Journal of Scientific Computing, Vol. 18, No. 1, February 2003 (© 2003)

On the Conservation and Convergence to Weak Solutions of Global Schemes

Mark H. Carpenter,[1] David Gottlieb,[2] and Chi-Wang Shu[2]

Received October 15, 2001; accepted (in revised form) January 20, 2002

In this paper we discuss the issue of conservation and convergence to weak solutions of several global schemes, including the commonly used compact schemes and spectral collocation schemes, for solving hyperbolic conservation laws. It is shown that such schemes, if convergent boundedly almost everywhere, will converge to weak solutions. The results are extensions of the classical Lax–Wendroff theorem concerning conservative schemes.

KEY WORDS: Conservation laws; conservation; weak solutions; convergence.

1. INTRODUCTION

We are interested in numerical solutions to the conservation laws:

$$u_t + f(u)_x = 0, \qquad u(x, 0) = u^0(x), \qquad -1 \leqslant x \leqslant 1 \tag{1.1}$$

Here we have written (1.1) in the one dimensional form, but the results of this paper are also valid for multi dimensions.

The purpose of this paper is not to study the issue of convergence. We actually *assume* that the numerical solution converges boundedly a.e. (almost everywhere), to a certain function $u(x, t)$. More precisely, for a numerical scheme defined at the (uniform or nonuniform) grid points x_j, $0 \leqslant j \leqslant N$, with $\Delta x = \max(x_{j+1} - x_j)$ and $v_j(t)$ as the numerical solution at $x = x_j$, we define the function $v_{\Delta x}(x, t)$ by

$$v_{\Delta x}(x, t) = v_j(t), \qquad x_j \leqslant x < x_{j+1} \tag{1.2}$$

[1] Computational Methods and Simulation Branch (CMSB), NASA Langley Research Center, Hampton, Virginia 23681-0001. E-mail: m.h.carpenter@larc.nasa.gov

[2] Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912. E-mail: {dig;shu}@cfm.brown.edu

and assume that $v_{\Delta x}(x, t)$ is uniformly bounded with respect to x, t, and Δx, and, as $\Delta x \to 0$, $v_{\Delta x}(x, t)$ converges pointwise a.e. to $u(x, t)$. See, e.g., [5, 16, 17, 20] for discussions, in the scalar case, of convergence of some of the schemes studied in this paper, under the L^∞ boundedness assumption. We will concentrate on the issue of whether the limit function $u(x, t)$ is a weak solution to (1.1), that is whether it satisfies

$$-\int_0^T \int_{-1}^1 (u(x, t)\, \phi_t(x, t) + f(u(x, t))\, \phi_x(x, t))\, dx\, dt - \int_{-1}^1 u^0(x)\, \phi(x, 0)\, dx = 0 \tag{1.3}$$

for any smooth function $\phi(x, t)$ which is compactly supported. Also, in this paper we only consider semidiscrete method-of-lines schemes, i.e., schemes which are discretized in the spatial variable(s) only.

The classical result in this area is the famous Lax–Wendroff Theorem [11]:

Theorem 1.1 (Lax and Wendroff). If the numerical solution of a conservative scheme:

$$(v_j)_t + \frac{1}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}) = 0 \tag{1.4}$$

where the numerical flux

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(v_{j-p}, \ldots, v_{j+q}) \tag{1.5}$$

is *local* (i.e., p and q are constants independent of Δx), Lipschitz continuous in every argument, and consistent with the physical flux $\hat{f}(v, \ldots, v) = f(v)$, converges boundedly a.e. (almost everywhere) to a function $u(x, t)$, then $u(x, t)$ is a weak solution to (1.1). □

The proof follows easily from a summation by parts and an application of the dominant convergence theorem. See LeVeque [13] for a slightly different version of this theorem and its proof.

The Lax–Wendroff Theorem, however, does not cover global schemes, i.e., schemes which can not be written in the form (1.4) with a *local* flux $\hat{f}_{j+\frac{1}{2}}$. Examples of global schemes include the compact schemes [12, 2, 3, 5], and various spectral Galerkin or collocation schemes (Fourier, Legendre, Chebyshev) [9, 1, 6]. We will extend the Lax Wendroff Theorem to these global schemes in this paper.

We remark that there are discussions in the literature about schemes which are not of the conservative form (1.1) but nevertheless still converge to weak solutions. One such example is the class of schemes for general curvilinear coordinates, see [19] for a proof that such schemes actually do converge to weak solutions.

In [8] the authors discussed conservation issues of Chebyshev methods. However, they only considered the mean of the solution, that is, they verified that the limit solution satisfies (1.3) with $\phi(x, t) = 1$.

The organization of the paper is as follows. In Sec. 2 we discuss compact schemes. Section 3 contains the Legendre collocation method, while Sec. 4 discusses the Chebyshev method. Section 5 discusses the Legendre approximation in the multi-domain case.

To close this section we mention that in this paper C (or c) is a generic constant.

2. COMPACT SCHEMES

Compact schemes are methods where the derivatives are approximated by rational function operators on the discrete solutions. We consider compact schemes defined on a uniform grid x_j, $0 \leqslant j \leqslant N$. For example, a fourth order central compact approximation to the derivative is [12]:

$$\tfrac{1}{6}\left((v_x)_{j-1} + 4(v_x)_j + (v_x)_{j+1}\right) = \tfrac{1}{2}\left(v_{j+1} - v_{j-1}\right) \tag{2.1}$$

and a third order upwind compact approximation to the derivative is [5]:

$$\tfrac{1}{3}\left(-(v_x)_{j-1} + 5(v_x)_j - (v_x)_{j+1}\right) = \tfrac{1}{2}\left(3v_j - 4v_{j-1} + v_{j-2}\right) \tag{2.2}$$

Adequate boundary conditions must be used for the compact schemes, to retain accuracy and stability, see [2, 3] for details. Together with boundary conditions, a compact scheme for (1.1) can be written as

$$Pv_t + Qf(v) = -\tau(v_B - g_B) \tag{2.3}$$

where $v = (v_0, \ldots, v_N)^*$ is the numerical solution, τ is a constant, $v_B = (v_0, 0, \ldots, 0, v_N)^*$ is the boundary part of the numerical solution, and $g_B = (g_0, 0, \ldots, 0, g_N)^*$ is the given boundary data. Depending on the wind direction, one or both of the first and last components of v_B and g_B may also be zero(s). The matrices P and Q satisfy the following conditions [3]:

- P is symmetric, and satisfies

$$P\phi - \phi = O(\Delta x) \tag{2.4}$$

Here and below, $\phi = (\phi(x_0), \ldots, \phi(x_N))^*$, and $\phi(x)$ is an arbitrary smooth (C^1 or smoother) function. $O(\Delta x)$ for a vector means that each component is bounded by a constant times Δx, and the constant depends only on the derivatives of $\phi(x)$;

- Q is "almost" anti-symmetric, that is:

$$Q + Q^* = R + S \tag{2.5}$$

where $R = (r_{ij})$, and $r_{ij} = 0$ except for r_{00} and r_{NN}. S is either identically 0 for the central compact schemes, or satisfies

$$S\phi = O(\Delta x) \tag{2.6}$$

for the upwind compact schemes, where ϕ is defined as before. Also, Q is at least a first order approximation to the derivative:

$$Q\phi - \phi_x = O(\Delta x) \tag{2.7}$$

We can easily verify that all the compact schemes in [12, 2, 3, 5] satisfy the above conditions for P and Q.

For such compact schemes we can state the following proposition:

Proposition 2.1. If the solution of the compact scheme (2.3) converges almost everywhere to a function $u(x, t)$, then $u(x, t)$ is a weak solution to (1.1).

Proof. For any compactly supported, C^2 function $\phi(x, t)$, we denote by $\phi = (\phi(x_0, t),..., \phi(x_N, t))^*$, left multiply (2.3) by ϕ^*, and integrate over $[0, T]$ to obtain:

$$\int_0^T \phi^*(Pv_t + Qf(v))\, dt = 0$$

due to the zero boundary conditions of ϕ. Now integrating by parts in t for the first term, taking a transpose of the equation (which is a scalar), and using the symmetry of P and condition (2.5) of Q, we obtain:

$$-\int_0^T (v^*P\phi_t + f(v)^*\, Q\phi)\, dt - (v^*P\phi)|_{t=0} = -\int_0^T f(v)^*\, S\phi\, dt$$

Or, considering (2.4), (2.6), and (2.7), and the uniform boundedness (with respect to the mesh size Δx) of v,

$$-\int_0^T (v^*\phi_t + f(v)^*\, \phi_x)\, dt - (v^*\phi)|_{t=0} = O(1) \tag{2.8}$$

where the constant term $O(1)$ results from a summation of $N = \frac{1}{O(\Delta x)}$ terms of $O(\Delta x)$ quantities; the constant depends only on the derivatives of $\phi(x)$.

Recalling the definition of the function $v_{\Delta x}(x, t)$ in (1.2), we can multiply (2.8) by Δx to obtain

$$-\int_0^T \int_{-1}^1 (v_{\Delta x}(x, t)\, \phi_t^{\Delta x}(x, t) + f(v_{\Delta x}(x, t))\, \phi_x^{\Delta x}(x, t))\, dx\, dt$$

$$-\int_{-1}^1 v_{\Delta x}(x, 0)\, \phi^{\Delta x}(x, 0)\, dx = O(\Delta x) \tag{2.9}$$

where

$$\phi^{\Delta x}(x, t) = \phi(x_j, t),\ \ \phi_x^{\Delta x}(x, t) = \phi_x(x_j, t),\ \ \phi_t^{\Delta x}(x, t) = \phi_t(x_j, t),\ \ x_j \leqslant x < x_{j+1} \tag{2.10}$$

By assumption, $v_{\Delta x}(x, t)$ converges to $u(x, t)$ boundedly a.e. There is no problem about the uniform convergence of $\phi^{\Delta x}(x, t)$, $\phi_x^{\Delta x}(x, t)$ and $\phi_t^{\Delta x}(x, t)$ due to the smoothness of ϕ. By the dominant convergence theorem, taking the limit as $\Delta x \to 0$ in (2.9), we obtain (1.3). This proves that $u(x, t)$ is a weak solution of (1.1). □

3. LEGENDRE SPECTRAL COLLOCATION SCHEMES

The Legendre collocation method can be written in the following way:

$$\frac{\partial u_N(x, t)}{\partial t} + \frac{\partial I_N f(u_N(x, t))}{\partial x} = SV(u_N(x, t)) + Bu_N(x, t) \tag{3.1}$$

where $u_N(x, t)$ is the numerical solution which is a polynomial of degree at most N in x, I_N is the Legendre interpolation operator, i.e., for any function $g(x)$, $I_N g(x)$ is the unique polynomial of degree at most N satisfying $I_N g(x_j) = g(x_j)$ at the $N+1$ Legendre Gauss–Lobatto points x_j, which are the zeros of the polynomial $(1-x^2)\, P'_N$, where P_N is the Legendre polynomial of degree N.

The term SV is the spectral viscosity term needed to stabilize the scheme and in order for the assumption "$v_{\Delta x}(x, t)$ converges boundedly a.e. to a function $u(x, t)$" to be realistic. We consider here the superviscosity term

$$SV(u_N) = \frac{\epsilon(-1)^s}{N^{2s-1}} \left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial}{\partial x} \right]^s u_N(x, t) \tag{3.2}$$

ϵ is the superviscosity coefficient, s is an integer growing with N [21, 10, 14, 15]. We remark that this superviscosity term is equivalent in practice to a low pass filter.

Finally, the boundary term $Bu_N(x, t)$ could be either 0, or

$$\tau(u_N(1, t)-g_1)(1+x)\, P'_N(x)$$

or

$$\tau(u_N(-1, t)-g_{-1})(1-x)\, P'_N(x)$$

or a combination, depending on the wind directions at the boundary points. Here τ is a constant chosen for stability and g_1 and g_{-1} are functions of the time only.

Let $\phi(x, t)$ be a test function in C_0^∞. Take $\phi_{N-1}(x, t) = I_{N-1}\phi(x, t)$, then clearly ϕ_{N-1} are polynomials of degree at most $N-1$ and vanish at both boundary points $x = \pm 1$. Also $\phi_{N-1}(x, t) \to \phi(x, t)$, $(\phi_{N-1})_x\,(x, t) \to \phi_x(x, t)$, and $(\phi_{N-1})_t\,(x, t) \to \phi_t(x, t)$ uniformly.

We denote now by

$$(f, g) = \int_{-1}^{1} f(x)\, g(x)\, dx$$

and by

$$(f, g)_N = \sum_{k=0}^{N} f(x_j)\, g(x_j)\, \omega_j$$

where $\omega_j > 0$ are the weights in the Gauss–Lobatto formula. We note that $(f, g)_N = (f, g)$ if fg is a polynomial of degree at most $2N-1$.

We first show that the boundary terms do not cause a problem:

Lemma 3.1.

$$(\phi_{N-1}, Bu_N) = 0 \tag{3.3}$$

Proof. We start by observing that

$$(\phi_{N-1}, Bu_N) = (\phi_{N-1}, Bu_N)_N$$

Bu_N vanishes for the inner Gauss–Lobatto points and ϕ_{N-1} vanishes at the boundaries and therefore the lemma is proven. □

With (3.3), we multiply (3.1) by $\phi_{N-1}(x, t)$, integrate over x, and integrate by parts for the second term to obtain

$$\int_{-1}^{1} \phi_{N-1}(x, t) \frac{\partial u_N(x, t)}{\partial t} dx - \int_{-1}^{1} \frac{\partial \phi_{N-1}(x, t)}{\partial x} I_N f(u_N(x, t)) dx$$
$$= \frac{\epsilon(-1)^s}{N^{2s-1}} \int_{-1}^{1} \phi_{N-1}(x, t) \left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial}{\partial x} \right]^s u_N(x, t) dx \tag{3.4}$$

We now estimate the right hand side of (3.4):

Lemma 3.2.

$$\lim_{N \to \infty} \frac{\epsilon(-1)^s}{N^{2s-1}} \int_{-1}^{1} \phi_{N-1}(x, t) \left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial}{\partial x} \right]^s u_N(x, t) dx = 0 \tag{3.5}$$

Also, the quantity under the limit sign is uniformly bounded with respect to t.

Proof. Since ϕ_{N-1} is a polynomial of degree $N-1$,

$$\phi_{N-1}(x, t) = \sum_{k=0}^{N-1} \hat{\phi}_{k, N}(t) P_k(x)$$

where $\hat{\phi}_{k, N}(t)$ are the collocation Legendre coefficients of the test function ϕ. Note that

$$\left[\frac{\partial}{\partial x} (1-x^2) \frac{\partial}{\partial x} \right]^s P_k(x) = (-1)^s k^s (k+1)^s P_k(x)$$

and therefore

$$(\phi_{N-1}, SV(u_N)) = (-1)^s \frac{\epsilon}{N^{2s-1}} \sum_{k=0}^{N} k^s (k+1)^s \hat{\phi}_{k, N}(t) \hat{u}_{k, N}(t) (P_k, P_k)$$

Here $\hat{u}_{k, N}$ are the Legendre collocation coefficients of u_N. We note that as a consequence of the uniform boundedness of $u_N(x_j)$ and the fact that ϕ is in C_0^∞,

$$|\hat{\phi}_{k, N}(t) \hat{u}_{k, N}(t)| \leqslant \frac{C}{k^3} \tag{3.6}$$

This implies (3.5) and the uniform boundedness of the quantity under the limit sign with respect to t. □

We thus only have to deal with the left hand side of (3.4). We integrate (3.4) in t, integrate by parts for the first term, and use Lemma 3.2 to obtain:

$$-\int_0^T \int_{-1}^1 \left(u_N(x, t) \frac{\partial \phi_{N-1}(x, t)}{\partial t} + I_N f(u_N(x, t)) \frac{\partial \phi_{N-1}(x, t)}{\partial x} \right) dx\, dt$$

$$-\int_{-1}^1 u_N(x, 0)\, \phi_{N-1}(x, 0)\, dx = o(1) \tag{3.7}$$

It looks like we can immediately take the limit as in the Lax–Wendroff Theorem. The trouble is that, we have only assumed the uniform boundedness of $u_N(x_j, t)$, hence of $f(u_N(x_j, t))$, but this *does not imply* the uniform boundedness of either $u_N(x, t)$ or $I_N f(u_N(x, t))$ due to the lack of regularity.

We need the following lemma:

Lemma 3.3. Let $v_{\Delta x}$ be the piecewise linear polynomial taking the values $u_N(x_j, t)$, then

$$\left(u_N(x, t), \frac{\partial \phi_{N-1}(x, t)}{\partial t} \right) = \left(v_{\Delta x}, \frac{\partial \phi_{N-1}(x, t)}{\partial t} \right) + o(1), \tag{3.8}$$

$$\left(I_N f(u_N(x, t)), \frac{\partial \phi_{N-1}(x, t)}{\partial x} \right) = \left(I_N f(v_{\Delta x}), \frac{\partial \phi_{N-1}(x, t)}{\partial x} \right) + o(1) \tag{3.9}$$

Proof. We will switch back and forth between integrals and quadrature summations:

$$\left(u_N(x, t), \frac{\partial \phi_{N-1}(x, t)}{\partial t} \right) = \left(u_N(x, t), \frac{\partial \phi_{N-1}(x, t)}{\partial t} \right)_N$$

$$= \sum_{j=0}^{N} u_N(x_j, t) \frac{\partial \phi_{N-1}(x_j, t)}{\partial t}\, \omega_j$$

$$= \int_{-1}^1 \theta(x)\, v_{\Delta x}(x, t) (\phi_{N-1})_t^{\Delta x}\, (x, t)\, dx$$

where

$$\theta(x) = \frac{\omega_j}{x_{j+1} - x_j}, \qquad x_j \leqslant x < x_{j+1}$$

In [18] it has been established that $\theta(x)$ is uniformly bounded and converges a.e. to 1 as $\Delta x \to 0$.

This proves (3.8). The proof for (3.9) is similar. □

We can state now

Theorem 3.4. If the function $v_{\Delta x}(x, t)$ defined in (1.2), obtained from the solution of the Legendre collocation scheme (3.1) at the Legendre Gauss–Lobatto points x_j, converges almost everywhere to a function $u(x, t)$, then $u(x, t)$ is a weak solution to (1.1).

Proof. By assumption, $v_{\Delta x}(x, t)$ converges to $u(x, t)$ boundedly a.e. Also, there is no problem about the uniform convergence of $\phi_{N-1}^{\Delta x}(x, t)$, $(\phi_{N-1})_x^{\Delta x}(x, t)$ and $(\phi_{N-1})_t^{\Delta x}(x, t)$ due to the smoothness of ϕ. Using the dominant convergence theorem, taking the limit as $\Delta x \to 0$ in (3.8), (3.9) and (3.7), we obtain (1.3). This proves that $u(x, t)$ is a weak solution of (1.1). □

We close this section by commenting on other spectral viscosity terms in (3.1) that stabilize the Legendre method. One such term is

$$\epsilon_N \frac{\partial}{\partial x} Q \frac{\partial u_N}{\partial x}$$

where the spectral viscosity operator Q is defined by

$$Q\phi = \sum_{k=0}^{N} \hat{Q}_k \hat{\phi}_k P_k(x)$$

where

$$\phi = \sum_{k=0}^{N} \hat{\phi}_k P_k(x)$$

and

$$\hat{Q}_k = 0, \qquad k \leqslant m_N$$

$$1 \geqslant \hat{Q}_k \geqslant 1 - \left(\frac{m_N}{k}\right)^4, \qquad k > m_N$$

with m_N growing with N.

We can establish also for this viscosity term that

$$(\phi_{N-1}, SV(u_N)) \to 0$$

and therefore the result above holds also for this kind of spectral viscosity.

4. CHEBYSHEV SPECTRAL COLLOCATION SCHEMES

In this section we consider the Chebyshev collocation schemes. These are more difficult to analyze than the Legendre method because of the weight function $\frac{1}{\sqrt{1-x^2}}$.

The Chebyshev collocation method can be written in the following way:

$$\frac{\partial u_N(x,t)}{\partial t}+\frac{\partial J_N f(u_N(x,t))}{\partial x}=\frac{\epsilon(-1)^s}{N^{2s-1}}\left[\sqrt{1-x^2}\,\frac{\partial}{\partial x}\right]^{2s} u_N(x,t)+Bu_N(x,t) \tag{4.1}$$

where again $u_N(x,t)$ is the numerical solution which is a polynomial of degree at most N in x, J_N is the Chebyshev interpolation operator, i.e., for any function $g(x)$, $J_N g(x)$ is the unique polynomial of degree at most N satisfying $J_N g(x_j)=g(x_j)$ at the $N+1$ Chebyshev Gauss–Lobatto points x_j. ϵ is the superviscosity coefficient, s is an integer growing with N [21, 14, 15]. We remark again that this superviscosity term, which in practice is equivalent to a low pass filter, or a similar vanishing viscosity term [16, 17], is needed in order for the assumption "$v_{\Delta x}(x,t)$ converges boundedly a.e. to a function $u(x,t)$" to be realistic. Finally, the boundary term $Bu_N(x,t)$ could be either 0, or

$$\tau(u_N(1,t)-g_1)(1+x)\,T'_N(x)$$

or

$$\tau(u_N(-1,t)-g_{-1})(1-x)\,T'_N(x)$$

or a combination, depending on the wind directions at the boundary points. Here τ is a constant chosen for stability and g_1 and g_{-1} are functions of the time only.

Let $\phi(x,t)$ be a test function in C_0^5, that is, all x derivatives of $\phi(x,t)$ up to order 5 vanish at the boundary points $x=\pm 1$. Such test functions are, of course, dense in C_0^1. It follows that $(1-x^2)^{-3/2}\,\phi(x,t)$ is in C_0^3. We denote the $(N-5)$-th degree Chebyshev interpolation polynomial of the function $(1-x^2)^{-3/2}\,\phi(x,t)$ by

$$\xi_{N-5}(x,t)=J_{N-5}((1-x^2)^{-3/2}\,\phi(x,t)) \tag{4.2}$$

and note that

$$\xi_{N-5}(x,t)\to(1-x^2)^{-3/2}\,\phi(x,t),$$

$$\frac{\partial \xi_{N-5}(x,t)}{\partial x}\to\frac{\partial((1-x^2)^{-3/2}\,\phi(x,t))}{\partial x},\quad \frac{\partial \xi_{N-5}(x,t)}{\partial t}\to\frac{\partial((1-x^2)^{-3/2}\,\phi(x,t))}{\partial t}$$

uniformly. We now take

$$\psi_{N-1}(x,t)=(1-x^2)^2\,\xi_{N-5}(x,t), \tag{4.3}$$

then ψ_{N-1} is a polynomial of degree at most $N-1$ and vanishes at both boundary points $x=\pm 1$ together with its first and second x derivatives. Moreover, it can be easily verified that

$$\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\to\phi(x,t),\quad \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\right)_x\to\phi_x(x,t),\quad \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\right)_t\to\phi_t(x,t) \tag{4.4}$$

uniformly.

We again first show that the boundary term does not cause a problem:

Lemma 4.1.

$$\int_{-1}^{1}(1+x)\,T_N'(x)\,\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\,dx=0,\qquad \int_{-1}^{1}(1-x)\,T_N'(x)\,\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\,dx=0 \tag{4.5}$$

Proof. We only prove the first equality. Zero boundary values of $\psi_{N-1}(x,t)$ and its first x derivative imply

$$\begin{aligned}
&\int_{-1}^{1}(1+x)\,T_N'(x)\,\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\,dx\\
&\quad=-\int_{-1}^{1}T_N(x)\left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}(1+x)\right)_x dx\\
&\quad=\int_{-1}^{1}\frac{T_N(x)}{\sqrt{1-x^2}}\left(\frac{\partial\psi_{N-1}(x,t)}{\partial x}(1+x)+\frac{\psi_{N-1}(x,t)}{1-x}\right)dx\\
&\quad=0
\end{aligned}$$

The last equality is due to the fact that

$$\begin{aligned}
&\frac{\partial\psi_{N-1}(x,t)}{\partial x}(1+x)+\frac{\psi_{N-1}(x,t)}{1-x}\\
&\quad=\frac{\partial\psi_{N-1}(x,t)}{\partial x}(1+x)+(1+x)(1-x^2)\,\xi_{N-5}(x,t)
\end{aligned}$$

is a polynomial of degree at most $N-1$, hence is orthogonal to $T_N(x)$ with the weight $\frac{1}{\sqrt{1-x^2}}$. □

With (4.5), we can now multiply (4.1) by $\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}$, integrate over x, and integrate by parts for the second term to obtain

$$\int_{-1}^{1} \frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \frac{\partial u_N(x,t)}{\partial t}\, dx - \int_{-1}^{1} \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}}\right)_x J_N f(u_N(x,t))\, dx$$

$$= \frac{\epsilon(-1)^s}{N^{2s-1}} \int_{-1}^{1} \frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \left[(1-x^2)\frac{\partial}{\partial x}\right]^{2s} u_N(x,t)\, dx \tag{4.6}$$

We now estimate the right hand side of (4.6):

Lemma 4.2.

$$\lim_{N\to\infty} \frac{\epsilon(-1)^s}{N^{2s-1}} \int_{-1}^{1} \frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \left[\sqrt{1-x^2}\frac{\partial}{\partial x}\right]^{2s} u_N(x,t)\, dx = 0 \tag{4.7}$$

Also, the quantity under the limit sign is uniformly bounded with respect to t.

Proof. Integrating by parts $2s$ times, and noticing that the boundary terms are always 0 because of the fact that $\psi_{N-1}(x,t)$ vanishes at the boundaries and because of the factor $\sqrt{1-x^2}$, we obtain

$$\int_{-1}^{1} \frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \left[\sqrt{1-x^2}\frac{\partial}{\partial x}\right]^{2s} u_N(x,t)\, dx$$

$$= \int_{-1}^{1} \frac{u_N(x,t)}{\sqrt{1-x^2}} \left[\sqrt{1-x^2}\frac{\partial}{\partial x}\right]^{2s} \psi_{N-1}(x,t)\, dx \tag{4.8}$$

Recalling the definition of $\xi_{N-5}(x,t)$ in (4.2), we have

$$\xi_{N-5}(x,t) = \sum_{k=0}^{N-5} \hat{\xi}_k(t)\, T_k(x)$$

where $\hat{\xi}_k(t)$ are the collocation Chebyshev coefficients of the C_0^3 function $(1-x^2)^{-3/2}\, \phi(x,t)$, hence

$$|\hat{\xi}_k(t)| \leqslant \frac{C}{k^3} \tag{4.9}$$

Now, by the relationship between ψ_{N-1} and ξ_{N-5} in (4.3):

$$\begin{aligned}
\psi_{N-1}(x,t) &= (1-x^2)^2 \sum_{k=0}^{N-5} \hat{\xi}_k(t)\, T_k(x) \\
&= \tfrac{1}{4}(1-T_2(x))^2 \sum_{k=0}^{N-5} \hat{\xi}_k(t)\, T_k(x) \\
&= \tfrac{1}{4} \sum_{k=0}^{N-5} \hat{\xi}_k(t)\, T_k(x) - \tfrac{1}{4} \sum_{k=0}^{N-5} \hat{\xi}_k(t)(T_{k+2}(x)+T_{k-2}(x)) \\
&\quad + \tfrac{1}{16} \sum_{k=0}^{N-5} \hat{\xi}_k(t)(T_{k+4}(x)+2T_k(x)+T_{k-4}(x)) \\
&= \tfrac{1}{16} \sum_{k=0}^{N-1} (\hat{\xi}_{k-4}(t) - 4\hat{\xi}_{k-2}(t) + 6\hat{\xi}_k(t) - 4\hat{\xi}_{k+2}(t) + \hat{\xi}_{k+4}(t))\, T_k(x) \\
&\equiv \sum_{k=0}^{N-1} \hat{\psi}_k(t)\, T_k(x)
\end{aligned}$$

where we take the convention that $\hat{\xi}_k(t) = 0$ for $k < 0$ or $k > N-5$. This, together with (4.9), clearly implies

$$|\hat{\psi}_k(t)| \leqslant \frac{C}{k^3} \tag{4.10}$$

We now use the equality

$$\left[\sqrt{1-x^2}\,\frac{\partial}{\partial x}\right]^{2s} \psi_{N-1}(x,t) = \sum_{k=0}^{N-1} (-1)^s\, k^{2s} \hat{\psi}_k(t)\, T_k(x) \tag{4.11}$$

and the integral-quadrature equivalence:

$$\begin{aligned}
&\int_{-1}^{1} \frac{u_N(x,t)}{\sqrt{1-x^2}} \left[\sqrt{1-x^2}\,\frac{\partial}{\partial x}\right]^{2s} \psi_{N-1}(x,t)\, dx \\
&\quad = \sum_{j=0}^{N} w_j u_N(x_j,t) \left(\left[\sqrt{1-x^2}\,\frac{\partial}{\partial x}\right]^{2s} \psi_{N-1}(x,t) \right)\Bigg|_{x=x_j}
\end{aligned}$$

where x_j and w_j are the nodes and weights of the Chebyshev Gauss–Lobatto quadrature formula, because the integrand

$$u_N(x,t) \left[\sqrt{1-x^2}\,\frac{\partial}{\partial x}\right]^{2s} \psi_{N-1}(x,t)$$

is a polynomial of degree at most $2N-1$. This, together with the uniform boundedness of $u_N(x_j, t)$ and (4.8), (4.10) and (4.11), implies (4.7) and the uniform boundedness of the quantity under the limit sign with respect to t. □

We thus only have to deal with the left hand side of (4.6). We integrate (4.6) in t, integrate by parts for the first term, and use Lemma 4.2 to obtain:

$$-\int_0^T \int_{-1}^1 \left(u_N(x,t) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_t + J_N f(u_N(x,t)) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_x \right) dx\, dt$$
$$-\int_{-1}^1 u_N(x,0) \frac{\psi_{N-1}(x,0)}{\sqrt{1-x^2}} \, dx = o(1) \tag{4.12}$$

Again, the difficulty is that we have only assumed the uniform boundedness of $u_N(x_j, t)$, hence of $f(u_N(x_j, t))$, not the uniform boundedness of either $u_N(x, t)$ or $J_N f(u_N(x, t))$. We again get around this by switching between integrals and quadrature summations:

$$\int_{-1}^1 \left(u_N(x,t) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_t + J_N f(u_N(x,t)) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_x \right) dx$$
$$= \sum_{j=0}^N w_j \sqrt{1-x^2} \left(u_N(x_j,t) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_t \Bigg|_{x=x_j} \right.$$
$$\left. + f(u_N(x_j,t)) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_x \Bigg|_{x=x_j} \right) \tag{4.13}$$

because we can easily verify that the integrand

$$\sqrt{1-x^2} \left(u_N(x,t) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_t + J_N f(u_N(x,t)) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_x \right)$$

is a polynomial of degree at most $2N-1$. Recalling the definition of the function $v_{\Delta x}(x, t)$ in (1.2) and that of $\phi^{\Delta x}(x, t)$ etc. in (2.10), we can use (4.13) to rewrite (4.12) as

$$-\int_0^T \int_{-1}^1 \theta(x) \left(v_{\Delta x}(x,t) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_t^{\Delta x} + f(v_{\Delta x}(x,t)) \left(\frac{\psi_{N-1}(x,t)}{\sqrt{1-x^2}} \right)_x^{\Delta x} \right) dx\, dt$$
$$-\int_{-1}^1 \theta(x)\, v_{\Delta x}(x,0) \left(\frac{\psi_{N-1}(x,0)}{\sqrt{1-x^2}} \right)^{\Delta x} dx = o(1) \tag{4.14}$$

where

$$\theta(x)=\frac{w_j\sqrt{1-x^2}}{x_{j+1}-x_j},\qquad x_j\leqslant x<x_{j+1}$$

Clearly, $\theta(x)$ is uniformly bounded and converges to 1 as $\Delta x\to 0$. By assumption, $v_{\Delta x}(x,t)$ converges to $u(x,t)$ boundedly a.e. Also, (4.4) guarantees the uniform convergence of the ψ_{N-1} related terms to the right limits. Using the dominant convergence theorem, taking the limit as $\Delta x\to 0$ in (4.14), we obtain (1.3). This proves that $u(x,t)$ is a weak solution of (1.1), i.e., we have proved the following

Proposition 4.3. If the function $v_{\Delta x}(x,t)$ defined in (1.2), obtained from the solution of the Chebyshev collocation scheme (4.1) at the Chebyshev Gauss–Lobatto points x_j, converges almost everywhere to a function $u(x,t)$, then $u(x,t)$ is a weak solution to (1.1). □

5. MULTI-DOMAIN LEGENDRE METHODS

In this section we will discuss stable and conservative interface boundary conditions for the multi-domain Legendre method applied to Eq. (1.1). We assume that the domain $-1\leqslant x\leqslant 1$ is divided into two domains, and for the sake of simplicity we assume that the interface point is $x=0$. We will denote by $u_N(x,t)$ the numerical approximation in $-1\leqslant x\leqslant 0$ and by $v_N(x,t)$ the solution at $0\leqslant x\leqslant 1$. The multi-domain Legendre method is given by

$$\begin{aligned}\frac{\partial u_N}{\partial t}+\frac{\partial}{\partial x}I_N^I f(u_N)&=B(u_N(-1,t))+\tau_1 Q^I(x)[f^+(u_N(0,t))-f^+(v_N(0,t))]\\&\quad+\tau_2 Q^I(x)[f^-(u_N(0,t))-f^-(v_N(0,t))]+SV(u_N)\end{aligned}\tag{5.1}$$

$$\begin{aligned}\frac{\partial v_N}{\partial t}+\frac{\partial}{\partial x}I_N^{II} f(v_N)&=\tau_3 Q^{II}(x)[f^+(v_N(0,t))-f^+(u_N(0,t))]\\&\quad+\tau_4 Q^{II}(x)[f^-(v_N(0,t))-f^-(u_N(0,t))]\\&\quad+SV(v_N)+B(v_N(1,t))\end{aligned}\tag{5.2}$$

Equation (5.1) holds in the interval $-1\leqslant x\leqslant 0$, and (5.2) holds in $0\leqslant x\leqslant 1$. $I_N^I f(u_N)$ interpolates $f(u_N)$ at the zeroes ξ_j of the polynomial

xQ^I and $I_N^{II} f(v_N)$ interpolates $f(v_N)$ at the zeroes η_j of the polynomial xQ^{II}, where

$$Q^I(x) = \frac{(1+x)\, P'_N(2x+1)}{P'_N(1)}, \qquad Q^{II}(x) = \frac{(1-x)\, P'_N(2x-1)}{P'_N(-1)}.$$

The spectral viscosities $SV(u_N)$ and $SV(v_N)$ are of the form

$$SV(u_N) = \epsilon \frac{(-1)^{s+1}}{N^{2s-1}} \left[\frac{\partial}{\partial x} x(x+1) \frac{\partial}{\partial x} \right]^s u_N, \tag{5.3}$$

$$SV(v_N) = \epsilon \frac{(-1)^{s+1}}{N^{2s-1}} \left[\frac{\partial}{\partial x} x(1-x) \frac{\partial}{\partial x} \right]^s v_N \tag{5.4}$$

At this point we stress that the results of this section are valid only for this form of spectral viscosity and not for the others discussed in Sec. 3. The reason for that will be evident in the proof.

Finally the boundary operators B at the ends of the interval $-1 \leqslant x \leqslant 1$ are left unspecified for now.

We will also denote the scalar product $(p, q)_N = \sum_{j=0}^{N} p^T(\xi_j)\, q(\xi_j)\, \omega_j$ if $p(x)$ and $q(x)$ are defined in $[-1, 0]$ and $(p, q)_N = \sum_{j=0}^{N} p^T(\eta_j)\, q(\eta_j)\, \omega_j$ if $p(x)$ and $q(x)$ are defined in $[0, 1]$, and ω_j are the weights in the Gauss–Lobatto Legendre quadrature formula. Note that if pq is a polynomial of degree at most $2N-1$ defined in $[-1, 0]$ then

$$(p, q)_N = (p, q) = \int_{-1}^{0} p^T(x)\, q(x)\, dx$$

A similar formula holds in the interval $[0, 1]$.

Our aim in this section is to show that the choice of the parameters τ_i, $i = 1, 4$ that leads to linear stability is sufficient for proving conservation, i.e., if the numerical solution $u_N(x, t)$, $v_N(x, t)$ converges boundedly a.e. to functions $u(x, t)$, $v(x, t)$, then the solution w defined by $w(x, t) = u(x, t)$ if $-1 \leqslant x \leqslant 0$ and $w(x, t) = v(x, t)$ if $0 \leqslant x \leqslant 1$ converges to the weak solution of (1.1).

We will discuss first the stability of (5.1)–(5.2). We state

Proposition 5.1. The boundary operators are dissipative, i.e.,

$$(u_N, B(u_N(-1, t)))_N + \tfrac{1}{2}\, u_N^T(-1, t)\, Au_n(-1, t) \leqslant 0, \tag{5.5}$$

$$(v_N, B(v_N(1, t)))_N - \tfrac{1}{2}\, v_N^T(-1, t)\, Av_N(1, t) \leqslant 0 \quad \square \tag{5.6}$$

Proposition 5.1 implies that the boundary treatment at the end-points of the interval is stable. Example of such operators are given in [4].

We are ready to state the stability theorem for the linear constant coefficient case. In this case there is no need for the spectral viscosity terms and we will ignore them. We assume that $f = Au$ where A is symmetric in equation (1.1) and in the same way $f^+ = A^+u$, $f^- = A^-u$ where the eigenvalues of A^+ are nonnegative and those of A^- are nonpositive.

Theorem 5.2. Let u_N, v_N be the solutions of (5.1)–(5.2). Define

$$E(t) = (u_N(x,t), u_N(x,t))_N + (v_N(x,t), v_N(x,t))_N$$

then

$$E(t) \leqslant E(0)$$

provided that

$$\tau_1 \leqslant \frac{1}{2\omega_0}, \qquad \tau_2 \geqslant \frac{1}{2\omega_0}, \qquad \tau_3 \leqslant -\frac{1}{2\omega_0}, \qquad \tau_4 \geqslant -\frac{1}{2\omega_0},$$
$$\tau_1 - \tau_3 = \frac{1}{\omega_0}, \qquad \tau_2 - \tau_4 = \frac{1}{\omega_0} \tag{5.7}$$

Proof. The proof follows from multiplying (5.1), (5.2) by u_N^T, v_N^T and taking the scalar product. We use Proposition 5.1 and the following notation

$$u_0 = u_N(0,t), \qquad v_0 = v_N(0,t), \qquad \alpha_0^{\pm} = u_0 A^{\pm} u_0,$$
$$\beta_0^{\pm} = v_0 A^{\pm} v_0, \qquad \gamma_0^{\pm} = u_0 A^{\pm} v_0$$

to get

$$\frac{1}{2\omega_0}\frac{d}{dt}E(t) \leqslant \left(\tau_1 - \frac{1}{2\omega_0}\right)\alpha_0^+ - (\tau_1 + \tau_3)\,\gamma_0^+ + \left(\tau_3 + \frac{1}{2\omega_0}\right)\beta_0^+$$
$$+ \left(\tau_2 - \frac{1}{2\omega_0}\right)\alpha_0^- - (\tau_2 + \tau_4)\,\gamma_0^- + \left(\tau_4 + \frac{1}{2\omega_0}\right)\beta_0^- \tag{5.8}$$

The conditions stated above for the τ_i's guarantee that the right hand side of (5.8) is nonpositive and the proof is completed. □

Remark. The Discontinuous Galerkin method applied to this problem leads to the upwinding choice $\tau_1 = \tau_4 = 0$, $\tau_2 = \tau_3 = -\frac{1}{\omega_0}$. Another attractive choice that involves no splitting of the fluxes is $\tau_1 = \tau_2 = -\tau_3 = -\tau_4 = \frac{1}{2\omega_0}$.

We turn now to the main purpose of this section, namely the proof of convergence in the nonlinear case of the numerical solution to the correct entropy solution.

We first show that the spectral superviscosity terms do not create any problems: consider a compactly supported (in $[-1, 1]$) test function $\Psi(x, t)$ in $C^m[-1, 1]$, m is to be specified later.

Lemma 5.3. Let $\phi_{N-1}(x, t)$ and $\psi_{N-1}(x, t)$ be the Legendre interpolation polynomials of $\Psi(x, t)$ in the intervals $[-1, 0]$ (with collocation points ξ_j), and $[0, 1]$ (with collocation points η_j), respectively. Then

$$\lim_{N \to \infty} (\phi_{N-1}, SV(u_N)) = 0, \tag{5.9}$$

$$\lim_{N \to \infty} (\psi_{N-1}, SV(v_N)) = 0 \tag{5.10}$$

where the spectral superviscosities are defined in (5.3) and (5.4).

Proof. Since u_N is a polynomial of degree N it can be represented as

$$u_N(x) = \sum_{k=0}^{N} \hat{u}_{k,N} P_k(2x+1)$$

Therefore from (5.3)

$$\begin{aligned} SV(u_N) &= \epsilon \frac{(-1)^{s+1}}{N^{2s-1}} \left[\frac{\partial}{\partial x} x(x+1) \frac{\partial}{\partial x} \right]^s u_N \\ &= -\epsilon \frac{1}{N^{2s-1}} \sum_{k=0}^{N} k^s (k+1)^s \hat{u}_{k,N} P_k(2x+1) \end{aligned}$$

Also the test function ϕ_{N-1} can be represented as

$$\phi_{N-1} = \sum_{k=0}^{N} \hat{\phi}_{k,N} P_k(2x+1)$$

From the orthogonality of the Legendre polynomials it follows that

$$|(\phi_{N-1}, SV(u_N))| = \epsilon \frac{1}{N^{2s-1}} \sum_{k=0}^{N} \hat{\phi}_{k,N} \hat{u}_{k,N} k^s (k+1)^s \left(P_k(2x+1), P_k(2x+1)\right)$$

We can choose m large enough such that

$$|\hat{\phi}_{k,N}\hat{u}_{k,N}| \leqslant \frac{C}{k^3}$$

and since $\hat{u}_k$ and $(P_k(2x+1), P_k(2x+1))$ are bounded, the proof is established. The proof for (5.10) is similar. □

It is self evident that the form of the spectral viscosity SV is crucial. In fact the factor $1-x^2$ is necessary in the proof. Note that

$$(\phi_{N-1}, SV(u_N)) = (SV(\phi_{N-1}), u_N) = (SV(\phi_{N-1}), u_N)_N \tag{5.11}$$

We basically proved that the first argument in the scalar product in the right hand side of (5.11) tends to zero whereas the second argument is bounded. The relation (5.11) is not true for other forms of the spectral viscosity where the factor $1-x^2$ does not appear.

Lemma 5.4. Let τ_i satisfy (5.8), then

$$\left(\phi_{N-1}, \frac{\partial u_N}{\partial t}\right)_N - \left(f(u_N), \frac{\partial \phi_{N-1}}{\partial x}\right)_N + \left(\psi_{N-1}, \frac{\partial v_N}{\partial t}\right)_N - \left(f(v_N), \frac{\partial \psi_{N-1}}{\partial x}\right)_N$$

$$= (\phi_{N-1}, SV(u_N)) + (\psi_{N-1}, SV(v_N)) \tag{5.12}$$

Proof. Taking the scalar product of Eq. (5.1) with ϕ_{N-1} and (5.2) with ψ_{N-1} and denoting by $f(0, t) = f_0$, for all the quantities one gets

$$\left(\phi_{N-1}, \frac{\partial u_N}{\partial t}\right)_N + \left(\phi_{N-1}, \frac{\partial I^I_N f(u_N)}{\partial x}\right)_N$$

$$= \tau_1 \phi_0 (f^+(u_0) - f^+(v_0))\, \omega_0 \tau_2 \phi_0 (f^-(u_0) - f^-(v_0))\, \omega_0 + (\phi_{N-1}, SV(u_N))_N \tag{5.13}$$

$$\left(\psi_{N-1}, \frac{\partial v_N}{\partial t}\right)_N + \left(\psi_{N-1}, \frac{\partial I^{II}_N f(v_N)}{\partial x}\right)_N$$

$$= \tau_3 \psi_0 (f^+(v_0) - f^+(u_0))\, \omega_0 \tau_4 \psi_0 (f^-(v_0) - f^-(u_0))\, \omega_0 + (\psi_{N-1}, SV(v_n))_N) \tag{5.14}$$

Here we used the fact that $\phi(-1, t) = \psi(1, t) = 0$.

Now,

$$\left(\phi_{N-1}, \frac{\partial I^I_N f(u_N)}{\partial x}\right)_N = \left(\phi_{N-1}, \frac{\partial I^I_N f(u_N)}{\partial x}\right)$$
$$= \phi_0 f(u_0) - \left(I^I_N f(u_N), \frac{\partial \phi_{N-1}}{\partial x}\right)$$
$$= \phi_0 f(u_0) - \left(f(u_N), \frac{\partial \phi_{N-1}}{\partial x}\right)_N \tag{5.15}$$

and by the same token

$$\left(\psi_{N-1}, \frac{\partial I^{II}_N f(v_N)}{\partial x}\right)_N = -\phi_0 f(v_0) - \left(f(v_N), \frac{\partial \psi_{N-1}}{\partial x}\right)_N \tag{5.16}$$

Using (5.15)–(5.16) in (5.13)–(5.14) one gets

$$\left(\phi_{N-1}, \frac{\partial u_N}{\partial t}\right) - \left(f(u_N), \frac{\partial \phi_{N-1}}{\partial x}\right)_N + \left(\psi_{N-1}, \frac{\partial v_N}{\partial t}\right) - \left(f(v_N), \frac{\partial \psi_{N-1}}{\partial x}\right)_N$$
$$= (f^+(u_0) - f^+(v_0))[\tau_1\omega_0 - \tau_3\omega_0 - 1]\,\phi_0$$
$$+ (f^-(u_0) - f^-(v_0))[\tau_2\omega_0 - \tau_4\omega_0 - 1]\,\phi_0$$
$$+ (\phi_{N-1}, SV(u_N)) + (\psi_{N-1}, SV(v_N))$$

Taking (5.8) into account, the lemma is proven. □

We integrate now (5.12) with respect to time to get

$$-\int_0^T \left\{\left(u_N, \frac{\partial \phi_{N-1}}{\partial t}\right) - \left(f(u_N), \frac{\partial \phi_{N-1}}{\partial x}\right)_N + \left(v_N, \frac{\partial \psi_{N-1}}{\partial t}\right) - \left(f(v_N), \frac{\partial \psi_{N-1}}{\partial x}\right)_N\right\} dt$$
$$= -(u_N(t=0), \phi_{N-1}(t=0)) - (v_N(t=0), \psi_{N-1}(t=0))$$

We now use Lemma 3.3 in Sec. 3 to convert u_N, v_N to $u_{\Delta x}$ $v_{\Delta x}$, which are defined in (1.2) as the piecewise polynomials having the values of u_N, v_N at the grid points. Combining then with Lemma 5.3 it follows:

Theorem 5.5. Let u_N and v_N be the multi-domain Legendre approximation (5.1)–(5.2) to (1.1). Assume that the functions $u_{\Delta x}$ and $v_{\Delta x}$ defined in (1.2) converge boundedly a.e., then the limit function is a weak solution of (1.1).

ACKNOWLEDGMENTS

Research of the second and third authors is supported by AFOSR Grant F49620-99-1-0077, ARO Grant DAAD19-00-1-0405, DOE Grant DE-FG02-98ER25346, NSF Grant DMS-9804985, NASA Grant NAG1-1145 and NASA contract NAS1-19480 while in residence at ICASE, NASA Langley Research Center, Hampton, Virginia 23681-0001.

REFERENCES

1. Canuto, C., Hussaini, M. Y., Quarternoni, A., and Zang, T. A. (1988). *Spectral Methods in Fluid Dynamics*, Springer-Verlag.
2. Carpenter, M., Gottlieb, D., and Abarbanel, S. (1993). Stable and accurate boundary treatments for compact, high order finite difference schemes. *Appl. Numer. Math.* **12**, 55–87.
3. Carpenter, M., Gottlieb, D., and Abarbanel, S. (1994). Time stable boundary conditions for finite difference schemes solving hyperbolic systems. *J. Comput. Phys.* **11**, 220–236.
4. Carpenter, M., Nordstrom, J., and Gottlieb, D. (1999). A stable and conservative interface treatment of arbitrary spatial accuracy. *J. Comput. Phys.* **48**, 341–365.
5. Cockburn, B., and Shu, C.-W. (1994). Nonlinearly stable compact schemes for shock calculations. *SIAM J. Numer. Anal.* **31**, 607–627.
6. Don, W.-S., and Gottlieb, D. (1994). The Chebyshev–Legendre method: Implementing Legendre methods on Chebyshev points. *SIAM J. Numer. Anal.* **31**, 1519–1534.
7. Funaro, D. (1992). *Polynomial Approximation of Differential Equations*, Lecture Notes in Physics, Vol. 8, Springer-Verlag.
8. Gottlieb, D., Lustman, L., and Orszag, S. (1981). Spectral calculations of one dimensional inviscid flows with shocks. *SIAM J. Sci. Statist. Comput.* **2**, 296–310.
9. Gottlieb, D., and Orszag, S. (1977). *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM-CBMS, Philadelphia.
10. Guo, B.-Y., Ma, H.-P., and Tadmor, E. Spectral vanishing viscosity method for nonlinear conservation laws. *SIAM J. Numer. Anal.*, to appear.
11. Lax, P. D., and Wendroff, B. (1960). Systems of conservation laws, *Comm. Pure Appl. Math.* **13**, 217–237.
12. Lele, S. (1992). Compact finite difference schemes with spectral-like resolution. *J. Comput. Phys.* **103**, 16–42.
13. LeVeque, R. J. (1990). *Numerical Methods for Conservation Laws*, Birkhauser Verlag, Basel.
14. Ma, H. (1998). Chebyshev Legendre spectral viscosity method for nonlinear conservation laws. *SIAM J. Numer. Anal.* **35**, 869–892.
15. Ma, H. (1998). Chebyshev Legendre spectral super viscosity method for conservation laws. *SIAM J. Numer. Anal.* **35**, 893–908.
16. Maday, Y., and Tadmor, E. (1989). Analysis of the spectral vanishing viscosity method for periodic conservation laws. *SIAM J. Numer. Anal.* **26**, 854–870.
17. Maday, Y., Ould Kaber, S., and Tadmor, E. (1993). Legendre pseudospectral viscosity method for nonlinear conservation laws. *SIAM J. Numer. Anal.* **30**, 321–342.
18. Parter, S. (1999). Legendre-Gauss–Lobatto points and weights, *J. Sci. Comput.* **14**, 347–355.
19. Osher, S., and Chakravarthy, S. (1983). Upwind schemes and boundary conditions with applications to Euler equations in general geometries. *J. Comput. Phys.* **50**, 447–481.

20. Tadmor, E. (1989). Convergence of spectral methods for nonlinear conservation laws. *SIAM J. Numer. Anal.* **26**, 30–44.
21. Tadmor, E. (1993). Super viscosity and spectral approximations of nonlinear conservation laws. In Baines, M. J., and Morton, K. W. (eds.), *Numerical Methods for Fluid Dynamics IV*, Proceedings of the 1992 Conference on Numerical Methods for Fluid Dynamics, Clarendon Press, Oxford, pp. 69–82.