t \text{ for } k = 0, 1, \ldots, N, \tag{3.2}$$

lying on the lattices

$$\triangle := \{x_j : j = 0, 1 \ldots, M\} \text{ and } \triangle_T := \{(x_j, t_k) : x_j \in \triangle, k = 0, 1 \ldots, N\}. \tag{3.3}$$

The value of our numerical approximation at the point (x_j, t_k) will be written U_j^k. Lattice functions belonging to $S^h := l^\infty(\triangle)$ will be denoted, for example, by $\mathbf{V} := \{V_0, \ldots, V_M\}$, where V_j is the value of $\mathbf{V}$ at x_j.
We define an interpolation operator $I^h : C(\bar{\Omega}) \to S^h$ such that

$$(I^h v)_j = v(x_j) \text{ for } v \in C(\bar{\Omega}).$$

The discrete versions of the operators $\mathcal{F}, \mathcal{K}$ (to be defined in Section (5.1)) will be denoted by $\mathcal{F}^h, \mathcal{K}^h$ respectively, and we set $q_\infty^h := I^h q_\infty$ although q_∞^h could be chosen differently, e.g. a numerical solution for q_∞ from (1.2).
In order to reduce clutter, in what follows we write

$$\Delta_+ V_j = V_{j+1} - V_j, \text{ and } D_+^h V_j = \frac{\Delta_+ V_j}{h}, \tag{3.4}$$

and use the following norms:

$$\|v\| = \sup_{x \in \overline{\Omega}} |v(x)|, \quad v \in L^\infty(\bar{\Omega}) \text{ and } \|\mathbf{V}\|_h = \sup_{0 \leqslant j \leqslant M} |V_j|, \quad \mathbf{V} \in S^h.$$

3.2 *Definition of the schemes*

We create a $\mathbf{U}^k \in S^h$, for $k \geqslant 0$, so that U_j^k approximates $u(x_j, t_k)$, by a scheme of the form

$$\mathbf{U}^0 = I^h u_0 \text{ and } \mathbf{U}^k = \vec{G}(\mathbf{U}^{k-1}) \text{ for } k = 1, 2, \ldots, \tag{3.5}$$

where $\vec{G} : S^h \to S^h$.
$\vec{G}$ is defined in terms of the following functions:

$$G_0, G_M : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \quad G : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, \quad \mathcal{K}^h : S^h \to S^h,$$

so that

$$\left.\begin{aligned}\vec{G}_0(\mathbf{V}) &= G_0(a_0; V_0, V_1) := G(a_0; V_{-1}, V_0, V_1)\\ \vec{G}_j(\mathbf{V}) &= G(a_j; V_{j-1}, V_j, V_{j+1}), \quad j = 1, \ldots, M-1,\\ \vec{G}_M(\mathbf{V}) &= G_M(a_M; V_{M-1}, V_M) := G(a_M; V_{M-1}, V_M, V_{M+1})\end{aligned}\right\} \tag{3.6}$$

where

$$\boldsymbol{a} := \mathcal{F}^h(\mathbf{V}) \tag{3.7}$$

and V_{-1}, V_{M+1}, which depend on $\mathbf{V}$, are such that for a given value of $\theta \in [0, 1]$,

$$V_{-1} = \theta V_0 + (1-\theta)V_1 \text{ and } V_{M+1} = \theta V_M + (1-\theta)V_{M-1}. \tag{3.8}$$

This gives us

$$|\Delta_+ V_{-1}| \leqslant |\Delta_+ V_0| \text{ and } |\Delta_+ V_M| \leqslant |\Delta_+ V_{M-1}|, \tag{3.9}$$

and allows for the 'natural' choices of these functions, i.e. $\theta = 0$,

$$\Rightarrow V_{-1} = V_1 \text{ and } V_{M+1} = V_{M-1}$$

for central difference schemes, such as the schemes S1 and S2 given in Section 5.2, and $\theta = 1$,

$$\Rightarrow V_{-1} = V_0 \text{ and } V_{M+1} = V_M$$

for the upwind and max schemes, S3 and S4 respectively, also given in Section 5.2.
In order to proceed, and prove the results we want, the following assumptions have to be made concerning the properties of G and $\mathcal{K}^h$.

Property $G1$

The function G can be written in *differenced form*, that is, there exists a mapping g called the *numerical Hamiltonian*

$$g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R},$$

such that

$$G(a; v_{-1}, v_0, v_1) = v_0 - \Delta t g(a; D_+^h v_{-1}, D_+^h v_0). \tag{3.10}$$

Property $G2$

The function G is *monotone* on $\{\mathrm{A}, \mathcal{L}\}$, that is, for $\underline{v} := (v_{-1}, v_0, v_1)$ and $\underline{w} := (w_{-1}, w_0, w_1)$

$$\underline{v} \leqslant \underline{w} \Rightarrow G(a; \underline{v}) \leqslant G(a; \underline{w}) \tag{3.11}$$

provided $|a| \leqslant \mathrm{A}$ and $|D_+^h v_{-1}|, |D_+^h v_0|, |D_+^h w_{-1}|, |D_+^h w_0| \leqslant \mathcal{L}$.

Property G3

The numerical Hamiltonian g is *consistent* with the Hamiltonian H, i.e.

$$g(a;\alpha,\alpha) = H(a,\alpha), \quad \forall a,\alpha \in \mathbb{R}. \tag{3.12}$$

Property G4

The numerical Hamiltonian g has the local Lipschitz property (the discrete analogue of (1.9)) that there exist non-negative constants $C_g(A)$, $\tilde{C}_g(B)$ such that

$$\begin{aligned}\mathcal{L}_f(\tilde{C}_g(B)[b-a]^- - C_g(A)(|\alpha-\gamma|+|\beta-\delta|)) &\leqslant g(a;\alpha,\beta) - g(b;\gamma,\delta) \\ &\leqslant \mathcal{L}_f(\tilde{C}_g(B)[b-a]^+ + C_g(A)(|\alpha-\gamma|+|\beta-\delta|)),\end{aligned} \tag{3.13}$$

where $A = \max\{|a|,|b|\}$ and $B = \max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}$.

Property $\mathcal{K}1$

Define $\mathbf{e} \in S^h$ by

$$\mathbf{e} := (1,1,\ldots,1). \tag{3.14}$$

For all $\mathbf{V} \in S^h$ we require that the linear mapping $\mathcal{K}^h : S^h \to S^h$ satisfies

(1)

$$\mathbf{e}\min_{0\leqslant j\leqslant M} V_j \leqslant \mathcal{K}^h\mathbf{V} \leqslant \mathbf{e}\max_{0\leqslant j\leqslant M} V_j; \tag{3.15}$$

(2) $\exists C_{\mathcal{K}} \in \mathbb{R}^+$ such that

$$|(\mathcal{K}^h\mathbf{V})_i - (\mathcal{K}^h\mathbf{V})_j| \leqslant C_{\mathcal{K}}\|\mathbf{V}\|_h|i-j|h, \quad \forall\, 0 \leqslant i,j \leqslant M; \tag{3.16}$$

(3)

$$\mathcal{K}^h(\mathbf{V}+r\mathbf{e}) = \mathcal{K}^h\mathbf{V} + r\mathbf{e}, \quad \forall r \in \mathbb{R}. \tag{3.17}$$

Property $\mathcal{K}2$

$\mathcal{K}^h$ is consistent with $\mathcal{K}$, in the sense that there exists $C > 0$ such that

$$\|\mathcal{K}^h I^h v - I^h\mathcal{K}v\|_h \leqslant Ch\|v\|_{W^{1,\infty}(\Omega)} \quad \forall v \in W^{1,\infty}(\Omega). \tag{3.18}$$

Here $\mathcal{K}1$ is analogous to the properties (1.12)–(1.14) of the solution operator K and $\mathcal{K}2$ is required because we are dealing with a system.

REMARK The monotonicity of G on $\{\mathrm{A}, \mathcal{L}\}$ implies that $g(a;\alpha,\beta)$ is

(1) monotone non-decreasing in α for all $|a| \leqslant \mathrm{A}$ and $|\alpha|, |\beta| \leqslant \mathcal{L}$;

(2) monotone non-increasing in β for all $|a| \leqslant \mathrm{A}$ and $|\alpha|, |\beta| \leqslant \mathcal{L}$.

Therefore property G4 implies

$$\begin{aligned}\mathcal{L}_f(\tilde{C}_g(B)[b-a]^- + C_g(A)([\alpha-\gamma]^- + [\delta-\beta]^-)) &\leqslant g(a;\alpha,\beta) - g(b;\gamma,\delta)\\ &\leqslant \mathcal{L}_f(\tilde{C}_g(B)[b-a]^+ + C_g(A)([\alpha-\gamma]^+ + [\delta-\beta]^+)), \quad (3.19)\end{aligned}$$

where A, B, C_g, $\tilde{C}_g$ are as defined in property G4.

LEMMA 3.1 Suppose that G is monotone on $\{\mathrm{A}, \mathcal{L}\}$ and

$$\lambda \leqslant \frac{1}{\mathcal{L}_f(\mathsf{B}h + 2\mathsf{A})}, \tag{3.20}$$

where A, B are defined by

$$\mathsf{A} := C_g(\mathrm{A}) \text{ and } \mathsf{B} := \tilde{C}_g(\mathcal{L}); \tag{3.21}$$

then

$$a + v_0 \leqslant b + w_0 \text{ and } \underline{v} \leqslant \underline{w} \Rightarrow G(a;\underline{v}) \leqslant G(b;\underline{w}), \tag{3.22}$$

provided $|a|, |b| \leqslant \mathrm{A}$ and $|D_+^h v_{-1}|, |D_+^h v_0|, |D_+^h w_{-1}|, |D_+^h w_0| \leqslant \mathcal{L}$.

Proof. For $|a|, |b| \leqslant \mathrm{A}$ and $|D_+^h v_{-1}|, |D_+^h v_0|, |D_+^h w_{-1}|, |D_+^h w_0| \leqslant \mathcal{L}$ we can use (3.19) to write

$$\begin{aligned}G(a;\underline{v}) - G(b;\underline{w}) &= v_0 - w_0 - \Delta t\{g(a; D_+^h v_{-1}, D_+^h v_0) - g(b; D_+^h w_{-1}, D_+^h w_0)\}\\ &\leqslant v_0 - w_0 + \Delta t \mathcal{L}_f\{\mathsf{B}[a-b]^+\\ &\quad + \mathsf{A}([D_+^h w_{-1} - D_+^h v_{-1}]^+ + [D_+^h v_0 - D_+^h w_0]^+)\}\\ &\leqslant v_0 - w_0 + \lambda\mathcal{L}_f\{\mathsf{B}h[w_0 - v_0]^+\\ &\quad + \mathsf{A}([(w_0 - v_0) - (w_{-1} - v_{-1})]^+ + [(w_0 - v_0) - (w_1 - v_1)]^+)\}\\ &\leqslant v_0 - w_0 + \lambda\mathcal{L}_f\{\mathsf{B}h(w_0 - v_0)\\ &\quad + \mathsf{A}(2[w_0 - v_0]^+ - [w_{-1} - v_{-1}]^- - [w_1 - v_1]^-)\}\\ &= (v_0 - w_0)(1 - \lambda\mathcal{L}_f(\mathsf{B}h + 2\mathsf{A})) \leqslant 0 \text{ by condition (3.20).}\end{aligned}$$

□

3.3 *Stability of the scheme*

Later results will rely on the properties of $\vec{G}$ below. The proof depends on the structure of the Hamiltonian and on the properties of $\mathcal{K}^h$.

PROPOSITION 3.1 Let $\mathcal{R}, \mathcal{L} \geqslant 0$ and define $S_{\mathcal{R},\mathcal{L}} \subset S^h$ by

$$S_{\mathcal{R},\mathcal{L}} := \left\{\mathbf{U} \in S^h : \|\mathbf{U}\|_h \leqslant \mathcal{R} \text{ and } \sup_{0 \leqslant j \leqslant M-1} |D_+^h U_j| \leqslant \mathcal{L}\right\}. \tag{3.23}$$

Let λ be as at (3.20).

Suppose that G is monotone on $\{2\mathcal{R} + Q_0, \mathcal{L}\}$. Then, for all $\mathbf{V}, \mathbf{W} \in S_{\mathcal{R},\mathcal{L}}$,

(1) $\mathbf{V} \leqslant \mathbf{W} \Rightarrow \vec{G}(\mathbf{V}) \leqslant \vec{G}(\mathbf{W})$;
(2) $\vec{G}(\mathbf{V} + r\mathbf{e}) = \vec{G}(\mathbf{V}) + r\mathbf{e}, \quad \forall r \in \mathbb{R}$;
(3) $\|\vec{G}(\mathbf{V}) - \vec{G}(\mathbf{W})\|_h \leqslant \|\mathbf{V} - \mathbf{W}\|_h$.

Proof. (1) Property (3.15) gives $\mathcal{F}^h\mathbf{V} + \mathbf{V} \leqslant \mathcal{F}^h\mathbf{W} + \mathbf{W}$, whence Lemma (3.1) implies the result.

(2) For $j \in \{0, \ldots, M\}$, we have

$$\begin{aligned}\vec{G}_j(\mathbf{V} + r\mathbf{e}) &= V_j + r - \Delta t g(\mathcal{F}^h(\mathbf{V} + r\mathbf{e})_j;\, D_+^h(\mathbf{V} + r\mathbf{e})_{j-1},\, D_+^h(\mathbf{V} + r\mathbf{e})_j)\\ &= V_j + r - \Delta t g(\mathcal{F}^h(\mathbf{V})_j;\, D_+^h V_{j-1},\, D_+^h V_j)(\text{using (3.17)})\\ &= \vec{G}_j(\mathbf{V}) + r.\end{aligned}$$

(3) Let $r = \|\mathbf{V} - \mathbf{W}\|_h$, so we have that $\mathbf{V} \leqslant \mathbf{W} + r\mathbf{e}$, and therefore

$$\begin{aligned}\vec{G}(\mathbf{V}) &\leqslant \vec{G}(\mathbf{W} + r\mathbf{e}) \text{ by } \mathbf{(1)}\\ &= \vec{G}(\mathbf{W}) + r\mathbf{e} \text{ by } \mathbf{(2)}\\ \Rightarrow \vec{G}(\mathbf{V}) - \vec{G}(\mathbf{W}) &\leqslant r\mathbf{e}\\ &\leqslant \|\mathbf{V} - \mathbf{W}\|_h\mathbf{e}.\end{aligned}$$

Now interchange $\mathbf{V}$ and $\mathbf{W}$ and the result follows. □

We would hope that the $\mathbf{U}^k$ created by the process (3.5) would have properties analogous to those proved for u in Theorem 2.1, and this is indeed the case, the results being summed up in the following theorem.

THEOREM 3.1 Define

$$\mathcal{A} := 2\mathcal{R} + Q_0, \tag{3.24}$$

$$\mathcal{L} := \max\{\mathcal{L}_0, C_{\mathcal{K}}\mathcal{R} + Q_1\}. \tag{3.25}$$

Suppose that $\vec{G}$ is monotone on $\{\mathcal{A}, \mathcal{L}\}$ and that

$$\lambda \leqslant \min\left\{\frac{1}{\mathcal{L}_f((C_{\mathcal{K}}\mathcal{R} + \mathcal{L} + Q_1)h + 2\mathsf{A})}, \frac{1}{\mathcal{L}_f(\mathsf{B}h + 2\mathsf{A})}\right\}, \tag{3.26}$$

where A, B are as defined at (3.21). Then, for $\mathbf{U} \in S_{\mathcal{R},\mathcal{L}}$,

(1) $\|\vec{G}(\mathbf{U})\|_h \leqslant \|\mathbf{U}\|_h$;
(2) $\max_{0 \leqslant j \leqslant M-1} |D_+^h \vec{G}_j^k(\mathbf{U})| \leqslant \mathcal{L}$ for $k = 0, 1, \ldots$;
(3) $\|\mathbf{U} - \vec{G}(\mathbf{U})\|_h \leqslant \Delta t \mathcal{L}_f(\mathsf{B}\mathcal{A} + 2\mathsf{A}\mathcal{L})$.

Proof. (1) Since we have $\lambda \leqslant \frac{1}{\mathcal{L}_f(\mathsf{B}h + 2\mathsf{A})}$ this is just a trivial consequence of Proposition 3.1.3.

(2) It is convenient to set

$$w_j^k := D_+^h G_j^k(\mathbf{U}), \text{ for } j = -1, \ldots, M$$

(where U^k_{-1} and U^k_{M+1} are defined as at (3.8)),

$$\mathcal{F}^k_j := \mathcal{F}^h(\mathbf{U}^k)_j, \text{ for } j = 0, \dots, M,$$

and

$$\theta^k_{i,j} = \begin{cases} 1 & \text{if } w^k_i - w^k_j > 0 \\ 0 & \text{if } w^k_i - w^k_j \leqslant 0. \end{cases}$$

We are required to prove that

$$|w^k_j| \leqslant \mathcal{L}, \quad j \in \{-1, \dots, M\}, \quad \text{for } k = 0, 1, \dots \tag{3.27}$$

and proof will be by induction, so assume (3.27) to be true for some $k = n \in \mathbb{N}$. We have that

$$|\mathcal{F}^n_j| = |\mathcal{K}^h(\mathbf{U}^n)_j - U^n_j + q_\infty(x_j)| \leqslant 2\mathcal{R} + Q_0 = \mathcal{A}$$

and

$$\begin{aligned} \mathcal{K}^h(\mathbf{U}^n)_{j+1} - \mathcal{K}^h(\mathbf{U}^n)_j + q_\infty(x_{j+1}) - q_\infty(x_j) &\leqslant (C_\mathcal{K}\mathcal{R} + Q_1)h \\ &\leqslant \mathcal{L}h \text{ by (3.25).} \end{aligned} \tag{3.28}$$

The following argument proves that $w^{n+1}_j \leqslant \mathcal{L}$ for $j \in \{0, \dots, M-1\}$, and hence by (3.8) $w^{n+1}_{-1}, w^{n+1}_M \geqslant -\mathcal{L}$.
We can write (using (3.19) and the consistency of g), for $j \in \{0, \dots, M-1\}$,

$$\begin{aligned} w^{n+1}_j &= \frac{1}{h}\{G(\mathcal{F}^n_{j+1}; U^n_j, U^n_{j+1}, U^n_{j+2}) - G(\mathcal{F}^n_j; U^n_{j-1}, U^n_j, U^n_{j+1})\} \\ &= w^n_j + \lambda\{g(\mathcal{F}^n_j; w^n_{j-1}, w^n_j) - g(\mathcal{F}^n_{j+1}; w^n_j, w^n_{j+1})\} \\ &= w^n_j + \lambda\{g(\mathcal{F}^n_j; w^n_{j-1}, w^n_j) - g(\mathcal{F}^n_j; w^n_j, w^n_j)\cdot \\ &\quad + H(\mathcal{F}^n_j, w^n_j) - H(\mathcal{F}^n_{j+1}, w^n_j) \\ &\quad + g(\mathcal{F}^n_{j+1}; w^n_j, w^n_j) - g(\mathcal{F}^n_{j+1}; w^n_j, w^n_{j+1})\} \\ &\leqslant w^n_j + \lambda\mathcal{L}_f\{\mathsf{A}\theta^n_{j-1,j}(w^n_{j-1} - w^n_j)\cdot \\ &\quad + |w^n_j|[\mathcal{F}^n_{j+1} - \mathcal{F}^n_j]^+ + \mathsf{A}\theta^n_{j+1,j}(w^n_{j+1} - w^n_j)\} \\ &= (1 + \lambda\mathcal{L}_f\{[\mathcal{F}^n_{j+1} - \mathcal{F}^n_j]^+ - \mathsf{A}(\theta^n_{j-1,j} + \theta^n_{j+1,j})\})[w^n_j]^+ \\ &\quad + (\lambda\mathcal{L}_f\mathsf{A}\theta^n_{j-1,j})[w^n_{j-1}]^+ + (\lambda\mathcal{L}_f\mathsf{A}\theta^n_{j+1,j})[w^n_{j+1}]^+ \\ &\quad + (1 - \lambda\mathcal{L}_f\{[\mathcal{F}^n_{j+1} - \mathcal{F}^n_j]^+ + \mathsf{A}(\theta^n_{j-1,j} + \theta^n_{j+1,j})\})[w^n_j]^- \\ &\quad + (\lambda\mathcal{L}_f\mathsf{A}\theta^n_{j-1,j})[w^n_{j-1}]^- + (\lambda\mathcal{L}_f\mathsf{A}\theta^n_{j+1,j})[w^n_{j+1}]^-. \end{aligned}$$

Our choice of λ in (3.26) ensures that the coefficient of $[w^n_j]^-$ is non-negative, so we may discard the non-positive terms and continue thus:

$$\begin{aligned} w^{n+1}_j &\leqslant (1 + \lambda\mathcal{L}_f\{[\mathcal{F}^n_{j+1} - \mathcal{F}^n_j]^+ - \mathsf{A}(\theta^n_{j-1,j} + \theta^n_{j+1,j})\})[w^n_j]^+ \\ &\quad + (\lambda\mathcal{L}_f\mathsf{A}\theta^n_{j-1,j})[w^n_{j-1}]^+ + (\lambda\mathcal{L}_f\mathsf{A}\theta^n_{j+1,j})[w^n_{j+1}]^+ \qquad (1) \\ &=: a[w^n_j]^+ + b[w^n_{j-1}]^+ + c[w^n_{j+1}]^+. \qquad (2) \end{aligned}$$

First note that if $w_j^n \leqslant 0$ the result follows from (1) since

$$\begin{aligned} w_j^{n+1} &\leqslant (\lambda \mathcal{L}_f \mathsf{A} \theta_{j-1,j}^n)[w_{j-1}^n]^+ + (\lambda \mathcal{L}_f \mathsf{A} \theta_{j+1,j}^n)[w_{j+1}^n]^+, \\ &\leqslant 2\lambda \mathcal{L}_f \mathsf{A} \mathcal{L} \\ &\leqslant \mathcal{L} \quad \text{by (3.26).} \end{aligned}$$

We now assume that $w_j^n > 0$. The identity $a + b + c = 1 + \lambda \mathcal{L}_f [\mathcal{F}_{j+1}^n - \mathcal{F}_j^n]^+$ combined with our inductive assumption transforms (2) into the following inequality:

$$w_j^{n+1} \leqslant a w_j^n + \mathcal{L}(1 + \lambda \mathcal{L}_f [\mathcal{F}_{j+1}^n - \mathcal{F}_j^n]^+ - a). \tag{3.29}$$

If $[\mathcal{F}_{j+1}^n - \mathcal{F}_j^n]^+ = 0$ we see that (3.29) implies that $w_j^{n+1} \leqslant \mathcal{L}$ for all $j \in \{0, \ldots, M-1\}$ as required.
Suppose instead that $[\mathcal{F}_{j+1}^n - \mathcal{F}_j^n]^+ > 0$. Then rearranging (3.29) gives

$$\begin{aligned} w_j^{n+1} &\leqslant w_j^n (a - \Delta t \mathcal{L}_f \mathcal{L}) + \mathcal{L}\left(1 - a + \Delta t \mathcal{L}_f \frac{(\mathcal{K}_{j+1}^n - \mathcal{K}_j^n + q_{j+1}^\infty - q_j^\infty)}{h}\right) \\ &\leqslant w_j^n (a - \Delta t \mathcal{L}_f \mathcal{L}) + \mathcal{L}(1 - a + \Delta t \mathcal{L}_f \mathcal{L}) \quad \text{by (3·28).} \end{aligned}$$

Noting that $\mathsf{B} \geqslant \mathcal{L}$ and using the restrictions placed on λ by (3.26) we find that $a - \Delta t \mathcal{L}_f \mathcal{L} \geqslant 0$. Thus $w_j^{n+1} \leqslant \mathcal{L}$ for all $j \in \{0, \ldots, M-1\}$.
A similar argument to the above yields the corresponding lower bound $w_j^{n+1} \geqslant -\mathcal{L}$ for all $j \in \{0, \ldots, M-1\}$, and hence by (3.8) $w_{-1}^{n+1}, w_M^{n+1} \leqslant \mathcal{L}$; thus

$$|w_j^{n+1}| \leqslant \mathcal{L} \text{ for } j \in \{-1, \ldots, M\}.$$

Now, since (3.27) is plainly true for $k = 0$, the proof is complete.

(3)

$$\begin{aligned} \|\mathbf{U} - \vec{G}(\mathbf{U})\|_h &= \max_{0 \leqslant j \leqslant M} |U_j - (U_j - \Delta t g(\mathcal{F}(\mathbf{U})_j; D_+^h U_{j-1}, D_+^h U_j))| \\ &\leqslant \Delta t \mathcal{L}_f (\mathsf{B}\mathcal{A} + 2\mathsf{A}\mathcal{L}) \text{ using (3.13).} \end{aligned}$$

□

REMARK Note that Theorem 3.1 combined with Proposition 3.1.3 is the discrete analogue of Theorem 2.1 and that the constants in the discrete estimate are independent of time.

4. Error bound

THEOREM 4.1 Let $\mathcal{A}, \mathcal{L}, \lambda$ be as defined in Theorem 3·1. Suppose that $\vec{G}$ is monotone on $\{\mathcal{A}, \mathcal{L}^*\}$ where $\mathcal{L}^* > \mathcal{L}$. Let u be the viscosity solution of (1.5) and define $\mathbf{U}^k$ by (3.5) for $k = 1, \ldots, N$. Then, for λ fixed and Δt sufficiently small, there exists a constant $C = C(u_0, q_\infty, g, T)$ such that

$$\sup_{0 \leqslant j \leqslant M} |u(x_j, t_k) - U_j^k| \leqslant C\sqrt{\Delta t} \tag{4.1}$$

for $0 \leqslant k \leqslant N$.

Proof. Suppose that

$$\sup_{0\leqslant j\leqslant M,\ 0\leqslant k\leqslant N} |u(x_j,t_k)-U_j^k| = \sup_{0\leqslant j\leqslant M,\ 0\leqslant k\leqslant N} \{u(x_j,t_k)-U_j^k\} =: \sigma. \tag{4.2}$$

Let $(x_J,t_K)\in\triangle_T$ be such that $\sigma = u(x_J,t_K)-U_J^K$; we now seek an upper bound for σ. If our assumption had not been true and instead

$$\sup_{0\leqslant j\leqslant M,\ 0\leqslant k\leqslant N} |u(x_j,t_k)-U_j^k| = \sup_{0\leqslant j\leqslant M,\ 0\leqslant k\leqslant N} \{U_j^k-u(x_j,t_k)\},$$

then we would have interchanged u and $\mathbf{U}$ and used a similar argument.

We will re-use the functions d, Γ, Ψ defined at the beginning of Section (2.2) but now redefine the function $\Phi:\overline{\Omega}_T\times\triangle_T\to\mathbb{R}$, by

$$\Phi(x,t,y_j,s_k) := u(x,t)-U_j^k-\frac{\sigma}{4T}(t+s_k)-\frac{1}{\alpha}(t-s_k)^2-\Psi(x,y_j). \tag{4.3}$$

Again Γ is used in the definition of Ψ in order to handle the Neumann boundary condition. Let $(\hat{x},\hat{t},y_{\hat{j}},s_{\hat{K}})$ be such that

$$\Phi(\hat{x},\hat{t},y_{\hat{j}},s_{\hat{K}}) = \sup_{(x,t,y_j,s_k)\in\overline{\Omega}_T\times\triangle_T} \Phi(x,t,y_j,s_k). \tag{4.4}$$

The existence of such a maximiser follows from the fact that $\overline{\Omega}_T$ is compact, u and Ψ are continuous, and the U_j^k are bounded. Note the following lower bound for Φ evaluated at $(\hat{x},\hat{t},\hat{y},\hat{s})$:

$$\begin{aligned}\Phi(\hat{x},\hat{t},\hat{y},\hat{s}) &\geqslant \Phi(x_J,t_K,x_J,t_K)\\ &= \sigma-\frac{\sigma}{2T}t_K-2\beta d(x_J)\\ &\geqslant \frac{\sigma}{2}.\end{aligned} \tag{4.5}$$

To prove Theorem 4.1 it is convenient to consider three possible cases for $\hat{t}$ and $\hat{K}$, these being (1) $\hat{t}\geqslant 0$, $\hat{K}=0$; (2) $\hat{t}=0$, $\hat{K}>0$ and finally (3) $\hat{t}>0$, $\hat{K}>0$.

Case 1. $\hat{t}\geqslant 0$, $\hat{K}=0$. Rearranging (4.5) gives

$$u(\hat{x},\hat{t})-u(\hat{x},0)+u(\hat{x},0)-u(y_{\hat{j}},0) \geqslant \frac{\sigma}{2}+\frac{\sigma}{4T}\hat{t}+\frac{\hat{t}^2}{\alpha}+\Psi(\hat{x},y_{\hat{j}}) \tag{4.6}$$

and hence

$$-\beta\Gamma(\hat{x},y_{\hat{j}})+\tilde{\mathcal{L}}_u\hat{t}+\mathcal{L}_0|\hat{x}-y_{\hat{j}}| \geqslant \frac{\sigma}{2}+\frac{\sigma}{4T}\hat{t}+\frac{\hat{t}^2}{\alpha}+\frac{|\hat{x}-y_{\hat{j}}|^2}{\alpha}. \tag{4.7}$$

The aim now is to obtain upper bounds for $\tilde{\mathcal{L}}_u\hat{t}$ and $\mathcal{L}_0|\hat{x}-y_{\hat{j}}|$ in terms of $\frac{\hat{t}^2}{\alpha}$ and $\frac{|\hat{x}-y_{\hat{j}}|^2}{\alpha}$ respectively. Using the standard inequality,

$$ab\leqslant\frac{a^2}{2}+\frac{b^2}{2}, \tag{4.8}$$

we obtain the estimates

$$\tilde{\mathcal{L}}_u \hat{t} \leqslant \frac{\hat{t}^2}{2\alpha} + \frac{\hat{\mathcal{L}}_u^2 \alpha}{2} \text{ and } \mathcal{L}_0 |\hat{x} - y_{\hat{J}}| \leqslant \frac{|\hat{x} - y_{\hat{J}}|^2}{2\alpha} + \frac{\mathcal{L}_0^2 \alpha}{2}. \tag{4.9}$$

Hence, letting $C_1 := \max\{\mathcal{L}_0^2, \tilde{\mathcal{L}}_u^2\}$ we have that (4.7) implies

$$L\beta + 2C_1\alpha \geqslant \sigma + \frac{\sigma}{2T}\hat{t} + \frac{\hat{t}^2}{\alpha} + \frac{|\hat{x} - y_{\hat{J}}|^2}{\alpha} \geqslant \sigma. \tag{4.10}$$

Now by our choice of $\alpha, \beta \leqslant \sqrt{\Delta t}$ we obtain the result, i.e. $\sigma \leqslant C\sqrt{\Delta t}$.

Case 2. $\hat{t} = 0, \hat{K} > 0$. Rearranging (4.5) gives,

$$u(\hat{x}, 0) - u(y_{\hat{J}}, 0) + U_{\hat{J}}^0 - U_{\hat{J}}^{\hat{K}} \geqslant \frac{\sigma}{2} + \frac{\sigma}{4T}s_{\hat{K}} + \frac{s_{\hat{K}}^2}{\alpha} + \Psi(\hat{x}, y_{\hat{J}}) \tag{4.11}$$

and hence, by Theorem 3·1.3,

$$-\beta\Gamma(\hat{x}, y_{\hat{J}}) + \mathcal{L}_0|\hat{x} - y_{\hat{J}}| + Cs_{\hat{K}} \geqslant \frac{\sigma}{2} + \frac{\sigma}{4T}s_{\hat{K}} + \frac{s_{\hat{K}}^2}{\alpha} + \frac{|\hat{x} - y_{\hat{J}}|^2}{\alpha}. \tag{4.12}$$

To obtain upper bounds for $\mathcal{L}_0|\hat{x} - y_{\hat{J}}|$ and $Cs_{\hat{K}}$ in terms of $\frac{|\hat{x}-y_{\hat{J}}|^2}{\alpha}$ and $\frac{s_{\hat{K}}^2}{\alpha}$ respectively, we appeal to (4.8), as in case (1). So, letting $C_2 := \max\{\mathcal{L}_0^2, C^2\}$, we see that (4.12) implies

$$L\beta + 2C_2\alpha \geqslant \sigma + \frac{\sigma}{2T}s_{\hat{K}} + \frac{s_{\hat{K}}^2}{\alpha} + \frac{|\hat{x} - y_{\hat{J}}|^2}{\alpha} \geqslant \sigma. \tag{4.13}$$

Now, as before, since $\alpha, \beta \leqslant \sqrt{\Delta t}$ this yields the result $\sigma \leqslant C\sqrt{\Delta t}$.

Case 3. $\hat{t} > 0, \hat{K} > 0$. First consider the function $(x, t) \to \Phi(x, t, y_{\hat{J}}, s_{\hat{K}})$. This function has a maximum at $(\hat{x}, \hat{t}) \in \overline{\Omega}_T$ and therefore, defining $\phi : \overline{\Omega}_T \to \mathbb{R}$ by

$$\phi(x, t) := \frac{\sigma}{4T}t + \frac{1}{\alpha}(t - s_{\hat{K}})^2 + \Psi(x, y_{\hat{J}}),$$

we see that

$$u - \phi := u(x, t) - \frac{\sigma}{4T}t - \frac{1}{\alpha}(t - s_{\hat{K}})^2 - \Psi(x, y_{\hat{J}}) \tag{4.14}$$

also has a maximum at $(\hat{x}, \hat{t}) \in \overline{\Omega}_T$. Now since u is a viscosity solution of (1.5), and $\phi \in C^1(\overline{\Omega}_T)$, it follows from Definition 2.1 that

$$\left.\begin{aligned} &\frac{2}{\alpha}(\hat{t} - s_{\hat{K}}) + \frac{\sigma}{4T} + H(\mathcal{F}(u)(\hat{x}, \hat{t}), D_x\Psi(\hat{x}, y_{\hat{J}})) \leqslant 0 \quad \hat{x} \in \Omega \\ &\min\left\{\begin{matrix} \frac{2}{\alpha}(\hat{t} - s_{\hat{K}}) + \frac{\sigma}{4T} + H(\mathcal{F}(u)(\hat{x}, \hat{t}), D_x\Psi(\hat{x}, y_{\hat{J}})), \\ D_x\Psi(\hat{x}, y_{\hat{J}}) \cdot \nu_{\hat{x}} \end{matrix}\right\} \leqslant 0 \quad \hat{x} \in \partial\Omega. \end{aligned}\right\} \tag{4.15}$$

However, by construction, $D_x\Psi(\hat{x}, y_{\hat{J}})\cdot \nu_{\hat{x}} \geqslant \beta > 0$ when $\hat{x}\in\partial\Omega$ and therefore (4.15) reduces to the inequality

$$\frac{2}{\alpha}(\hat{t}-s_{\hat{K}}) + \frac{\sigma}{4T} + H(\mathcal{F}(u)(\hat{x},\hat{t}), D_x\Psi(\hat{x}, y_{\hat{J}})) \leqslant 0. \tag{4.16}$$

Here we have made use of the properties of Γ. Next consider the function $(y_j, s_k) \to \Phi(\hat{x},\hat{t}, y_j, s_k)$. This function has a maximum at $(y_{\hat{J}}, s_{\hat{K}}) \in \triangle_T$ thus, defining $\psi : \triangle_T \to \mathbb{R}$ by

$$\psi(y_j, s_k) := -\frac{\sigma}{4T}s_k - \frac{1}{\alpha}(\hat{t}-s_k)^2 - \Psi(\hat{x}, y_j)$$

we have that

$$U_j^k - \psi := U_j^k + \frac{\sigma}{4T}s_k + \frac{1}{\alpha}(\hat{t}-s_k)^2 + \Psi(\hat{x}, y_j), \tag{4.17}$$

has a minimum at $(y_{\hat{J}}, s_{\hat{K}})$. Hence

$$U_j^k + \frac{\sigma}{4T}s_k + \frac{1}{\alpha}(\hat{t}-s_k)^2 + \Psi(\hat{x}, y_j) \geqslant U_{\hat{J}}^{\hat{K}} + \frac{\sigma}{4T}s_{\hat{K}} + \frac{1}{\alpha}(\hat{t}-s_{\hat{K}})^2 + \Psi(\hat{x}, y_{\hat{J}}) \tag{4.18}$$

for $j = 0,\dots,M$ and $k = 0,\dots,N$.
Rearranging yields

$$U_j^k \geqslant U_{\hat{J}}^{\hat{K}} + \frac{\sigma}{4T}(s_{\hat{K}} - s_k) + \frac{1}{\alpha}((\hat{t}-s_{\hat{K}})^2 - (\hat{t}-s_k)^2) + \Psi(\hat{x}, y_{\hat{J}}) - \Psi(\hat{x}, y_j) \tag{4.19}$$

for $j = 0,\dots,M$ and $k = 0,\dots,N$.

Step (i). Let $k = \hat{K}-1$ in (4.19) to obtain

$$\begin{aligned} U_j^{\hat{K}-1} &\geqslant U_{\hat{J}}^{\hat{K}} + \frac{\sigma\Delta t}{4T} + \frac{\Delta t}{\alpha}(2(s_{\hat{K}} - \hat{t}) - \Delta t) + \Psi(\hat{x}, y_{\hat{J}}) - \Psi(\hat{x}, y_j) \\ &=: r - \Psi(\hat{x}, y_j) \end{aligned} \tag{4.20}$$

for $j = 0,\dots,M$, where $r = r(\hat{x},\hat{t}, y_{\hat{J}}, s_{\hat{K}})$ is a constant independent of j.

Step (ii). In order to proceed with step (ii) the following preliminary results will be required. The details of the proof of Lemma 4.1 can be found in Briggs *et al.* (1999).

LEMMA 4.1

(1)

$$\frac{|\hat{x} - y_{\hat{J}}|}{\alpha} \leqslant \frac{\mathcal{L}}{2} + C\beta \text{ for } \hat{x}\in\overline{\Omega},\ \hat{J}\in\{0,\dots,M\}; \tag{4.21}$$

(2)

$$\frac{|\hat{t} - s_{\hat{K}}|}{\alpha} \leqslant \frac{\tilde{\mathcal{L}}_u}{2} + \frac{\sigma}{8T} \text{ for } \hat{t}\in(0,T],\ \hat{K}\in\{0,\dots,N\}; \tag{4.22}$$

(3) Let $\Psi_j := \Psi(\hat{x}, y_j)$ for $j = -1, \dots, M+1$, with Ψ_{-1} and Ψ_{M+1} being calculated in accordance with the scheme, that is to say the rule at (3.8). Then

$$\max\{|D_+^h \Psi_{\hat{J}-1}|, |D_+^h \Psi_{\hat{J}}|\} \leqslant \mathcal{L}^* \text{ for } \hat{x} \in \overline{\Omega},\ \hat{J} \in \{0, \dots, M\}. \tag{4.23}$$

By definition

$$\begin{aligned} U_{\hat{J}}^{\hat{K}} &= G(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; U_{\hat{J}-1}^{\hat{K}-1}, U_{\hat{J}}^{\hat{K}-1}, U_{\hat{J}+1}^{\hat{K}-1}) \\ &\geqslant G(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; (r,r,r) - \underline{\Psi}_{\hat{J}}) \text{ using (4.20), (4.23), and monotonicity} \\ &= r - \Psi(\hat{x}, y_{\hat{J}}) - \Delta t g(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; -D_+^h \Psi_{\hat{J}-1}, -D_+^h \Psi_{\hat{J}}) \\ &= U_{\hat{J}}^{\hat{K}} + \frac{\sigma \Delta t}{4T} + \frac{\Delta t}{\alpha}(2(s_{\hat{K}} - \hat{t}) - \Delta t) \\ &\quad - \Delta t g(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; -D_+^h \Psi_{\hat{J}-1}, -D_+^h \Psi_{\hat{J}}). \end{aligned}$$

Rearranging yields,

$$\frac{\sigma}{4T} + \frac{1}{\alpha}(2(s_{\hat{K}} - \hat{t}) - \Delta t) \leqslant g(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; -D_+^h \Psi_{\hat{J}-1}, -D_+^h \Psi_{\hat{J}}). \tag{4.24}$$

Consider the error incurred when $g(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; -D_+^h \Psi_{\hat{J}-1}, -D_+^h \Psi_{\hat{J}})$ is considered as an approximation to $H(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}, -D_y \Psi(\hat{x}, y_{\hat{J}}))$. We have to exploit the structure of H and the properties of $\mathcal{K}^h$.
When $\hat{J} \in \{1, \dots, M-1\}$ we have the following:

$$\begin{aligned} &|g(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; -D_+^h \Psi_{\hat{J}-1}, -D_+^h \Psi_{\hat{J}}) - H(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}, -D_y \Psi(\hat{x}, y_{\hat{J}}))| \\ &\quad = \left| \begin{array}{l} g\left(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; \dfrac{-\Delta_+ \Psi_{\hat{J}-1}}{h}, \dfrac{-\Delta_+ \Psi_{\hat{J}}}{h}\right) \\ \qquad -g(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}; -D_y \Psi(\hat{x}, y_{\hat{J}}), -D_y \Psi(\hat{x}, y_{\hat{J}})) \end{array} \right| \\ &\quad \leqslant C\frac{h}{\alpha}. \end{aligned} \tag{4.25}$$

When $\hat{J} = 0$ we use the positivity of $-D_y \Psi(\hat{x}, 0) \geqslant \beta > 0$ which implies that, for h sufficiently small,

$$-D_+^h \Psi_0 \geqslant 0, \tag{4.26}$$

hence we can rewrite $g(a; -D_+^h \Psi_{-1}, -D_+^h \Psi_0)$ in terms of G and then make use of its monotonicity:

$$\begin{aligned} g(a; -D_+^h \Psi_{-1}, -D_+^h \Psi_0) &= -\frac{1}{\Delta t}(\Psi_0 + G(a;\ \Psi_{-1}, -\Psi_0, -\Psi_1)) \\ &= -\frac{1}{\Delta t}(\Psi_0 + G(a; -\theta\Psi_0 - (1-\theta)\Psi_1, -\Psi_0, -\Psi_1)) \text{ by (3.8)} \\ &= -\frac{1}{\Delta t}(\Psi_0 + G(a; -\Psi_0 - (1-\theta)\Delta_+\Psi_0, -\Psi_0, -\Psi_1)) \\ &\leqslant -\frac{1}{\Delta t}(\Psi_0 + G(a; -\Psi_0 + \Delta_+\Psi_0, -\Psi_0, -\Psi_1)) \text{ by (4·26)} \\ &\quad \text{and the monotonicity of } G \\ &= g(a; -D_+^h\Psi_0, -D_+^h\Psi_0) \\ &= H(a; -D_+^h\Psi_0) \text{ by consistency.} \end{aligned}$$

Hence $H(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}};\, D_+^h \Psi_0)$ can be compared with $H(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}, -D_y \Psi(\hat{x}, 0))$ as in (4.25) and the same error is obtained. Similarly, for $\hat{J} = M$ we can use the negativity of $-D_y \Psi(\hat{x}, L)$ along with (3.8) to extend (4.25) to include the right-hand boundary.

Now (4.24) and (4.25) allow us to write

$$\frac{\sigma}{4T} + \frac{2}{\alpha}(s_{\hat{K}} - \hat{t}) \leqslant H(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}, -D_y \Psi(\hat{x}, y_{\hat{J}})) + \frac{1}{\alpha}(Ch + \Delta t). \tag{4.27}$$

Adding (4.16) and (4.27) gives

$$\frac{\sigma}{2T} \leqslant H(\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}, -D_y \Psi(\hat{x}, y_{\hat{J}})) - H(\mathcal{F}(u)(\hat{x}, \hat{t}), D_x \Psi(\hat{x}, y_{\hat{J}})) + \frac{1}{\alpha}(Ch + \Delta t).$$

Using (1.9), this becomes

$$\frac{\sigma}{2T} \leqslant \mathcal{L}_f\{B[\mathcal{F}(u)(\hat{x}, \hat{t}) - \mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}]^+ + A|D_x \Psi(\hat{x}, y_{\hat{J}}) + D_y \Psi(\hat{x}, y_{\hat{J}})|\} + \frac{1}{\alpha}(Ch + \Delta t),$$

where

$$A = \max\{|\mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}|, |\mathcal{F}(u)(\hat{x}, \hat{t})|\} \text{ and } B = \max\{|D_x \Psi(\hat{x}, y_{\hat{J}})|, |D_y \Psi(\hat{x}, y_{\hat{J}})|\}.$$

(Note that $B \leqslant \mathcal{L}^*$ for β sufficiently small by Lemma 4.1.1). Since $D_x \Psi(x, y) + D_y \Psi(x, y) = \beta(d'(x) + d'(y))$, we obtain

$$\frac{\sigma}{2T} \leqslant \mathcal{L}_f B[\mathcal{F}(u)(\hat{x}, \hat{t}) - \mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}]^+ + C\left(\beta + \frac{h}{\alpha} + \Delta t\right). \tag{4.28}$$

We now consider this difference in $\mathcal{F}$ and $\mathcal{F}^h$,

$$
\begin{aligned}
&[\mathcal{F}(u)(\hat{x},\hat{t}) - \mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}]^+ \\
&\quad \leqslant |\mathcal{F}(u)(\hat{x},\hat{t}) - \mathcal{F}(u)(y_{\hat{J}},\hat{t})| + |\mathcal{F}(u)(y_{\hat{J}},\hat{t}) - \mathcal{F}^h(I^h u(\cdot,\hat{t}))_{\hat{J}}| \\
&\qquad + |\mathcal{F}^h(I^h u(\cdot,\hat{t}))_{\hat{J}} - \mathcal{F}^h(I^h u(\cdot, s_{\hat{K}-1}))_{\hat{J}}| + |\mathcal{F}^h(I^h u(\cdot, s_{\hat{K}-1}))_{\hat{J}} - \mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}| \\
&\quad \leqslant C_1\{|\hat{x} - y_{\hat{J}}| + h\|u\|_{L^\infty(0,T;L^\infty(\Omega))} + |\hat{t} - s_{\hat{K}-1}|\} \\
&\qquad + |\mathcal{F}^h(I^h u(\cdot, s_{\hat{K}-1}))_{\hat{J}} - \mathcal{F}^h(\mathbf{U}^{\hat{K}-1})_{\hat{J}}| \qquad (4.29)\\
&\quad \leqslant C_2\{\alpha\beta + \alpha + h\} + |\mathcal{K}^h(I^h u(\cdot, s_{\hat{K}-1}) - \mathbf{U}^{\hat{K}-1})_{\hat{J}} - (I^h u(\cdot, s_{\hat{K}-1}) - \mathbf{U}^{\hat{K}-1})_{\hat{J}}| \qquad (4.30)\\
&\quad \leqslant C_2\{\alpha\beta + \alpha + h\} + 2\sigma. \qquad (4.31)
\end{aligned}
$$

Note that (4.29) follows from the Lipschitz continuity of $\mathcal{K}u$ and $\mathcal{K}^h\mathbf{U}$ and the consistency of $\mathcal{K}^h$ as an approximation to $\mathcal{K}$; (4.30) follows from (4.21), Theorem 2·1, and Lemma 4.1.2. By (4.2) we have that $|u(x_j, t_k) - U_j^k| \leqslant \sigma$ for $0 \leqslant j \leqslant M$ and $0 \leqslant k \leqslant N$ and hence we obtain (4.31).

Thus, combining (4.28) and (4.31) yields

$$\sigma\left(\frac{1}{2T} - 2\mathcal{L}_f B\right) \leqslant C\left\{\alpha\beta + \alpha + h + \beta + \frac{h}{\alpha} + \Delta t\right\}. \qquad (4.32)$$

Plainly if we choose $T < \frac{1}{4\mathcal{L}_f B}$ then we have an upper bound for σ of the form

$$\sigma \leqslant C\left\{\alpha\beta + \alpha + h + \beta + \frac{h}{\alpha} + \Delta t\right\}.$$

Since $\alpha, \beta \leqslant \sqrt{\Delta t}$ and $h = \frac{\Delta t}{\lambda}$ we have the result $\sigma < C\sqrt{\Delta t}$, and the theorem is proved for any $T < \frac{1}{4\mathcal{L}_f B}$.

Suppose instead that $T \geqslant \frac{1}{4\mathcal{L}_f B}$. Define $T_1 = \frac{1}{8\mathcal{L}_f B}$ and over the time interval $[0, T_1]$ we have the desired error bound. Now define two new problems over the time interval $[T_1, 2T_1]$ where $\Omega_{T_1} := \Omega \times (T_1, 2T_1]$:

$$\left.\begin{aligned}
u_t^{(1)} + H(\mathcal{F}(u^{(1)}), u_x^{(1)}) &= 0 && (x,t) \in \Omega_{T_1} \\
u_x^{(1)} &= 0 && (x,t) \in \partial\Omega_{T_1} \\
u^{(1)}(x, T_1) &= u(x, T_1) && x \in \overline{\Omega}
\end{aligned}\right\} \qquad (4.33)$$

where u is the viscosity solution of (1.5), and

$$\left.\begin{aligned}
v_t + H(\mathcal{F}(v), v_x) &= 0 && (x,t) \in \Omega_{T_1} \\
v_x &= 0 && (x,t) \in \partial\Omega_{T_1} \\
v(x, T_1) &= u_I(x) && x \in \overline{\Omega}
\end{aligned}\right\} \qquad (4.34)$$

where $u_I \in C(\bar{\Omega})$ is an interpolation of $\mathbf{U}^N$ such that $\|u(x, T_1) - u_I(x)\|_{L\infty(\Omega)} \leqslant C\sqrt{\Delta t}$. We require the following bound:

$$\sup_{0 \leqslant j \leqslant M,\ N \leqslant k \leqslant 2N} |u(x_j, t_k) - U_j^k| \leqslant C\sqrt{\Delta t}. \qquad (4.35)$$

This can be rewritten as

$$\sup_{0\leqslant j\leqslant M,\ N\leqslant k\leqslant 2N} |u(x_j,t_k) - U_j^k| \leqslant \sup_{0\leqslant j\leqslant M,\ N\leqslant k\leqslant 2N} |u^{(1)}(x_j,t_k) - v(x_j,t_k)| + \sup_{0\leqslant j\leqslant M,\ N\leqslant k\leqslant 2N} |v(x_j,t_k) - U_j^k|.$$

Note that the first supremum will be bounded by the difference in the initial data $\|u(x,T_1) - u_I(x)\|_{L^\infty(\Omega)}$ (by Theorem 2.1) which itself is bounded above by $C\sqrt{\Delta t}$. The second supremum is equivalent to (4.1) and thus is also less than $C\sqrt{\Delta t}$. Hence we obtain

$$\sup_{0\leqslant j\leqslant M,\ N\leqslant k\leqslant 2N} |u(x_j,t_k) - U_j^k| \leqslant C\sqrt{\Delta t}. \tag{4.36}$$

Now repeat for the time interval $(2T_1, 3T_1]$ and so on. □

5. Example schemes

In this section we give examples of possible numerical schemes fitting the criteria given in Section 3.2.

5.1 *The operator $\mathcal{K}^h$ and its properties*

DEFINITION 5.1 The operator $\mathcal{K}^h$ is a discrete approximation of $\mathcal{K}$, where $\hat{q}^h := \mathcal{K}^h\mathbf{V}$ solves the system of equations:

$$\begin{aligned} -2\mu\frac{\hat{q}_1^h - \hat{q}_0^h}{h^2} + \hat{q}_0^h &= V_0, \\ -\mu\frac{\hat{q}_{j-1}^h - 2\hat{q}_j^h + \hat{q}_{j+1}^h}{h^2} + \hat{q}_j^h &= V_j, \quad j \in \{1,\ldots,M-1\}, \\ -2\mu\frac{\hat{q}_{M-1}^h - \hat{q}_M^h}{h^2} + \hat{q}_M^h &= V_M. \end{aligned} \tag{5.1}$$

PROPOSITION 5.1 $\mathcal{K}^h$ satisfies (a) property $\mathcal{K}1$ and (b) property $\mathcal{K}2$.

Proof of (a). Let $\mathbf{V} \in S^h$.

(1)
$$\hat{q}_j^h \leqslant \max_{0\leqslant j\leqslant M} \hat{q}_j^h = \hat{q}_{J_{\max}}^h \leqslant \mathbf{V}_{J_{\max}} \geqslant \max_{0\leqslant j\leqslant M} \mathbf{V}_j.$$

The lower bound follows in the same way.

(2) We define $z_j := D_+^h\hat{q}_j$, substitute it into (5.1) and then sum the resulting equations from $j = 0$ to $k \leqslant M-1$ to obtain the following:

$$z_k = -z_0 + \frac{h}{\mu}\sum_{j=0}^{k} \hat{q}_j - V_j.$$

Thus by (1) we find

$$|z_k| \leqslant |z_0| + \frac{h}{\mu}\sum_{i=0}^{k} |\hat{q}_j - V_j| \leqslant C_{\mathcal{K}} \|V\|_h$$

where $C_{\mathcal{K}} = \frac{h+2L}{\mu}$.

(3) Let $r \in \mathbb{R}$. Clearly $\mathcal{K}^h(r\mathbf{e}) = r\mathbf{e}$ since $r\mathbf{e}$ is constant. Thus by the linearity of $\mathcal{K}^h$ the property follows.

To simplify notation we define the operator $\mathcal{F}^h$ to be the discrete approximation of $\mathcal{F}$ which is defined in terms of $\mathcal{K}^h$ and $I^h q_\infty$:

$$\mathcal{F}^h(\mathbf{V}) := \mathcal{K}^h\mathbf{V} - \mathbf{V} + I^h q_\infty. \tag{5.2}$$

Proof of (b). First define the following notation. For any functions η, χ set

$$a(\eta, \chi) := \int_\Omega \eta_x \chi_x, \quad (\eta, \chi) := \int_\Omega \eta\chi, \quad (\eta, \chi)^h := \int_\Omega I^h(\eta\chi)$$

and

$$|\eta|_0 := (\eta, \eta)^{\frac{1}{2}}, \quad |\eta|_1 := (\eta_x, \eta_x)^{\frac{1}{2}}, \quad |\eta|_h := \{(\eta, \eta)^h\}^{\frac{1}{2}}.$$

Let $v \in W^{1,\infty}(\Omega)$; then $\hat{q}$ is the solution of (1.4) where u is replaced by v. Also let $v_I, \hat{q}_I \in S^h$ be the piecewise linear interpolants of v and $\hat{q}$ respectively. Define $\hat{q}^h \in S^h$ to be the solution of the following problem:

$$a(\hat{q}^h, \chi) + (\hat{q}^h, \chi)^h = (v_I, \chi)^h \quad \text{for all } \chi \in S^h. \tag{5.3}$$

This is equivalent to (5.1) with $V_j = v_I(x_j)$.

To prove consistency it is sufficient to bound $\|\hat{q}^h - \hat{q}_I\|_{L^\infty(\Omega)}$. In order to do this we introduce $\hat{q}^h_* \in S^h$ which solves

$$a(\hat{q}^h_*, \chi) + (\hat{q}^h_*, \chi) = (v_I, \chi) \quad \text{for all } \chi \in S^h. \tag{5.4}$$

Note that

$$a(\hat{q}, \chi) = a(\hat{q}_I, \chi) \quad \text{for all } \chi \in S^h,$$

$$\|\hat{q} - \hat{q}_I\|_{L^\infty(\Omega)} \leqslant Ch^2\|\hat{q}_{xx}\|_{L^\infty(\Omega)} \leqslant Ch^2\|v\|_{L^\infty(\Omega)},$$

$$|v - v_I|_0 \leqslant Ch\|v\|_{W^{1,\infty}(\Omega)}.$$

It follows from (5.4) and the definition of $\hat{q}$ that

$$a(\hat{q} - \hat{q}^h_*, \chi) + (\hat{q} - \hat{q}^h_*, \chi) = (v - v_I, \chi) \quad \text{for all } \chi \in S^h.$$

Rearranging, we obtain

$$a(\hat{q}_I - \hat{q}^h_*, \chi) + (\hat{q}_I - \hat{q}^h_*, \chi) = (v - v_I, \chi) + (\hat{q}_I - \hat{q}, \chi) \quad \text{for all } \chi \in S^h.$$

Letting $\chi = \hat{q}_I - \hat{q}_*^h$ and using Cauchy–Schwarz and Young's inequality we obtain

$$|\hat{q}_I - \hat{q}_*^h|_1^2 + |\hat{q}_I - \hat{q}_*^h|_0^2 \leqslant |v - v_I|_0^2 + |\hat{q}_I - \hat{q}|_0^2 + \tfrac{1}{2}|\hat{q}_I - \hat{q}_*^h|_0^2.$$

Thus $\|\hat{q}_I - \hat{q}_*^h\|_{H^1(\Omega)} \leqslant Ch\|v\|_{W^{1,\infty}(\Omega)}$ and since $H^1(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ we have

$$\|\hat{q}_I - \hat{q}_*^h\|_{L^\infty(\Omega)} \leqslant Ch\|v\|_{W^{1,\infty}(\Omega)}. \tag{5.5}$$

Next we consider (5.4), (5.3) which yields

$$a(\hat{q}_*^h - \hat{q}^h, \chi) + (\hat{q}_*^h, \chi) - (\hat{q}^h, \chi)^h = (v_I, \chi) - (v_I, \chi)^h \quad \text{for all } \chi \in S^h.$$

Rearranging gives

$$\begin{aligned} a(\hat{q}_*^h - \hat{q}^h, \chi) + (\hat{q}_*^h, \chi)^h - (\hat{q}^h, \chi)^h = (v_I, \chi) - (v_I, \chi)^h + (\hat{q}_*^h, \chi)^h \\ - (\hat{q}_*^h, \chi) \forall\ \chi \in S^h. \end{aligned}$$

Putting $\chi = \hat{q}_*^h - \hat{q}^h$ yields

$$\begin{aligned} |\hat{q}_*^h - \hat{q}^h|_1^2 + |\hat{q}_*^h - \hat{q}^h|_h^2 \leqslant Ch^2|\hat{q}_*^h - \hat{q}^h|_1(|v_I|_1 + |\hat{q}_*^h|_1) \\ (\text{by } |(\eta, \chi) - (\eta, \chi)^h| \leqslant Ch^2|\eta|_1|\chi|_1\ \forall\ \eta, \chi \in S^h). \end{aligned} \tag{5.6}$$

Note that $|\hat{q}_*^h|_1 \leqslant |v_I|_0$, $|v_I|_1 \leqslant \mathcal{L}_v$ and $|\cdot|_h$ is uniformly equivalent to $|\cdot|_0$. Thus by dividing (5.6) by $|\hat{q}_*^h - \hat{q}^h|_1$ we obtain

$$|\hat{q}_*^h - \hat{q}^h|_1 \leqslant Ch^2(|v_I|_0 + \mathcal{L}_v),$$

and hence we also have the same bound for $|\hat{q}_*^h - \hat{q}^h|_0$ and once again we can infer from the $H^1(\Omega)$ bound that

$$\|\hat{q}_*^h - \hat{q}^h\|_{L^\infty(\Omega)} \leqslant Ch^2(|v_I|_0 + \mathcal{L}_v). \tag{5.7}$$

Combining (5.5) and (5.7) we have the following:

$$\begin{aligned} \|\hat{q} - \hat{q}^h\|_{L^\infty(\Omega)} &\leqslant \|\hat{q} - \hat{q}_I\|_{L^\infty(\Omega)} + \|\hat{q}_I - \hat{q}_*^h\|_{L^\infty(\Omega)} + \|\hat{q}_*^h - \hat{q}^h\|_{L^\infty(\Omega)} \\ &\leqslant Ch\|v\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

5.2 *Schemes*

We now define four different schemes S1–S4 and demonstrate that they each possess properties G1–G4. Schemes S1 and S2 are finite-difference approximations to the standard parabolic regularisation of (1.5). S1 is the Lax–Friedrich scheme (Osher & Shu, 1991). S3 is an upwind scheme in the sense that the Hamiltonian is expanded as

$$\begin{aligned} H(a, u_x) &= -f(a)|u_x| \\ &= -[f(a)]^+[u_x]^+ + [f(a)]^+[u_x]^- - [f(a)]^-[u_x]^+ + [f(a)]^-[u_x]^- \end{aligned}$$

and appropriate one-sided differences are used to approximate u_x. S4 is a maximum scheme which uses the identity

$$\begin{aligned} H(a, u_x) &= -f(a)|u_x| \\ &= -[f(a)]^+ \max\{[u_x]^+, -[u_x]^-\} - [f(a)]^- \max\{[u_x]^+, -[u_x]^-\}. \end{aligned}$$

This modification prevents possible double contributions that occur with the upwind scheme S3 at nodes where there is a strict local minimum. S4 is of Godunov type (Osher & Shu, 1991).

The schemes are defined, for $j = 0, \dots M$, by the following:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \begin{cases} f_j^n\left(\dfrac{|D_+^h U_j^n + D_+^h U_{j-1}^n|}{2}\right) + \dfrac{\epsilon}{h}(D_+^h U_j^n - D_+^h U_{j-1}^n), & \text{(S1)} \\ f_j^n\left(\dfrac{|D_+^h U_j^n| + |D_+^h U_{j-1}^n|}{2}\right) + \dfrac{\epsilon}{h}(D_+^h U_j^n - D_+^h U_{j-1}^n), & \text{(S2)} \\ \{[f_j^n]^+([D_+^h U_j^n]^+ - [D_+^h U_{j-1}^n]^-) & \\ \quad +[f_j^n]^-([D_+^h U_{j-1}^n]^+ - [D_+^h U_j^n]^-)\}, & \text{(S3)} \\ \{[f_j^n]^+ \max\{[D_+^h U_j^n]^+, -[D_+^h U_{j-1}^n]^-\} & \\ \quad +[f_j^n]^- \max\{[D_+^h U_{j-1}^n]^+, -[D_+^h U_j^n]^-\}\}, & \text{(S4)} \end{cases}$$

with $\theta = 0$ for S1 and S2 and $\theta = 1$ for S3 and S4 in (3.8). Note that $f_j^n := f((\mathcal{F}^h \mathbf{U}^n)_j)$.

5.2.1 *Property* G1. Schemes S1–S4 are of difference form with

$$\text{(S1)} \quad g(a; \alpha, \beta) = -f(a)\left(\frac{|\alpha + \beta|}{2}\right) - \frac{\epsilon}{h}(\beta - \alpha),$$

$$\text{(S2)} \quad g(a; \alpha, \beta) = -\left(\frac{|\alpha| + |\beta|}{2}\right) f(a) - \frac{\epsilon}{h}(\beta - \alpha),$$

$$\text{(S3)} \quad g(a; \alpha, \beta) = -[f(a)]^+(\beta^+ - \alpha^-) - [f(a)]^-(\alpha^+ - \beta^-),$$

$$\text{(S4)} \quad g(a; \alpha, \beta) = -[f(a)]^+ \max\{-\alpha^-, \beta^+\} - [f(a)]^- \max\{-\beta^-, \alpha^+\}.$$

5.2.2 *Property* G2.

LEMMA 5.1 Schemes S1–S4 are monotone on $\{\mathrm{A}, \mathcal{L}\}$ if

$$\text{(S1)} \quad \lambda \leqslant \frac{h}{2\epsilon} \leqslant \frac{1}{\mathcal{L}_f \mathrm{A}}. \tag{5.8}$$

$$\text{(S2)} \quad \begin{cases} \lambda \leqslant \dfrac{1}{\frac{2\epsilon}{h} + \mathcal{L}_f \mathrm{A}} \\ \dfrac{2\epsilon}{h} \geqslant \mathcal{L}_f \mathrm{A}. \end{cases} \tag{5.9}$$

$$\text{(S3)} \quad \lambda \leqslant \frac{1}{2\mathcal{L}_f \mathrm{A}}. \tag{5.10}$$

$$\text{(S4)} \quad \lambda \leqslant \frac{1}{\mathcal{L}_f \mathrm{A}}. \tag{5.11}$$

Proof. Let $\underline{c} := (c_{-1}, c_0, c_1) \leqslant \underline{d} := (d_{-1}, d_0, d_1)$.
We require to prove that

$$G(a; \underline{c}) - G(a; \underline{d}) \leqslant 0 \tag{5.12}$$

provided $|a| \leqslant$ A and $|D_+^h c_{-1}|, |D_+^h c_0|, |D_+^h d_{-1}|, |D_+^h d_0| \leqslant \mathcal{L}$.
Set $\underline{w} = \underline{c} - \underline{d}$. We now consider the different schemes separately.

S1. We have

$$\begin{aligned}
G(a; c_{-1}, c_0, c_1) &- G(a; d_{-1}, d_0, d_1) = w_0 \\
&+ \frac{\lambda}{2} f(a)(|\Delta_+ c_0 + \Delta_+ c_{-1}| - |\Delta_+ d_0 + \Delta_+ d_{-1}|) + \lambda \frac{\epsilon}{h}(w_1 - 2w_0 + w_{-1}) \\
&= w_0 + \frac{\lambda}{2} f(a) S((c_1 - c_{-1}) - (d_1 - d_{-1})) \\
&\quad + \lambda \frac{\epsilon}{h}(w_1 - 2w_0 + w_{-1}) \ \ \text{(for some } S \text{ where } |S| \leqslant 1) \\
&= w_0 \left(1 - 2\lambda \frac{\epsilon}{h}\right) + w_1 \lambda \left(\frac{\epsilon}{h} + S \frac{f(a)}{2}\right) + w_{-1} \lambda \left(\frac{\epsilon}{h} - S \frac{f(a)}{2}\right) \leqslant 0,
\end{aligned}$$

since $\underline{w} \leqslant \underline{0}$ and the coefficients are non-negative because of (5.8).

S2.

$$\begin{aligned}
G(a; \underline{c}) - G(a; \underline{d}) &= w_0 + \frac{\lambda}{2}(|\Delta_+ c_0| + |\Delta_+ c_{-1}| - |\Delta_+ d_0| - |\Delta_+ d_{-1}|) f(a) \\
&\quad + \frac{\epsilon}{h} \lambda (w_1 - 2w_0 + w_{-1}) \\
&= w_0 + \frac{\lambda}{2}(S_1(w_1 - w_0) + S_0(w_0 - w_{-1})) f(a) \\
&\quad + \frac{\epsilon}{h} \lambda (w_1 - 2w_0 + w_{-1}) \ \ \text{(for some } S_0, S_1 \text{ where } |S_0|, |S_1| \leqslant 1) \\
&= w_0 \left(1 - \lambda \left[\frac{2\epsilon}{h} + \frac{f(a)}{2}(S_1 - S_0)\right]\right) + w_1 \lambda \left(\frac{\epsilon}{h} + \frac{S_1 f(a)}{2}\right) \\
&\quad + w_{-1} \lambda \left(\frac{\epsilon}{h} - \frac{S_0 f(a)}{2}\right) \leqslant 0,
\end{aligned}$$

since $\underline{w} \leqslant \underline{0}$ and (5.9) ensures that the coefficients are non-negative.

S3. We will use the identities

$$A^+ - B^+ = \mu(A - B) \text{ and } C^- - D^- = \nu(C - D) \text{ for some } \mu, \nu \in [0, 1].$$

$$\begin{aligned}
G(a; \underline{c}) - G(a; \underline{d}) &= w_0 + \lambda\{[f(a)]^+((c_1 - c_0)^+ - (c_0 - c_{-1})^-)\\
&\quad + [f(a)]^-((c_0 - c_{-1})^+ - (c_1 - c_0)^-)\\
&\quad - [f(a)]^+((d_1 - d_0)^+ - (d_0 - d_{-1})^-)\\
&\quad + [f(a)]^-((d_0 - d_{-1})^+ - (d_1 - d_0)^-)\}\\
&= w_0 + \lambda[f(a)]^+(\mu_1(w_1 - w_0) - \nu_1(w_0 - w_{-1}))\\
&\quad + \lambda[f(a)]^-(\mu_2(w_0 - w_{-1}) - \nu_2(w_1 - w_0))\\
&\qquad \text{for some } \mu_1, \mu_2, \nu_1, \nu_2 \in [0, 1]\\
&= w_0(1 - \lambda([f(a)]^+(\mu_1 + \nu_1) - [f(a)]^-(\mu_2 + \nu_2)))\\
&\quad + w_1\lambda([f(a)]^+\mu_1 - [f(a)]^-\nu_2)\\
&\quad + w_{-1}\lambda([f(a)]^+\nu_1 - [f(a)]^-\mu_2) \leqslant 0,
\end{aligned}$$

since the coefficients of w_1 and w_{-1} are always non-negative, and the coefficient of w_0 is non-negative because of (5.10).

S4. Similarly,

$$\begin{aligned}
&G(a; \underline{c}) - G(a; \underline{d})\\
&\quad = w_0 + \lambda\{[f(a)]^+ \max\{-[c_0 - c_{-1}]^-, [c_1 - c_0]^+\}\\
&\qquad + [f(a)]^- \max\{-[c_1 - c_0]^-, [c_0 - c_{-1}]^+\}\\
&\qquad - [f(a)]^+ \max\{-[d_0 - d_{-1}]^-, [d_1 - d_0]^+\}\\
&\qquad - [f(a)]^- \max\{-[d_1 - d_0]^-, [d_0 - d_{-1}]^+\}\}\\
&\quad \leqslant w_0 + \lambda\{[f(a)]^+ \max\{[d_0 - d_{-1}]^- - [c_0 - c_{-1}]^-, [c_1 - c_0]^+ - [d_1 - d_0]^+\}\cdot\\
&\qquad - [f(a)]^- \max\{[c_1 - c_0]^- - [d_1 - d_0]^-, [d_0 - d_{-1}]^+ - [c_0 - c_{-1}]^+\}\}\\
&\quad \leqslant w_0 + \lambda\{[f(a)]^+ \max\{[w_{-1} - w_0]^+, [w_1 - w_0]^+\}\cdot\\
&\qquad - [f(a)]^- \max\{[w_1 - w_0]^+, [w_{-1} - w_0]^+\}\}\\
&\quad = w_0 + \lambda|f(a)| \max\{[w_{-1} - w_0]^+, [w_1 - w_0]^+\}\\
&\quad \leqslant w_0 + \lambda\mathcal{L}_f A \max\{[w_{-1}]^+ - [w_0]^-, [w_1]^+ - [w_0]^-\}\\
&\quad = w_0 - \lambda\mathcal{L}_f A[w_0]^- = [w_0]^-(1 - \lambda\mathcal{L}_f A) \leqslant 0,
\end{aligned}$$

since the coefficient of $[w_0]^-$ is non-negative because of (5.11).

5.2.3 *Property* G3. Schemes S1–S4 are consistent since in each case

$$g(a; \alpha, \alpha) = -f(a)|\alpha| = H(a, \alpha). \tag{5.13}$$

5.2.4 *Property* G4. Schemes S1–S4 all possess the Lipschitz property that is property G4.

S1. We make use of the triangle inequality,

$$||\alpha| - |\beta|| \leqslant |\alpha - \beta|. \tag{5.14}$$

We have that

$$\begin{aligned} g(a;\alpha,\beta) - g(b;\gamma,\delta) &= -f(a)\left(\frac{|\alpha+\beta|}{2}\right) - \frac{\epsilon}{h}(\beta-\alpha) + f(b)\left(\frac{|\gamma+\delta|}{2}\right) + \frac{\epsilon}{h}(\delta-\gamma) \\ &= \left(\frac{f(b)-f(a)}{2}\right)|\gamma+\delta| + \frac{f(a)}{2}(|\gamma+\delta| - |\alpha+\beta|) \\ &\quad + \frac{\epsilon}{h}(\alpha-\gamma) + \frac{\epsilon}{h}(\delta-\beta), \end{aligned}$$

or equally we could write

$$\begin{aligned} g(a;\alpha,\beta) - g(a;\gamma,\delta) &= \left(\frac{f(b)-f(a)}{2}\right)|\alpha+\beta| + \frac{f(b)}{2}(|\gamma+\delta| - |\alpha+\beta|) \\ &\quad + \frac{\epsilon}{h}(\alpha-\gamma) + \frac{\epsilon}{h}(\delta-\beta). \end{aligned}$$

Now (5.14) and the properties of f imply

$$\begin{aligned} \mathcal{L}_f\left(B[b-a]^- - \left(\frac{1}{2}A + \frac{\epsilon}{\mathcal{L}_f h}\right)(|\alpha-\gamma| + |\beta-\delta|)\right) &\leqslant g(a;\alpha,\beta) - g(b;\gamma,\delta) \\ \leqslant \mathcal{L}_f(B[b-a]^+ + \left(\frac{1}{2}A + \frac{\epsilon}{\mathcal{L}_f h}\right)(|\alpha-\gamma| + |\beta-\delta|)), & \end{aligned} \tag{5.15}$$

where $A = \max\{|a|, |b|\}$ and $B = \max\{|\alpha|, |\beta|, |\gamma|, |\delta|\}$; here (5.15) is the inequality (3.13) of property G4 with $C_g(A) = \frac{1}{2}A + \frac{\epsilon}{\mathcal{L}_f h}$ and $\tilde{C}_g(B) = B$.

S2. We have that

$$\begin{aligned} g(a;\alpha,\beta) - g(b;\gamma,\delta) &= -f(a)\left(\frac{|\alpha|+|\beta|}{2}\right) - \frac{\epsilon}{h}(\beta-\alpha) \\ &\quad + f(b)\left(\frac{|\gamma|+|\delta|}{2}\right) + \frac{\epsilon}{h}(\delta-\gamma) \\ &= \left(\frac{f(b)-f(a)}{2}\right)(|\gamma|+|\delta|) + \frac{f(a)}{2}(|\gamma| - |\alpha| + |\delta| - |\beta|) \\ &\quad + \frac{\epsilon}{h}(\alpha-\gamma) + \frac{\epsilon}{h}(\delta-\beta), \end{aligned}$$

or equally we could write

$$\begin{aligned} g(a;\alpha,\beta) - g(a;\gamma,\delta) &= \left(\frac{f(b)-f(a)}{2}\right)(|\alpha|+|\beta|) + \frac{f(b)}{2}(|\gamma| - |\alpha| + |\delta| - |\beta|) \\ &\quad + \frac{\epsilon}{h}(\alpha-\gamma) + \frac{\epsilon}{h}(\delta-\beta). \end{aligned}$$

Now (5.14) and the properties of f imply

$$\mathcal{L}_f\left(B[b-a]^- - \left(\frac{1}{2}A + \frac{\epsilon}{\mathcal{L}_f h}\right)(|\alpha-\gamma| + |\beta-\delta|)\right) \leqslant g(a;\alpha,\beta) - g(b;\gamma,\delta)$$
$$\leqslant \mathcal{L}_f\left(B[b-a]^+ + \left(\frac{1}{2}A + \frac{\epsilon}{\mathcal{L}_f h}\right)(|\alpha-\gamma| + |\beta-\delta|)\right), \qquad (5.16)$$

where $A = \max\{|a|,|b|\}$ and $B = \max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}$; here (5.16) is the inequality (3.13) of property G4 with $C_g(A) = \frac{1}{2}A + \frac{\epsilon}{\mathcal{L}_f h}$ and $\tilde{C}_g(B) = B$.

S3. We will make use of the inequalities,

$$(\alpha-\beta)^- \leqslant \alpha^+ - \beta^+ \leqslant (\alpha-\beta)^+ \text{ and } (\alpha-\beta)^- \leqslant \alpha^- - \beta^- \leqslant (\alpha-\beta)^+. \qquad (5.17)$$

We have that

$$\begin{aligned} g(a;\alpha,\beta) - g(b;\gamma,\delta) &= -[f(a)]^+(\beta^+ - \alpha^-) - [f(a)]^-(\alpha^+ - \beta^-) \\ &\quad + [f(b)]^+(\delta^+ - \gamma^-) + [f(b)]^-(\gamma^+ - \delta^-) \\ &= ([f(b)]^+ - [f(a)]^+)(\delta^+ - \gamma^-) \\ &\quad - (-[f(b)]^- + [f(a)]^-)(\gamma^+ - \delta^-) \\ &\quad + [f(a)]^+((\alpha^- - \gamma^-) + (\delta^+ - \beta^+)) \\ &\quad - [f(a)]^-((\alpha^+ - \gamma^+) + (\delta^- - \beta^-)), \end{aligned}$$

or equally we could write

$$\begin{aligned} g(a;\alpha,\beta) - g(b;\gamma,\delta) &= ([f(b)]^+ - [f(a)]^+)(\beta^+ - \alpha^-) \\ &\quad - (-[f(b)]^- + [f(a)]^-)(\alpha^+ - \beta^-) \\ &\quad + [f(b)]^+((\alpha^- - \gamma^-) + (\delta^+ - \beta^+)) \\ &\quad - [f(b)]^-((\alpha^+ - \gamma^+) + (\delta^- - \beta^-)). \end{aligned}$$

Now (5.17) and the properties of f imply,

$$\mathcal{L}_f(2B[b-a]^- - A(|\alpha-\gamma| + |\beta-\delta|)) \leqslant g(a;\alpha,\beta) - g(b;\gamma,\delta)$$
$$\leqslant \mathcal{L}_f(2B[b-a]^+ + A(|\alpha-\gamma| + |\beta-\delta|)), \qquad (5.18)$$

where $A = \max\{|a|,|b|\}$ and $B = \max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}$; here (5.18) is the inequality (3.13) of property G4 with $C_g(A) = A$ and $\tilde{C}_g(B) = 2B$.

S4. We make use of the following inequality:

$$\max\{A,B\} - \max\{C,D\} \leqslant \max\{A-C, B-D\}, \quad A,B,C,D \in \mathbb{R}. \qquad (5.19)$$

We have that

$$
\begin{aligned}
g(a; \alpha, \beta) &- g(b; \gamma, \delta) \\
&= -[f(a)]^+ \max\{-\alpha^-, \beta^+\} - [f(a)]^- \max\{-\beta^-, \alpha^+\} \\
&\quad + [f(b)]^+ \max\{-\gamma^-, \delta^+\} + [f(b)]^- \max\{-\delta^-, \gamma^+\} \\
&= ([f(b)]^+ - [f(a)]^+) \max\{-\gamma^-, \delta^+\} \\
&\quad + ([f(b)]^- - [f(a)]^-) \max\{-\delta^-, \gamma^+\} \\
&\quad + [f(a)]^+ (\max\{-\gamma^-, \delta^+\} + \max\{-\alpha^-, \beta^+\}) \\
&\quad - [f(a)]^- (\max\{-\beta^-, \alpha^+\} - \max\{-\delta^-, \gamma^+\}), \\
&\leqslant [f(b) - f(a)]^+ \max\{-\gamma^-, \delta^+\} + [f(b) - f(a)]^+ \max\{-\delta^-, \gamma^+\} \\
&\quad + [f(a)]^+ \max\{\alpha^- - \gamma^-, \delta^+ - \beta^+\} - [f(a)]^- \max\{\delta^- - \beta^-, \alpha^+ - \gamma^+\} \\
&\leqslant \mathcal{L}_f [b - a]^+ \{|\gamma| + |\delta|\} \\
&\quad + [f(a)]^+ \max\{\alpha^- - \gamma^-, \delta^+ - \beta^+\} - [f(a)]^- \max\{\delta^- - \beta^-, \alpha^+ - \gamma^+\},
\end{aligned}
$$

or equally we could write

$$
\begin{aligned}
g(a; \alpha, \beta) - g(b; \gamma, \delta) &\leqslant \mathcal{L}_f [b - a]^+ \{|\alpha| + |\beta|\} \\
&\quad + [f(b)]^+ \max\{\alpha^- - \gamma^-, \delta^+ - \beta^+\} - [f(b)]^- \max\{\delta^- - \beta^-, \alpha^+ - \gamma^+\}.
\end{aligned}
$$

Now (5.19) and the properties of f imply

$$
\begin{aligned}
\mathcal{L}_f (2B[b - a]^- - A(|\alpha - \gamma| + |\beta - \delta|)) &\leqslant g(a; \alpha, \beta) - g(b; \gamma, \delta) \\
&\leqslant \mathcal{L}_f (2B[b - a]^+ + A(|\alpha - \gamma| + |\beta - \delta|)), \quad (5.20)
\end{aligned}
$$

where $A = \max\{|a|, |b|\}$ and $B = \max\{|\alpha|, |\beta|, |\gamma|, |\delta|\}$; here (5.20) is the inequality (3.13) of property G4 with $C_g(A) = A$ and $\tilde{C}_g(B) = 2B$.

6. Numerics

In this section we display and discuss a variety of computations using the schemes outlined in Section 5. We present our results as Figs 1–5. We fix $\Omega = (0, 1)$ for all the computations and take the function $f : \mathbb{R} \to \mathbb{R}$ to be defined by

$$
f(r) = \mathcal{L}_f [|r| - J_p]^+ \mathrm{sgn}(r) \quad (6.21)
$$

where J_p is a given non-negative constant referred to as the pinning current in the superconductivity literature. It is a critical current which the current $J := q_{xx}$, induced by the applied magnetic field $H_{\mathrm{appl}} := (q_\infty)_x$, must exceed in order for the vorticity, ω, to move. (In all of our computations we take $\mathcal{L}_f = \frac{1}{\mu}$). u_0 is always taken to be a piecewise linear continuous function and the value of $(q_\infty)_x$ on $\partial\Omega$ is set to be 1. For a particular computation, once h has been specified, Δt and ϵ (where appropriate) can be chosen to satisfy the necessary conditions of Theorem 3.1 and property G2. The computations run in a loop, first calculating an update for $\mathbf{U}^n$ explicitly using the formulas given in Section 5

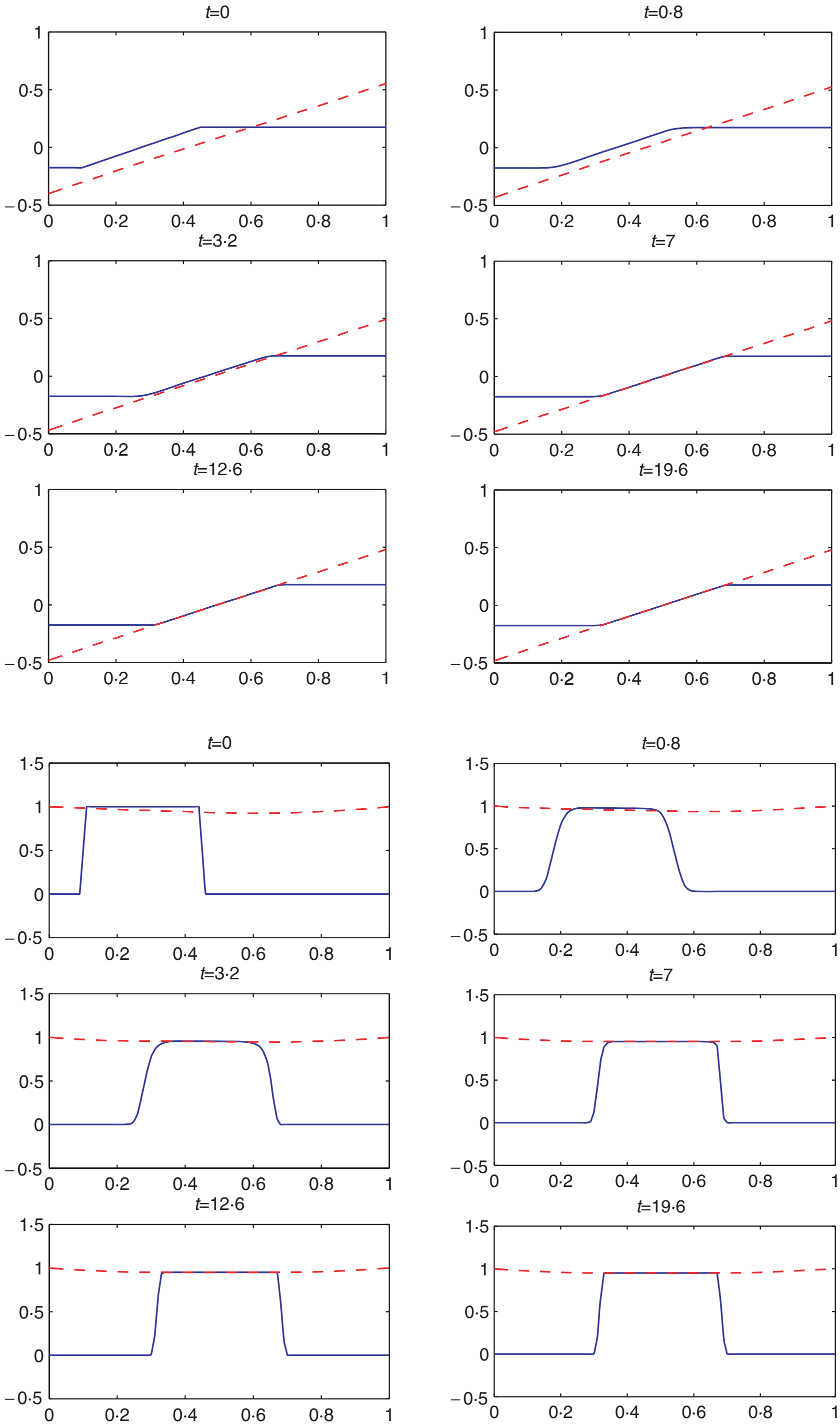

FIG. 1. Maximum scheme (S4); $h = 0{\cdot}01$, $\Delta t = 0{\cdot}0005$, $\mu = 1{\cdot}0$, $J_p = 0{\cdot}0$.

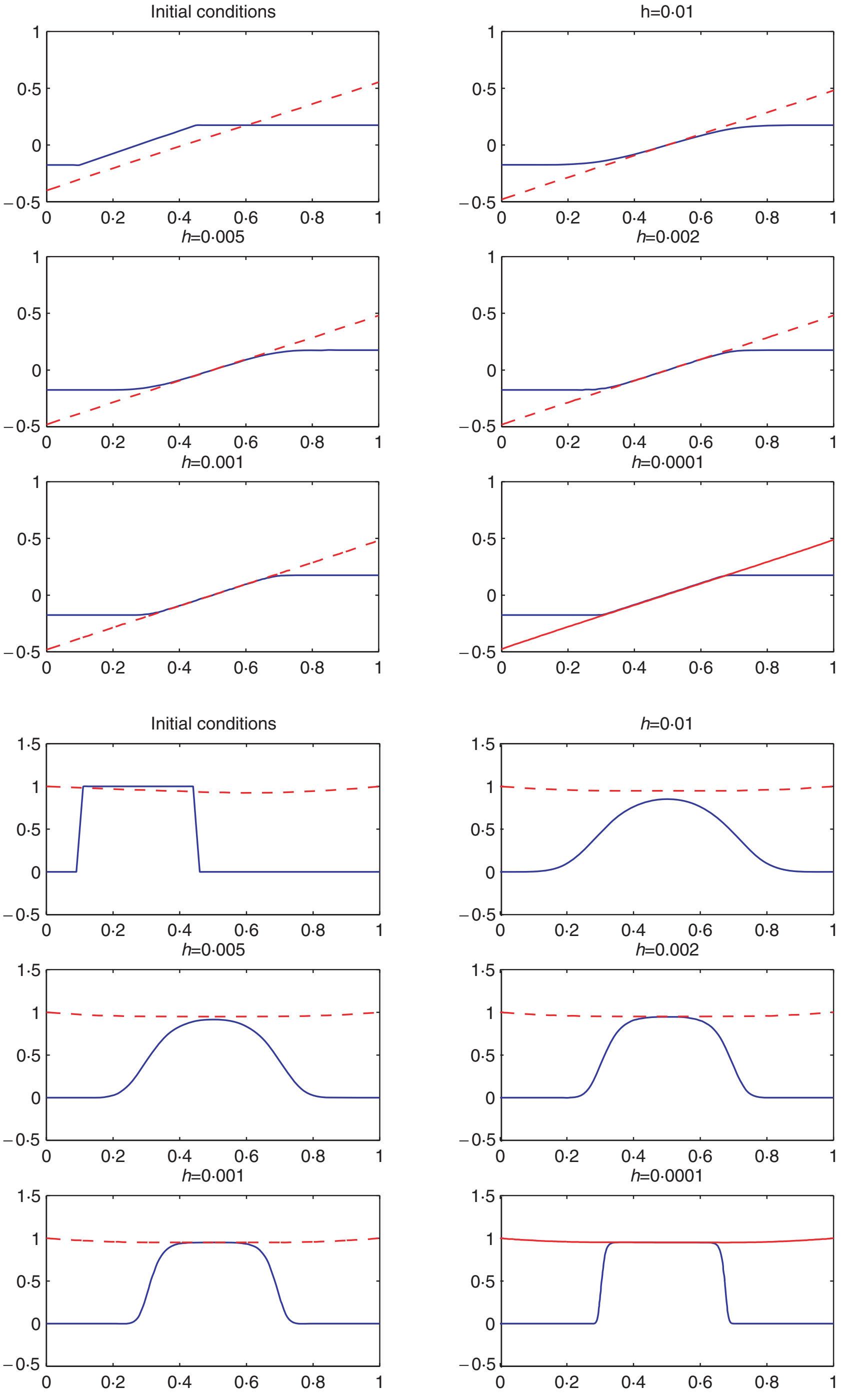

FIG. 2. Regularized scheme (S1); variable h, Δt and ϵ. $\mu = 1{\cdot}0$, $J_p = 0{\cdot}0$.

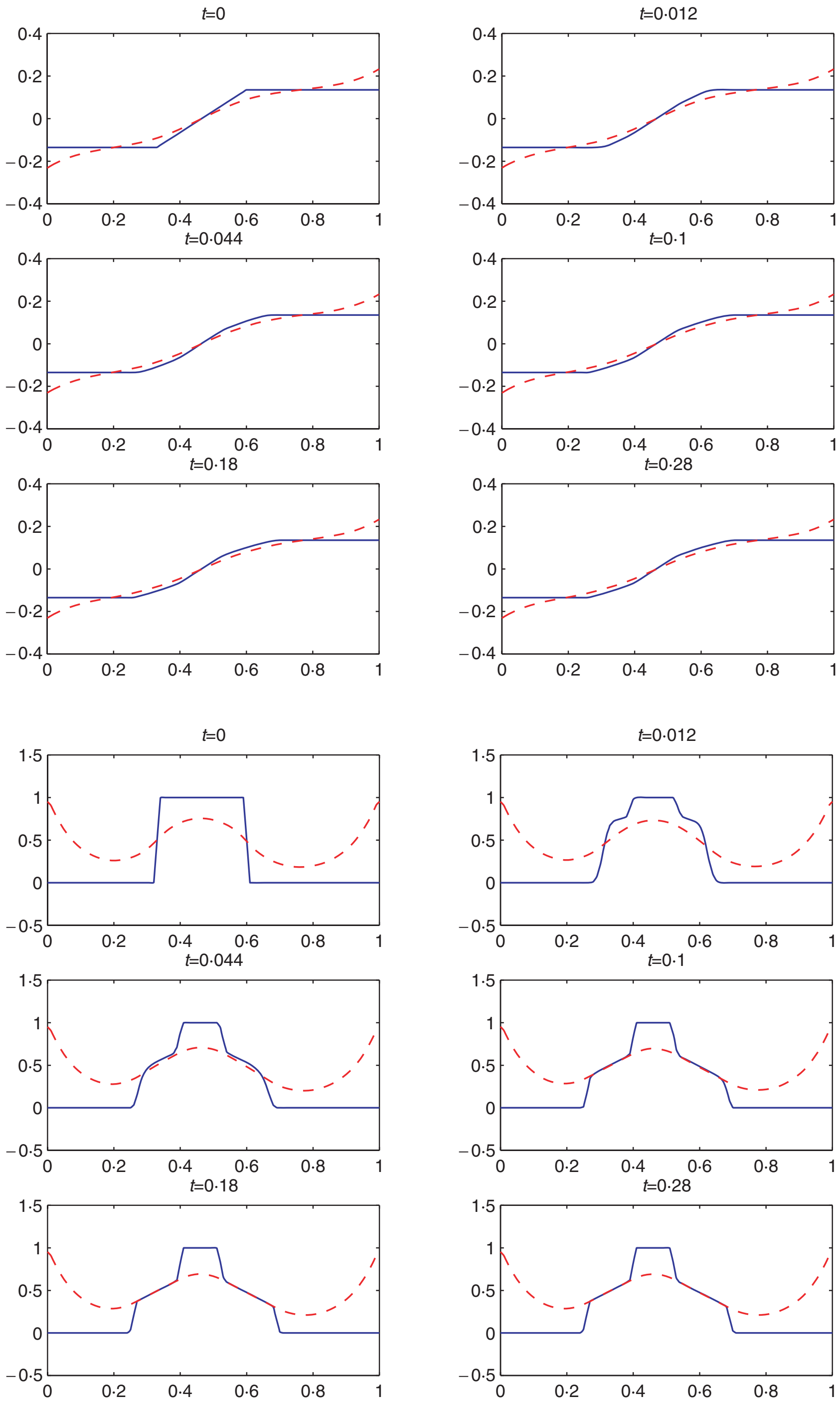

FIG. 3. Maximum scheme (S4); $h = 0{\cdot}01$, $\Delta t = 0{\cdot}0001$, $\mu = 0{\cdot}01$, $J_p = 2{\cdot}0$.

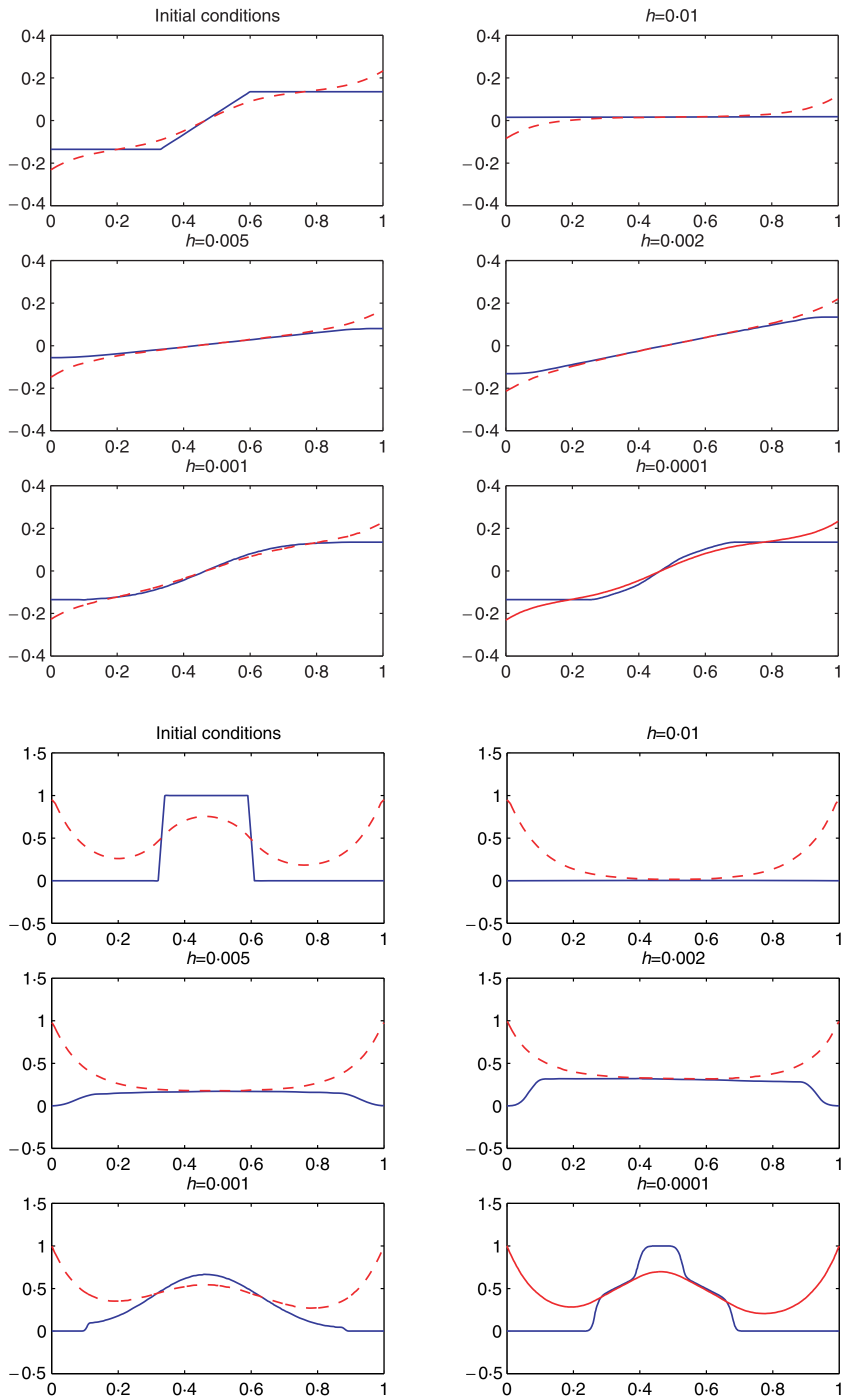

FIG. 4. Regularised scheme (S1); variable h, Δt and ϵ. $\mu = 0{\cdot}01$, $J_p = 2{\cdot}0$.

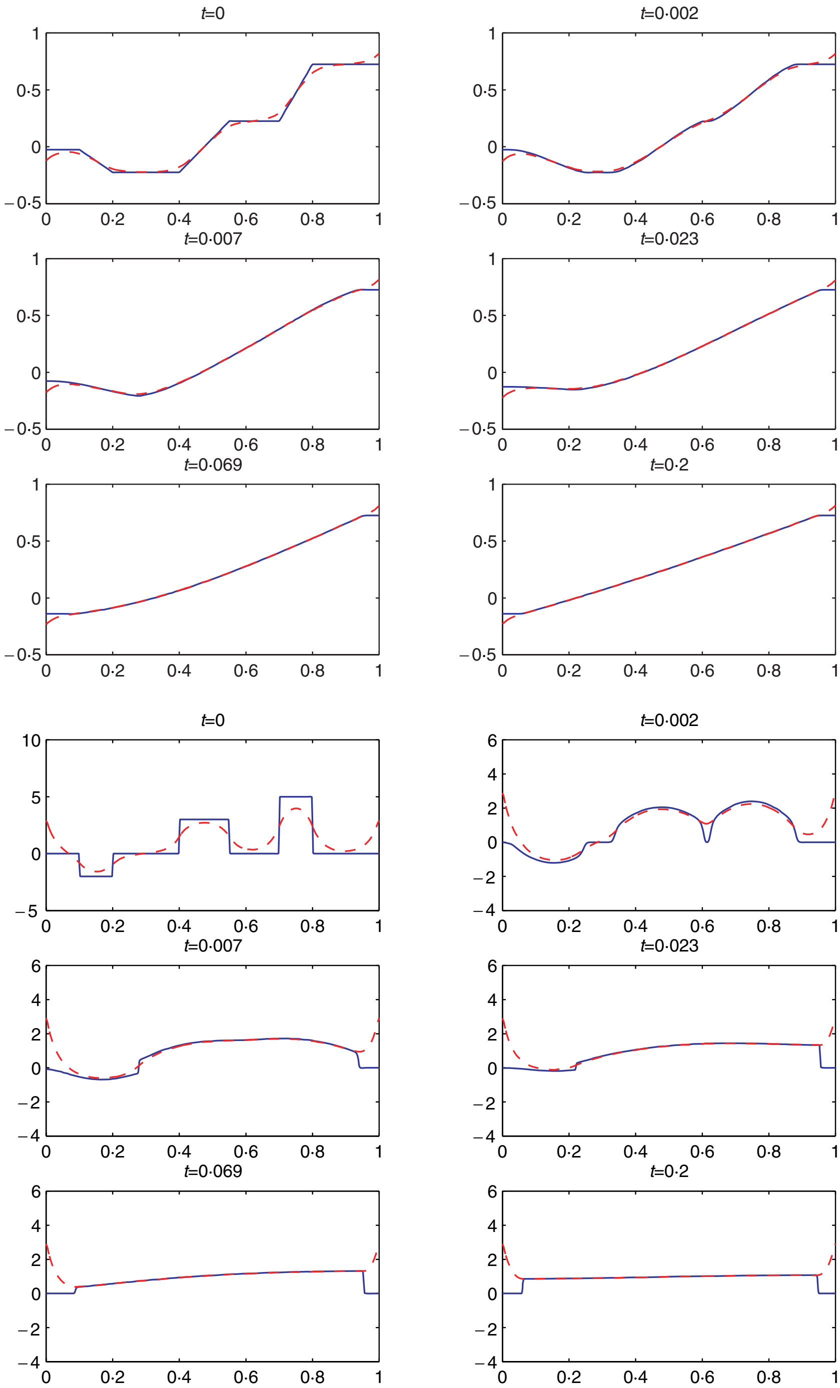

FIG. 5. Maximum scheme (S4); $h = 0{\cdot}002$, $\Delta t = 0{\cdot}000\,0005$, $\mu = 0{\cdot}001$, $J_p = 0{\cdot}0$.

and then using this new value of $\mathbf{U}^n$ to solve for $\hat{q}$ and q_∞ numerically in one combined step. Note that a standard numerical approximation to q, $q^h = \hat{q}^h + q^h_\infty$, solves a simple tridiagonal system.

Noting that the vorticity is given by $\omega = u_x$ and the magnetic field is given by $H = q_x$ it is straightforward to convert our numerical solutions for u and q into approximations of ω and H respectively using differences of the numerical solution. Thus the resulting plots can be compared with those produced by Elliott & Styles (2000, 2001) and Styles (1997). It is for this reason that all plots of u and q are replicated in terms of ω and H. (Note that the top six plots in each figure show u (solid curve) and q (dashed curve), and the bottom six plots show ω (solid curve) and H (dashed curve)).

Note that we do not include results for all four schemes as the results produced by the two regularized schemes S1, S2 and those by the upwind and maximum schemes S3, S4 are very similar. Thus we only consider numerical solutions obtained via the maximum scheme S4 and the regularized scheme S1.

Figures 1 and 2 have the same initial data u_0, and take $\mu = 1$ and $J_p = 0$. Figure 1 shows the time evolution of the numerical solution computed with the maximum scheme. We see the region of vorticity move to the centre of Ω where it reaches a steady state. The actual steady states of the continuous problem (in terms of ω and h) are piecewise constant. The numerical steady state is almost piecewise constant, the steep sides of the region of vorticity cover a spatial distance of $2h$. This agrees with the numerical results of Styles (1997). When the equivalent computation is run for the regularized scheme (a) we see the same motion towards the centre of Ω. However, the smoothing effect of the regularising term is severe. In order to obtain a more accurate solution with this method we consider progressively smaller values of $h(= 0{\cdot}005, 0{\cdot}002, 0{\cdot}001, 0{\cdot}0001)$. Figure 2 shows the initial condition followed by the five different steady states obtained for each of these values of h. Note that there is a continuum of solutions for the steady state problem. Although for a given initial condition the system converges to a particular steady state solution in long time. See Claisse (2000) for a proof.

Figure 3 takes $\mu = 0{\cdot}01$ and $J_p = 2{\cdot}0$. The introduction of pinning means that where the difference between u and q is sufficiently small the value of u should remain fixed. Figure 3 shows the time evolution of the numerical solution computed with the maximum scheme and once again this solution exhibits the same properties as those found by Elliott & Styles (2001).

Tables 1 and 4 give, for Figs 1 and 3 respectively, the maximum absolute errors between the numerical solution for a given value of h and the numerical solution for $h = 0{\cdot}0001$. The values in brackets are estimated orders of convergence ($EOC = \log(E_1/E_2)/\log(h_1/h_2)$). As we can see these EOC values mainly lie between 0·45 and 0·65 which fits with our main result that the schemes should be of at least order $\sqrt{h}$. The notable outliers occur when we halve the coarsest grid and hence can be disregarded. The L^1 errors for u are given in Tables 2 and 5. The *EOC* values are close to 1. These convergence rates are similar to those that hold for the classical Hamilton–Jacobi equation, see Lin & Tadmor (2001). This is to be expected as the regularity of the solution is similar in both cases. It would be interesting to prove an L^1 error bound for the superconductivity problem. Tables 3 and 6 show the L^∞ errors for q. Table 3 gives *EOC*s somewhat bigger than 1 while for Tables 6 the *EOC* values are slightly less than 1.

TABLE 1 L^{∞} *errors in u for Fig.* 1

	$t = 0{\cdot}8$	$t = 1{\cdot}6$	$t = 2{\cdot}4$	$t = 3{\cdot}2$
$h = 0{\cdot}04$	0·0122	0·0175	0·0136	0·0163
$h = 0{\cdot}02$	0·0121(0·0119)	0·0107(0·7097)	0·0115(0·2420)	0·0102(0·6763)
$h = 0{\cdot}01$	0·0082(0·5613)	0·0087(0·2985)	0·0074(0·6360)	0·0063(0·6951)
$h = 0{\cdot}005$	0·0055(0·5762)	0·0057(0·6101)	0·0052(0·5090)	0·0044(0·5178)
$h = 0{\cdot}0025$	0·0036(0·6114)	0·0038(0·5850)	0·0034(0·6130)	0·0030(0·5525)

TABLE 2 L^{1} *errors in u for Fig.* 1

	$t = 0{\cdot}8$	$t = 1{\cdot}6$	$t = 2{\cdot}4$	$t = 3{\cdot}2$
$h = 0{\cdot}04$	0·0020	0·0023	0·0019	0·0016
$h = 0{\cdot}02$	0·0011(0·8625)	0·0011(1·0641)	9·4939e^{-4}(1·0009)	7·2313e^{-4}(1·1457)
$h = 0{\cdot}01$	5·4631e^{-4}(1·0097)	5·9269e^{-4}(0·8922)	4·8637e^{-4}(0·9649)	3·5912e^{-4}(1·0098)
$h = 0{\cdot}005$	2·7474e^{-4}(0·9917)	3·0016e^{-4}(0·9815)	2·4594e^{-4}(0·9837)	1·7935e^{-4}(1·0017)
$h = 0{\cdot}0025$	1·3630e^{-4}(1·0113)	1·4924e^{-4}(1·0081)	1·2278e^{-4}(1·0022)	8·9886e^{-5}(0·9966)

TABLE 3 L^{∞} *errors in q for Fig.* 1

	$t = 0{\cdot}8$	$t = 1{\cdot}6$	$t = 2{\cdot}4$	$t = 3{\cdot}2$
$h = 0{\cdot}04$	6·1202e^{-4}	9·9165e^{-4}	0·0012	0·0012
$h = 0{\cdot}02$	1·4315e^{-4}(2·0961)	2·9127e^{-4}(1·7675)	3·7206e^{-4}(1·6894)	3·9571e^{-4}(1·6005)
$h = 0{\cdot}01$	5·4805e^{-5}(1·3851)	9·9482e^{-5}(1·5498)	1·2505e^{-4}(1·5730)	1·3476e^{-4}(1·5541)
$h = 0{\cdot}005$	2·7197e^{-5}(1·0109)	3·1040e^{-5}(1·6803)	3·9721e^{-5}(1·6545)	4·3726e^{-5}(1·6238)
$h = 0{\cdot}0025$	1·6514e^{-5}(0·7198)	1·4736e^{-5}(1·0748)	1·1894e^{-5}(1·7397)	1·3474e^{-5}(1·6983)

TABLE 4 L^{∞} *errors in u for Fig.* 3

	$t = 0{\cdot}008$	$t = 0{\cdot}016$	$t = 0{\cdot}024$	$t = 0{\cdot}032$
$h = 0{\cdot}04$	0·0058	0·0088	0·0084	0·0075
$h = 0{\cdot}02$	0·0062(−0·0962)	0·0063(0·4822)	0·0062(0·4381)	0·0052(0·5284)
$h = 0{\cdot}01$	0·0042(0·5619)	0·0042(0·5850)	0·0042(0·5619)	0·0038(0·4525)
$h = 0{\cdot}005$	0·0028(0·5850)	0·0031(0·4381)	0·0029(0·5343)	0·0028(0·4406)
$h = 0{\cdot}0025$	0·0019(0·5594)	0·0020(0·6323)	0·0020(0·5361)	0·0019(0·5594)

TABLE 5 L^{1} *errors in u for Fig.* 3

	$t = 0{\cdot}008$	$t = 0{\cdot}016$	$t = 0{\cdot}024$	$t = 0{\cdot}032$
$h = 0{\cdot}04$	6·2555e^{-4}	0·0011	0·0012	0·0013
$h = 0{\cdot}02$	4·0502e^{-4}(0·6271)	6·1139e^{-4}(0·8473)	7·1994e^{-4}(0·7371)	7·5944e^{-4}(0·7755)
$h = 0{\cdot}01$	2·0938e^{-4}(0·9519)	3·1589e^{-4}(0·9527)	3·7772e^{-4}(0·9306)	4·0699e^{-4}(0·8999)
$h = 0{\cdot}005$	1·0872e^{-4}(0·9455)	1·6680e^{-4}(0·9213)	1·9903e^{-4}(0·9243)	2·1590e^{-4}(0·9146)
$h = 0{\cdot}0025$	5·5376e^{-5}(0·9733)	8·5103e^{-5}(0·9708)	1·0172e^{-4}(0·9684)	1·1061e^{-4}(0·9649)

TABLE 6 L^∞ *errors in q for Fig.* 3

	$t = 0{\cdot}008$	$t = 0{\cdot}016$	$t = 0{\cdot}024$	$t = 0{\cdot}032$
$h = 0{\cdot}04$	0·0022	0·0023	0·0024	0·0024
$h = 0{\cdot}02$	$8{\cdot}0346e^{-4}$(1·4532)	0·0011(1·0641)	0·0013(0·8845)	0·0014(0·7776)
$h = 0{\cdot}01$	$4{\cdot}2682e^{-4}$(0·9126)	$6{\cdot}4184e^{-4}$(0·7772)	$7{\cdot}4767e^{-4}$(0·7980)	$7{\cdot}9173e^{-4}$(0·8223)
$h = 0{\cdot}005$	$2{\cdot}3413e^{-4}$(0·8663)	$3{\cdot}4904e^{-4}$(0·8788)	$4{\cdot}0555e^{-4}$(0·8825)	$4{\cdot}3023e^{-4}$(0·8799)
$h = 0{\cdot}0025$	$1{\cdot}2186e^{-4}$(0·9421)	$1{\cdot}8150e^{-4}$(0·9434)	$2{\cdot}1101e^{-4}$(0·9426)	$2{\cdot}2407e^{-4}$(0·9412)

TABLE 7 L^∞ *errors in u for Fig.* 5

	$t = 0{\cdot}002$	$t = 0{\cdot}004$	$t = 0{\cdot}006$	$t = 0{\cdot}008$
$h = 0{\cdot}04$	0·0190	0·0174	0·0156	0·0150
$h = 0{\cdot}02$	0·0136(0·4824)	0·0105(0·7286)	0·0094(0·7308)	0·0092(0·7053)
$h = 0{\cdot}01$	0·0090(0·5956)	0·0067(0·6482)	0·0058(0·6966)	0·0052(0·8231)
$h = 0{\cdot}005$	0·0061(0·5611)	0·0045(0·5742)	0·0034(0·7705)	0·0027(0·9456)
$h = 0{\cdot}0025$	0·0041(0·5732)	0·0028(0·6845)	0·0021(0·6951)	0·0016(0·7549)

TABLE 8 L^1 *errors in u for Fig.* 5

	$t = 0{\cdot}002$	$t = 0{\cdot}004$	$t = 0{\cdot}006$	$t = 0{\cdot}008$
$h = 0{\cdot}04$	0·0057	0·0058	0·0058	0·0059
$h = 0{\cdot}02$	0·0035(0·7036)	0·0036(0·6881)	0·0035(0·7287)	0·0035(0·7534)
$h = 0{\cdot}01$	0·0020(0·8074)	0·0021(0·7776)	0·0020(0·8074)	0·0020(0·8074)
$h = 0{\cdot}005$	0·0011(0·8625)	0·0012(0·8074)	0·0011(0·8625)	0·0011(0·8625)
$h = 0{\cdot}0025$	$5{\cdot}7701e^{-4}$(0·9308)	$6{\cdot}1421e^{-4}$(0·9662)	$5{\cdot}7370e^{-4}$(0·9391)	$5{\cdot}4681e^{-4}$(1·0084)

TABLE 9 L^∞ *errors in q for Fig.* 5

	$t = 0{\cdot}002$	$t = 0{\cdot}004$	$t = 0{\cdot}006$	$t = 0{\cdot}008$
$h = 0{\cdot}04$	0·0102	0·0119	0·0119	0·0117
$h = 0{\cdot}02$	0·0078(0·3870)	0·0077(0·6280)	0·0075(0·6660)	0·0072(0·7004)
$h = 0{\cdot}01$	0·0046(0·7618)	0·0046(0·7432)	0·0043(0·8026)	0·0041(0·8124)
$h = 0{\cdot}005$	0·0026(0·8231)	0·0027(0·7687)	0·0024(0·8413)	0·0021(0·9652)
$h = 0{\cdot}0025$	0·0014(0·8931)	0·0016(0·7549)	0·0012(1·0000)	0·0011(0·9329)

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