Optimal Time-Dependent Lower Bound on Density for Classical Solutions of 1-D Compressible Euler Equations

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ABSTRACT. For the compressible Euler equations, even when initial data are uniformly away from vacuum, solutions can approach vacuum in infinite time. Achieving sharp lower bounds of density is crucial in the study of Euler equations. In this paper, for the initial value problems of isentropic and full Euler equations in one space dimension, assuming the initial density has positive lower bound, we prove that density functions in classical solutions have positive lower bounds in the order of $O(1+t)^{-1}$ and $O(1+t)^{-1-\delta}$ for any $0 < \delta \ll 1$, respectively, where t is time. The orders of these bounds are optimal or almost optimal, respectively. Furthermore, for classical solutions in Eulerian coordinates $(y,t) \in \mathbb{R} \times [0,T)$, we show velocity u satisfies that $u_y(y,t)$ is uniformly bounded from above by a constant independent of T, although $u_y(y,t)$ tends to negative infinity when gradient blowup happens, that is, when shock forms, in finite time.

1. INTRODUCTION

The compressible Euler equations in Lagrangian coordinates in one space dimension are as follows:

$$\tau_t - u_x = 0, \tag{1.1}$$

$$u_t + p_x = 0, \tag{1.2}$$

$$\left(\tfrac{1}{2}u^2 + e\right)_t + (up)_x = 0, \tag{1.3}$$

where ρ is the density, $\tau = \rho^{-1}$ is the specific volume, p is the pressure, u is the velocity, e is the specific internal energy, $t \in \mathbb{R}^+$ is the time, and $x \in \mathbb{R}$ is the spatial coordinate. The compressible Euler equations are widely used, especially in the gas dynamics. The classical solutions for the compressible Euler equations in Lagrangian and Eulerian coordinates are equivalent [10].

For simplicity, in this paper, we only consider the case when the gas is ideal polytropic, in which

$$p = Ke^{S/c_v}\tau^{-\gamma} \quad \text{with adiabatic gas constant } \gamma > 1, \tag{1.4}$$

and

$$e = \frac{p\tau}{\gamma - 1},$$

where S is the entropy and K and c_v are positive constants (cf. [9] or [16]). For C^1 solutions, it follows that (1.3) is equivalent to the conservation of entropy [16]:

$$S_t = 0, \tag{1.5}$$

hence

$$S(x,t) \equiv S(x,0) \doteq S(x).$$

If the entropy is constant, the flow is isentropic, and then (1.1) and (1.2) become a closed system, known as the p-system:

$$\tau_t - u_x = 0, \tag{1.6}$$

$$u_t + p_x = 0, \tag{1.7}$$

with

$$p = K\tau^{-\gamma}, \quad \gamma > 1, \tag{1.8}$$

where, without loss of generality, we still use K to denote the constant in pressure.

We consider here the classical solutions of initial value problems for full Euler equations (1.1), (1.2), (1.4), and (1.5) with initial data $(u(x,0), \tau(x,0), S(x,0))$ and isentropic Euler equations (1.6)–(1.8) with initial data $(u(x,0), \tau(x,0))$. We consider the large data problem, which means that there is no restriction on the size of the solutions.

Toward a large data global existence of BV solutions for the compressible Euler equations, which is a major open problem in the field of hyperbolic conservation laws, one of the main challenges is the possible degeneracy when density approaches zero. In fact, a solution loses its strict hyperbolicity as density approaches zero. (See [1, 3, 14] for analysis and examples showing these difficulties.) Therefore, the sharp information on the time decay of density lower bound is critical in the study of compressible Euler equations. Furthermore, the time-dependent lower bound on density for classical solutions can be used to study the shock formation and life-span of classical solutions.

The study of lower bound of density for classical solutions can be traced back to Riemann's pioneer paper [15] in 1860, in which he considered a special wave interaction between two strong rarefaction waves. By studying Riemann's construction, Lipschitz continuous examples for isentropic Euler equations (1.6)–(1.8) were provided in Section 82 in [9], in which the function $\min_{x\in\mathbb{R}} \rho(x,t)$ was proved to decay to zero in an order of $O(1+t)^{-1}$ as $t \to \infty$, while the initial density is uniformly away from zero.[1] A relative detailed discussion can be found in [5], when the adiabatic constant $\gamma = (2\mathcal{N}+1)/(2\mathcal{N}-1)$ with any positive integer $\mathcal{N}$.

At that time, there were many articles dealing with a time-dependent lower bound on density for general classical solutions of isentropic Euler equations (1.6)–(1.8) under the assumption that initial density is uniformly positive. For rarefactive piecewise Lipschitz-continuous solutions, for any $\gamma > 1$, L. Lin first proved in [13] that the density has lower bound in the order of $O(1+t)^{-1}$ by introducing a polygonal scheme. A breakthrough for general classical solutions happens in a recent paper [4], in which R. Pan, S. Zhu, and the author found a lower bound of density in the order of $O(1+t)^{-4/(3-\gamma)}$ when $1 < \gamma < 3$. Using this result together with Lax's decomposition in [12], Pan, Zhu, and the author proved that gradient blowup of u and/or τ happens in finite time if and only if the initial data are forward or backward compressive somewhere. Next, for a general Lipschitz continuous solution, Pan, Zhu, and the author in [5] improved the lower bound on density from the order of $O(1+t)^{-4/(3-\gamma)}$ to the optimal order $O(1+t)^{-1}$ by introducing a polygonal scheme. The advantage of this method is that it works for not only classical solutions but also Lipschitz continuous solutions. Moreover, the scheme itself is of both analytical and numerical interest. However, the use of a polygonal scheme makes the proof very complex, and the method seems difficult to extend to full Euler equations. Another result on the lower bound of density for classical solution in the order of $O(1+t)^{-1}$ when $\gamma = 3$ was given by A. Bressan[2], where the proof relies on the study of Riccati equations established by Lax in [12].

For non-isentropic full Euler equations, before this paper, the only polynomial order upper bound of τ (lower bound of ρ) for a general classical solution was established by Pan, Zhu, and the author in [4]. More precisely, we showed that density has a lower bound in the order of $O(1+t)^{-4/(3-\gamma)}$ when $1 < \gamma < 3$.

In summary, before this paper, a lower bound of density in optimal order $O(1+t)^{-1}$ was still not available for isentropic Euler equations with $\gamma > 3$ and full nonisentropic Euler equations with $\gamma > 1$.

In this paper, we consider classical solutions of Cauchy problems of both isentropic Euler equations and nonisentropic Euler equations. We assume that

[1] The author thanks Helge Kristian Jenssen who first pointed out this result to him.

[2] The author became aware of this unpublished result through a private communication with A. Bressan.

initial density is uniformly positive, and give a short proof that density has time-dependent lower bound in optimal order $O(1+t)^{-1}$ for isentropic Euler equations (in Theorem 2.1) and in almost-optimal order $O(1+t)^{-1-\delta}$ for any $0<\delta<\frac{1}{3}$ for full Euler equations (in Theorem 2.3) in one space dimension, respectively.

Furthermore, for classical solutions, we prove that $u_x(x,t)$ for p-system and $\rho^\varepsilon u_x$ for any $0<\varepsilon<\frac{1}{4}$ for full Euler equations are uniformly bounded above by a constant, respectively, although they are unbounded from below when gradient blowup happens, that is, when shock forms. In Eulerian coordinates (y,t), we show for full Euler equations that $u_y(y,t)$ is uniformly bounded above by a constant.

The lower bounds of density achieved in this paper can give us more precise estimates of life span of classical solution than those achieved in [4], and motivate us in searching for a lower bound of density for BV solutions including shock waves, which is a major obstacle in establishing large BV existence theory for Euler equations. (Another interesting result on a time-dependent density lower bound for isentropic Euler-Poisson equations can be found in [17] by E. Tadmor and D. Wei.)

The rest of the paper is divided into three sections. In Section 2, we introduce the main results and ideas in this paper. In Section 3, we prove Theorem 2.1 for the p-system. In Section 4, we prove Theorem 2.3 for the full Euler equations.

2. Main Results and Ideas

We first introduce some variables and notation. For Euler equations (1.1)–(1.5), we use variables

$$m \doteq e^{S/(2c_v)} \quad \text{and} \quad \eta \doteq \frac{2\sqrt{K\gamma}}{\gamma-1}\tau^{-(\gamma-1)/2} \tag{2.1}$$

to take the roles of S and τ. We denote the Riemann invariants

$$s \doteq u+m\eta \quad \text{and} \quad r \doteq u-m\eta,$$

respectively, and gradient variables

$$\alpha \doteq u_x+m\eta_x+\frac{\gamma-1}{\gamma}m_x\eta \quad \text{and} \quad \beta \doteq u_x-m\eta_x-\frac{\gamma-1}{\gamma}m_x\eta. \tag{2.2}$$

For the isentropic Euler equations (p-system) (1.6)–(1.8), whose solutions are special solutions of full Euler equations (1.1)–(1.4) when we restrict our consideration to the classical solution, the Riemann invariants are

$$s=u+\eta \quad \text{and} \quad r=u-\eta \tag{2.3}$$

and

$$\alpha=u_x+\eta_x=s_x \quad \text{and} \quad \beta=u_x-\eta_x=r_x. \tag{2.4}$$

The main results in this paper are listed in Theorem 2.1 (for the p-system) and Theorem 2.3 (for full Euler equations).

Theorem 2.1. *Let $(u(x,t),\tau(x,t))$ be a C^1 solution of the isentropic Euler equations (1.6)–(1.8) in the region $(x,t)\in\mathbb{R}\times[0,T)$, where T can be any finite positive constant or infinity. Assume $u(x,0)$, $\tau(x,0)>0$, $\rho(x,0)=1/\tau(x,0)$, $\alpha(x,0)$, and $\beta(x,0)$ are all uniformly bounded, where α and β take the form in* (2.4).

Let M be an upper bound of $\alpha(x,0)$ and $\beta(x,0)$, that is,

$$\max_{x\in\mathbb{R}}\{\alpha(x,0),\beta(x,0)\}<M. \tag{2.5}$$

Then

$$\max_{(x,t)\in\mathbb{R}\times[0,T)}\{\alpha(x,t),\beta(x,t)\}<M. \tag{2.6}$$

This gives

$$\max_{(x,t)\in\mathbb{R}\times[0,T)}\{\tau_t\}=\max_{(x,t)\in\mathbb{R}\times[0,T)}\{u_x\}<M \tag{2.7}$$

by (2.4) *and* (1.6)*. Hence, there exist positive constants M_1 and M_2 independent of T such that*

$$\min_x \rho(x,t)\geq\frac{M_1}{M_2+t}. \tag{2.8}$$

The key step in the proof of Theorem 2.1 is to prove (2.6). In fact, suppose (2.6) is correct; then, by the conservation of mass (1.6) and (2.4), we can easily prove (2.7):

$$\tau_t=u_x=\frac{1}{2}(\alpha+\beta)<M, \tag{2.9}$$

which directly gives (2.8), together with the initial condition. To prove (2.6), we need to study the characteristic decomposition established by Lax in [12]. The key idea is to find an invariant domain on α and β.

One conclusion that we can draw from (2.5)–(2.6) is that although the variables α and β might increase along forward and backward characteristics, respectively, the function $\max_{x\in\mathbb{R}}\{\alpha(x,t),\beta(x,t)\}$ is not increasing with respect to t, which means that the maximum rarefaction of classical solution is not increasing. This result can be easily seen from the fact that (2.6) is still correct if we change 0 in (2.5) into any $t^*\in(0,t)$.

Remark 2.2. Under assumptions in Theorem 2.1, in Eulerian coordinates (y,t), we see that the inequality (2.7) gives that smooth solutions in the region $(y,t)\in\mathbb{R}\times[0,T)$ satisfy

$$\max_{(y,t)\in\mathbb{R}\times[0,T)}\left\{\frac{u_y}{\rho}\right\}<M,$$

where M is the constant given in (2.5), because $\rho u_x(x,t) = u_y(y,t)$. (See [16] for the transformation between Eulerian and Lagrangian coordinates.)

Since ρ is uniformly bounded above, which can be easily proved by the fact that Riemann invariants s and r are initially bounded and are constant along forward and backward characteristics, respectively, we know

$$\max_{(y,t)\in\mathbb{R}\times[0,T)}\{u_y\} < \bar{M},$$

for some constant $\bar{M}$ independent of T.

Then, we consider the full Euler equations.

Theorem 2.3. *Let $(u(x,t),\tau(x,t),S(x))$ be a C^1 solution of full Euler equations (1.1)–(1.4) in the region $(x,t)\in\mathbb{R}\times[0,T)$. Here, T can be any finite positive constant or infinity. Assume that initial data $u(x,0)$, $\tau(x,0)>0$, $\rho(x,0)=1/\tau(x,0)$, $S(x)$, $S'(x)$, $\alpha(x,0)$, and $\beta(x,0)$, are all uniformly bounded, and that total variation of $S(x)$ is finite, where α and β satisfy (2.2). Then, for any*

$$0<\varepsilon<\frac{1}{4},$$

there exists constant N_0 independent of T such that

$$\max_{(x,t)\in\mathbb{R}\times[0,T)}\{\rho^{\varepsilon}\cdot\tau_t\} = \max_{(x,t)\in\mathbb{R}\times[0,T)}\{\rho^{\varepsilon}\cdot u_x\} < N_0, \tag{2.10}$$

and there exist positive constants N_1 and N_2 independent of T such that

$$\min_x \rho(x,t) \ge \left(\frac{N_1}{N_2+t}\right)^{1+\delta}, \tag{2.11}$$

where $\delta=\varepsilon/(1-\varepsilon)>0$.

We first prove a result in Lemma 4.4 by taking a role much as in (2.6) in Theorem 2.8. In fact, we find uniform bounds on gradient variables $\rho^{\varepsilon}\alpha$ and $\rho^{\varepsilon}\beta$, using which we can easily prove (2.10) by (2.2) and (1.1):

$$\rho^{\varepsilon}\tau_t=\rho^{\varepsilon}u_x=\frac{1}{2}(\rho^{\varepsilon}\alpha+\rho^{\varepsilon}\beta)<\text{Constant}.$$

Then, we can show (2.11). The reason why we use $\rho^{\varepsilon}\alpha$ and $\rho^{\varepsilon}\beta$ instead of α and β is to control the lower-order terms in the Riccati equations produced by the varying entropy. The proof of Theorem 2.3 also relies on the uniform constant upper bound of density established in [8] by R. Young, Q. Zhang, and the author for classical solutions when total variation of initial entropy is finite.

Remark 2.4. Under assumptions in Theorem 2.3, in Eulerian coordinates (y,t), the inequality (2.7) gives that the classical solution that is in the region $(y,t)\in\mathbb{R}\times[0,T)$ satisfies

$$\max_{(y,t)\in\mathbb{R}\times[0,T)}\left\{\frac{u_y}{\rho^{1-\varepsilon}}\right\}<N_0.$$

Since ρ is uniformly bounded above under assumptions in Theorem 2.3, we know

$$\max_{(y,t)\in\mathbb{R}\times[0,T)}\{u_y\}<\bar{N}_0,$$

for some constant $\bar{N}_0$ independent of T.

(See [16] for the transformation between Eulerian and Lagrangian coordinates.) Since this result is a local result, we only need to assume that initial entropy is locally BV.

One direct application of Theorem 2.3 is that one can use (2.11) to improve the life-span estimates established in [4] when $1<\gamma<3$, which depends on the time-dependent lower bound of density. We leave this to the reader.

3. Lower Bound of Density for p-system: The Proof of Theorem 2.1

We first introduce the characteristic decompositions for the C^1 solution of p-system. For any classical solution of (1.6)–(1.8), the Riemann invariants s and r in (2.3) are constant along forward and backward characteristics, respectively:

$$\partial_+ s=0\quad\text{and}\quad\partial_- r=0 \tag{3.1}$$

with

$$\partial_+=\partial_t+c\,\partial_x\quad\text{and}\quad\partial_-=\partial_t-c\,\partial_x$$

and wave speed

$$c=\sqrt{-p_\tau}=\sqrt{K\gamma}\tau^{-(\gamma+1)/2}.$$

Furthermore, gradient variables $\alpha=s_x$ and $\beta=r_x$ defined in (2.4) satisfy the following Riccati equations.

Proposition 3.1 ([2]). *The classical solution in* (1.6)–(1.8) *satisfies*

$$\partial_+\alpha=k_1\{\alpha\beta-\alpha^2\}, \tag{3.2}$$

and

$$\partial_-\beta=k_1\{\alpha\beta-\beta^2\}, \tag{3.3}$$

where

$$k_1\doteq\frac{(\gamma+1)K_c}{2(\gamma-1)}\eta^{2/(\gamma-1)},$$

where K_c *is a positive constant given in* (4.1). *The function* $\eta>0$ *is defined in* (2.1).

Equations (3.2) and (3.3) are special examples of Lax's decompositions in [12] for general hyperbolic systems with two unknowns. (See the detailed derivation of (3.2) and (3.3) in [2].)

Remark 3.2. The idea for the proof of (2.6) can be seen from Figure 3.1.

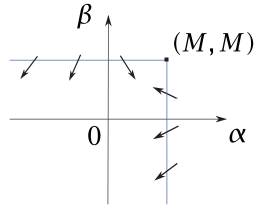

FIGURE 3.1. Here, $\max\{\alpha, \beta\} < M$ is an invariant domain. Note that α (or β) might increase.

Before the proof, we note that ρ, η, c, and k_1 are all bounded above by some constants if assumptions in Theorem 2.1 are satisfied. This can be easily obtained by (3.1), which says that s and r are constant along forward and backward characteristics. As a consequence, ρ, η, c, and function k_1 are all uniformly bounded from above. Denote

$$K_1 \doteq \max_{(x,t)\in\mathbb{R}\times[0,T)} k_1(x,t), \tag{3.4}$$

where K_1 is a constant only depending on γ and the initial condition.

Proof of Theorem 2.1. We first prove (2.6) by contradiction. Without loss of generality, assume that $\alpha(x_0, t_0) = M$ at some point (x_0, t_0). See Figure 3.2.

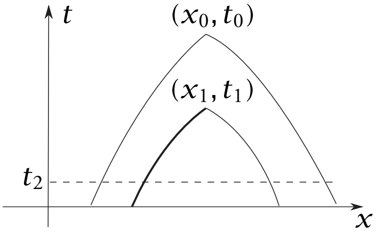

FIGURE 3.2. Proof of Theorems 2.1 and 2.3.

Because wave speed c is bounded above, we can find the characteristic triangle with vertex (x_0, t_0) and lower boundary on the initial line $t = 0$, denoted by Ω.

Then, we can find the first time t_1 such that $\alpha = M$ or $\beta = M$ in Ω. More precisely,

$$\max_{(x,t)\in\Omega,\, t<t_1} (\alpha(x,t), \beta(x,t)) < M,$$

and $\alpha(x_1,t_1) = M$ or/and $\beta(x_1,t_1) = M$ for some $(x_1,t_1) \in \Omega$. Without loss of generality, still assume $\alpha(x_1,t_1) = M$. The proof for another case is entirely the same. Let us denote the characteristic triangle with vertex (x_1,t_1) as $\Omega_1 \in \Omega$; then,

$$\max_{(x,t)\in\Omega_1,\, t<t_1} (\alpha(x,t), \beta(x,t)) < M, \tag{3.5}$$

and $\alpha(x_1,t_1) = M$. By the continuity of α, we could find a time $t_2 \in [0,t_1)$ such that,

$$\alpha(x,t) > 0, \quad \text{for any } (x,t) \in \Omega_1 \text{ and } t \geq t_2. \tag{3.6}$$

Next, we derive a contradiction. By (3.2), (3.4), and (3.5)–(3.6), along the forward characteristic segment through (x_1,t_1) when $t_2 \leq t < t_1$, we have

$$\partial_+\alpha = k_1\{\alpha\beta - \alpha^2\} \leq K_1\{M\alpha - \alpha^2\},$$

which gives, through integration along characteristic,

$$\begin{aligned} &\frac{\mathrm{d}\alpha}{(M-\alpha)\alpha} \leq K_1\,\mathrm{d}t \\ &\qquad \Longrightarrow \frac{1}{M}\ln\frac{\alpha(t)}{M-\alpha(t)} \leq \frac{1}{M}\ln\frac{\alpha(t_2)}{M-\alpha(t_2)} + K_1(t-t_2). \end{aligned}$$

As $t \to t_1-$, the left-hand side approaches infinity while the right-hand side approaches a finite number, which gives a contradiction. Hence, we prove that (2.6) is correct, that is, α and β are uniformly bounded above. Then, by the conservation of mass (1.6) and (2.4), we have (2.9), and then (2.7), which directly gives (2.8). Hence, we complete the proof of Theorem 2.1. ☐

Remark 3.3. Theorem 2.1 can be extended to the case with general pressure law $p = p(\tau)$ with $p_\tau < 0$, $p_{\tau\tau} > 0$, and some other suitable conditions on p. We leave this to the reader, referring him to [6] for the Riccati equations and the definitions of α and β. For full Euler equations, the extension of Theorem 2.3 to general pressure law is still not available because the current result on the uniform upper bound of density is only available for γ-law pressure.

4. Full Compressible Euler Equations

4.1. Equations and coordinates. We first introduce some notation and existing equations for C^1 solutions of full Euler equations (1.1)–(1.4). Recall we use new variables m and η to take the roles of S and τ, respectively:

$$m = e^{S/(2c_v)} \tag{4.1}$$

and

$$\eta = \frac{2\sqrt{K\gamma}}{\gamma - 1}\tau^{-(\gamma-1)/2}. \tag{4.2}$$

Without confusion, we still use c to denote the nonlinear Lagrangian wave speed for full Euler equations, where

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma}\tau^{-(\gamma+1)/2}e^{S/(2c_v)}.$$

The forward and backward characteristics are described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -c,$$

and we denote the corresponding directional derivatives along these characteristics by

$$\partial_+ := \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \quad \text{and} \quad \partial_- := \frac{\partial}{\partial t} - c\frac{\partial}{\partial x},$$

respectively.

It follows that

$$\begin{aligned} \tau &= K_\tau \eta^{-2/(\gamma-1)}, \\ p &= K_p m^2 \eta^{2\gamma/(\gamma-1)}, \\ c &= c(\eta, m) = K_c m \eta^{(\gamma+1)/(\gamma-1)}, \end{aligned}$$

with positive constants

$$K_\tau := \left(\frac{2\sqrt{K\gamma}}{\gamma - 1}\right)^{2/(\gamma-1)},$$

$$K_p := K K_\tau^{-\gamma}, \quad K_c := \sqrt{K\gamma} K_\tau^{-(\gamma+1)/2}, \tag{4.3}$$

so that also

$$K_p = \frac{\gamma-1}{2\gamma}K_c \quad \text{and} \quad K_\tau K_c = \frac{\gamma - 1}{2}.$$

In these coordinates, for C^1 solutions, equations (1.1)–(1.4) are equivalent to

$$\eta_t + \frac{c}{m}u_x = 0, \tag{4.4}$$

$$u_t + mc\eta_x + 2\frac{p}{m}m_x = 0, \tag{4.5}$$

$$m_t = 0, \tag{4.6}$$

where the last equation comes from (1.5), which is equivalent to (1.3) (cf. [16]). Note that, while the solution remains C^1, $m = m(x)$ is given by the initial data and can be regarded as a stationary quantity.

Recall that we denote the Riemann invariants by

$$r := u - m\eta \quad \text{and} \quad s := u + m\eta.$$

Different from the isentropic case (m constant), for general non-isentropic flow, s and r vary along characteristics. Also recall that we denote gradient variables

$$\alpha = u_x + m\eta_x + \frac{\gamma-1}{\gamma} m_x \eta, \tag{4.7}$$

$$\beta = u_x - m\eta_x - \frac{\gamma-1}{\gamma} m_x \eta, \tag{4.8}$$

which satisfy the following Riccati equations. (See the detailed derivation in [2].)

Proposition 4.1 ([2]). *The classical solutions for* (1.1)–(1.3) *satisfy*

$$\partial_+ \alpha = k_1\{k_2(3\alpha + \beta) + \alpha\beta - \alpha^2\}, \tag{4.9}$$

and

$$\partial_- \beta = k_1\{-k_2(\alpha + 3\beta) + \alpha\beta - \beta^2\}, \tag{4.10}$$

where

$$k_1 = \frac{(\gamma+1)K_c}{2(\gamma-1)} \eta^{2/(\gamma-1)}, \quad k_2 = \frac{\gamma-1}{\gamma(\gamma+1)} \eta m_x.$$

Proposition 3.1 is in fact a corollary of Proposition 4.1 for the isentropic case in which $m_x \equiv 0$.

4.2. Uniform upper bound on density. In this part, we review a result on the uniform upper bounds of $|u|$ and ρ established by the author, R. Young, and Q. Zhang in [7], for later references.

In this section, we always assume all initial conditions in Theorem 2.3 are satisfied. This means that

$$V := \frac{1}{2c_v} \int_{-\infty}^{+\infty} |S'(x)| \, \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{|m'(x)|}{m(x)} \, \mathrm{d}x < \infty,$$

while also, by (4.1), we have $0 < M_L < m(\cdot) < M_U$ for some constants M_L and M_U. Also, there exist positive constants M_s and M_r such that, in the initial data, $|s(\cdot, 0)| < M_s$ and $|r(\cdot, 0)| < M_r$.

In the following proposition established in [7], $|u|$ and ρ are shown to be uniformly bounded above.

Proposition 4.2 ([7]). *Assume all initial conditions in Theorem 2.3 are satisfied, and assume system* (1.1)–(1.4) *has a C^1 solution when $t \in [0, T)$. Then, one has the uniform bounds*

$$|u(x,t)| \le \frac{L_1 + L_2}{2} {M_U}^{1/(2\gamma)} \quad \text{and} \quad \eta(x,t) \le \frac{L_1 + L_2}{2} {M_L}^{1/(2\gamma)-1},$$

where

$$L_1 := M_s + \bar{V} M_r + \bar{V}(\bar{V} M_s + \bar{V}^2 M_r) e^{\bar{V}^2},$$
$$L_2 := M_r + \bar{V} M_s + \bar{V}(\bar{V} M_r + \bar{V}^2 M_s) e^{\bar{V}^2},$$

and

$$\bar{V} := \frac{V}{2\gamma}.$$

Constants L_1 and L_2 both clearly depend only on the initial data and γ. Here, T can be any positive number or infinity; the bounds are independent of T.

4.3. Proof of Theorem 2.3. Similar to Theorem 2.1 for the p-system, the key idea is still to get the uniform upper bound of some gradient variables measuring rarefaction.

However, we cannot directly get the uniform upper bound of α and β. In fact, in comparison to (3.2)–(3.3), equations (4.9)–(4.10) include some first-order terms in the right-hand side. In order to cope with these, we introduce some new gradient variables

$$\alpha_\varepsilon = \eta^{2\varepsilon/(\gamma-1)} \alpha \quad \text{and} \quad \beta_\varepsilon = \eta^{2\varepsilon/(\gamma-1)} \beta. \tag{4.11}$$

Using (4.4), we have

$$\begin{aligned} \partial_+ \eta = \eta_t + c\eta_x &= -\frac{c}{m} u_x + c\eta_x \\ &= -K_c \eta^{(\gamma+1)/(\gamma-1)} \beta - \frac{\gamma-1}{\gamma} K_c \eta^{2\gamma/(\gamma-1)} m_x \end{aligned}$$

and

$$\begin{aligned} \partial_- \eta = \eta_t - c\eta_x &= -\frac{c}{m} u_x - c\eta_x \\ &= -K_c \eta^{(\gamma+1)/(\gamma-1)} \alpha + \frac{\gamma-1}{\gamma} K_c \eta^{2\gamma/(\gamma-1)} m_x. \end{aligned}$$

Then, it is easy to prove the next lemma by Proposition 4.1.

Lemma 4.3. *The classical solutions in* (1.1)–(1.3) *satisfy*

$$\partial_+ \alpha_\varepsilon = k_{1\varepsilon} \left\{ k_{2\varepsilon} (3\alpha_\varepsilon - 4\varepsilon\alpha_\varepsilon + \beta_\varepsilon) + \left(1 - \frac{4\varepsilon}{\gamma+1}\right) \alpha_\varepsilon \beta_\varepsilon - \alpha_\varepsilon^2 \right\} \tag{4.12}$$

and

$$\partial_- \beta_\varepsilon = k_{1\varepsilon}\left\{-k_{2\varepsilon}(\alpha_\varepsilon + 3\beta_\varepsilon - 4\varepsilon\beta_\varepsilon) + \left(1 - \frac{4\varepsilon}{\gamma+1}\right)\alpha_\varepsilon\beta_\varepsilon - \beta_\varepsilon^2\right\},$$

where

$$k_{1\varepsilon} = \frac{(\gamma+1)K_c}{2(\gamma-1)}\eta^{(2/(\gamma-1))(1-\varepsilon)},$$
$$k_{2\varepsilon} = \frac{\gamma-1}{\gamma(\gamma+1)}\eta^{1+(2/\gamma-1)\varepsilon}m_x$$

and

$$0 < \varepsilon < \frac{1}{4}. \tag{4.13}$$

Note that, for any C^1 solutions in $(x,t) \in \mathbb{R} \times [0,T)$ satisfying initial conditions in Theorem 2.3, using Proposition 4.2, for any ε satisfying (4.13), we know $|k_{1\varepsilon}(x,t)|$ and $|k_{2\varepsilon}(x,t)|$ are both uniformly bounded above:

$$|k_{1\varepsilon}(x,t)| < \hat{K}_1 \quad \text{and} \quad |k_{2\varepsilon}(x,t)| < \hat{K}_2, \tag{4.14}$$

where constants $\hat{K}_1$ and $\hat{K}_2$ only depend on initial conditions and γ, but are independent of ε.

Next, we give the key lemma, which will be proved later.

Lemma 4.4. *Suppose the initial conditions in Theorem 2.3 are satisfied. For any ε satisfying (4.13), let N be an upper bound of $\alpha_\varepsilon(x,0)$ and $\beta_\varepsilon(x,0)$, that is,*

$$\max_{x\in\mathbb{R}}\{\alpha_\varepsilon(x,0), \beta_\varepsilon(x,0)\} < N,$$

where the constant N also satisfies

$$N > \max\left\{\frac{4(\gamma+1)\hat{K}_2}{\varepsilon}, \frac{2\hat{K}_2}{1-4\varepsilon/(\gamma+1)}\right\}. \tag{4.15}$$

Then,

$$\max_{(x,t)\in\mathbb{R}\times[0,T)}\{\alpha_\varepsilon(x,t), \beta_\varepsilon(x,t)\} < N.$$

Proof of Theorem 2.3. We only need show Lemma 4.4. In fact, if Lemma 4.4 is proved, then by the conservation of mass (1.1) and definitions of α_ε and β_ε in (4.11) and (4.7)–(4.8), we have

$$\eta^{2\varepsilon/(\gamma-1)}\tau_t = \eta^{2\varepsilon/(\gamma-1)}u_x = \frac{1}{2}(\alpha_\varepsilon + \beta_\varepsilon) < N,$$

which gives that, by (4.2), $\tau = 1/\rho$, and the fact that the initial density has positive lower bound, there exist positive constants N_1 and N_2 such that

$$\rho > \left(\frac{N_1}{N_2 + t}\right)^{1+\delta} \quad \text{where } \delta = \frac{\varepsilon}{1-\varepsilon}.$$

Then, it is easy to see that all results in Theorem 2.3 are correct.

We now prove Lemma 4.4 by contradiction. We still use Figure 3.2. Without loss of generality, assume that $\alpha_\varepsilon(x_0, t_0) = N$, at some point (x_0, t_0).

Because wave speed c is bounded above, we can find the characteristic triangle with vertex (x_0, t_0) and lower boundary on the initial line $t = 0$, denoted by Ω.

Then, we can find the first time t_1 such that $\alpha_\varepsilon = N$ or $\beta_\varepsilon = N$ in Ω. More precisely,

$$\max_{(x,t)\in\Omega,\, t<t_1} (\alpha_\varepsilon(x,t), \beta_\varepsilon(x,t)) < N,$$

and $\alpha_\varepsilon(x_1, t_1) = N$ and/or $\beta_\varepsilon(x_1, t_1) = N$ for some $(x_1, t_1) \in \Omega$. Without loss of generality, still assume $\alpha_\varepsilon(x_1, t_1) = N$. The proof for the other case is entirely the same. Let us denote the characteristic triangle with vertex (x_1, t_1) as $\Omega_1 \in \Omega$; then,

$$\max_{(x,t)\in\Omega_1,\, t<t_1} (\alpha_\varepsilon(x,t), \beta_\varepsilon(x,t)) < N,$$

and $\alpha_\varepsilon(x_1, t_1) = N$.

We then divide the problem into two cases:

(I) $N \geq \beta_\varepsilon(x_1, t_1) > -N/2$;

(II) $\beta_\varepsilon(x_1, t_1) \leq -N/2$.

In case (I), by the continuity of α_ε and β_ε and our construction, we can find a time $t_2 \in [0, t_1)$ such that

$$\frac{N}{2} < \alpha_\varepsilon(x,t) < N \text{ and } |\beta_\varepsilon| < N, \quad \forall\, (x,t) \in \Omega_1 \text{ and } t_2 \leq t < t_1. \tag{4.16}$$

Then, using (4.12), (4.14), (4.15), and (4.16), along the forward characteristic segment through (x_1, t_1), when $t_2 \leq t < t_1$, we have

$$\partial_+ \alpha_\varepsilon \leq k_{1\varepsilon}\left(1 - \frac{4\varepsilon}{\gamma+1}\right)(\alpha_\varepsilon\beta_\varepsilon - \alpha_\varepsilon^2) \leq \breve{K}_1(N\alpha_\varepsilon - \alpha_\varepsilon^2)$$

with

$$\breve{K}_1 \doteq \hat{K}_1\left(1 - \frac{4\varepsilon}{\gamma+1}\right),$$

which gives, through integration along characteristic,

$$\frac{\mathrm{d}\alpha_\varepsilon}{(N-\alpha_\varepsilon)\alpha_\varepsilon} \le \tilde{K}_1\,\mathrm{d}t$$
$$\Longrightarrow \frac{1}{N}\ln\frac{\alpha_\varepsilon(t)}{N-\alpha_\varepsilon(t)} \le \frac{1}{N}\ln\frac{\alpha_\varepsilon(t_2)}{N-\alpha_\varepsilon(t_2)} + \tilde{K}_1(t-t_2).$$

As $t \to t_1-$, the left-hand side approaches infinity while the right-hand side approaches a finite number, which gives a contradiction.

In case (II), by the continuity of α_ε, we could find a time $t_3 \in [0, t_1)$ such that

$$\frac{N}{2} < \alpha_\varepsilon(x,t) < N \quad \text{and} \quad \beta_\varepsilon(x,t) < -\frac{N}{4}, \tag{4.17}$$

for any $(x,t) \in \Omega_1$ and $t_3 \le t < t_1$, which gives, by (4.15),

$$\left(k_{2\varepsilon} + \left(1 - \frac{4\varepsilon}{\gamma+1}\right)\alpha_\varepsilon\right)\beta_\varepsilon < 0.$$

Hence, by (4.15), (4.13), and (4.17), we have

$$\partial_+\alpha_\varepsilon < k_{1\varepsilon}\{k_{2\varepsilon}(3-4\varepsilon)\alpha_\varepsilon - \alpha_\varepsilon^2\} < 0.$$

As a consequence, α_ε decreases on t along the forward characteristic line through (x_1,t_1), when $t_3 \le t < t_1$, which contradicts the idea that $\alpha_\varepsilon(x_1,t_1) = N$ while $\alpha_\varepsilon(x,t) < N$ when $(x,t) \in \Omega_1$ and $t_3 \le t < t_1$. Hence, Lemma 4.4 is proved, and this completes the proof of Theorem 2.3. □

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