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RECOVERY OF EDGES FROM SPECTRAL DATA WITH NOISE—A NEW PERSPECTIVE*

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Abstract. We consider the problem of detecting edges—jump discontinuities in piecewise smooth functions from their N-degree spectral content, which is assumed to be corrupted by noise. There are three scales involved: the "smoothness" scale of order $1/N$, the noise scale of order $\sqrt{\eta}$, and the $\mathcal{O}(1)$ scale of the jump discontinuities. We use concentration factors which are adjusted to the standard deviation of the noise $\sqrt{\eta} \gg 1/N$ in order to detect the underlying $\mathcal{O}(1)$-edges, which are separated from the noise scale $\sqrt{\eta} \ll 1$.

Key words. piecewise smoothness, edge detection, noisy data, concentration kernels, constrained optimization, separation of scales

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1. Introduction and statement of main results.

We consider the detection of jump discontinuities in piecewise smooth data from its Fourier projection

$$S_N f(x) = \sum_{|k| \le N} \widehat{f}(k) e^{ikx}, \qquad \widehat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{-i\xi x}.$$

Our approach for edge detection—we use the terminology of edges or jump discontinuities interchangeably—is based on the technique of *concentration kernels* advocated in [3, 4] and the closely related techniques described in [2]. The technique presented makes use of the fact that, if f is discontinuous, its Fourier coefficients contain a slowly decaying part associated with the jumps of f. This part decays much more slowly than the rapidly decaying smooth part of f. For example, if f is smooth except for a single jump discontinuity at $x = z$ of size $[f](z) := f(z+) - f(z-)$, the jump discontinuity is associated with slowly decaying Fourier coefficients (here and below we use $X \lesssim Y$ to denote the estimate $X \le CY$, where C is a constant which is independent of the variables $k, N, \dots$)

$$\widehat{f}(k) = [f](z) \frac{e^{-ikz}}{2\pi i k} + \widehat{s}(k), \qquad |\widehat{s}(k)| \lesssim \frac{1}{|k|^2}, \quad [f](x) := f(x+) - f(x-),$$

where $\widehat{s}(k)$ are the Fourier coefficients of s, the smooth part of f; the smoother s is, the faster is the decay of $\widehat{s}(k)$. Concentration kernels succeed in separating the two sets of coefficients. To this end one computes

$$\mathcal{K}_N^\sigma * (S_N f)(x) = \pi i \sum_{|k| \le N} \mathrm{sgn}(k) \sigma_N\left(\frac{|k|}{N}\right) \widehat{f}(k) e^{ikx}, \qquad \mathrm{sgn}(k) := \begin{cases} k/|k| & k \neq 0, \\ 0 & k = 0. \end{cases}$$

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Here, $\sigma_N(\xi)$ can be drawn from a large family of properly normalized concentration factors that is at our disposal. The resulting function tends to zero in regions in which f is smooth and tends to the amplitude of the jumps at points where the function has jump discontinuities, e.g., [8, section 4],

$$\mathcal{K}_N^\sigma * (S_N f)(x) = [f](x) + \mathcal{O}\left(\frac{\log N}{N}\right).$$

Thus, jump discontinuities, or edges, are detected by *separation of scales*: the quantity on the right is of a vanishing order $\lesssim (\log N)/N$, in regions of smoothness of f, and it is $\gg (\log N)/N$, in the neighborhood of jump discontinuities of f.

In this paper we utilize concentration kernels to detect edges from spectral information which is corrupted by *white noise*. In this context we observe that there are three scales involved—edges of order $\mathcal{O}(1)$, noise with variance $\mathcal{O}(\eta)$, and the smooth part of f which is resolved within order $\mathcal{O}(1/N)$ or smaller. Here, we can separate the noisy part $n(x)$ from the smooth part

$$s(x) \to s(x) + n(x), \qquad E(|\widehat{n}(k)|^2) = \eta;$$

if $\sqrt{\eta} \lesssim 1/N$, then the noisy part could be identified with (or below) the $\mathcal{O}(1/N)$-variation of the smooth part of f. In this case, there are essentially two scales, and edges can be detected using the usual framework of concentration kernels advocated in [3, 4]. Thus, our main focus in this paper is when the smoothness scale is dominated by the scale of the noise which is still well separated from the $\mathcal{O}(1)$-scale of the jumps,[1]

$$\frac{1}{N} \lesssim \sqrt{\eta} \ll 1.$$

The spectral information is now corrupted by white noise, affecting both low and high frequencies:

$$\widehat{f}(k) = [f](z)\frac{e^{-ikz}}{2\pi ik} + \widehat{s}(k) + \widehat{n}(k).$$

In order to separate edges from the noisy scale, the edge detector $K_N^\sigma f(x)$ must be properly adapted to the presence of white noise. We show how to design edge detectors that optimally compensate for noise and for the effects of the smooth part of the signal.

The paper is organized as follows. In section 2 we discuss the general framework of edge detection based on concentration kernels. We revisit the results of [4, 8], providing a simpler proof for the concentration property for a large family of concentration factors. In particular, we trace the precise dependence of the error on the regularity of the associated concentration factor. This will prove useful when we deal with noisy data in section 3. Here, we introduce our new perspective, where concentration factors are derived by a constrained minimization while taking into account the two main ingredients of our data—jump discontinuities and the noisy parts of the data. Numerical results are demonstrated in section 4. In section 5 we extend our construction of concentration factors to include the *three* ingredients of the data—taking into account the smooth part of the data in addition to the edges and noisy parts; numerical results are presented in section 6.

[1]Here and below, $X \ll Y$ indicates that X is an "order of magnitude smaller" than Y, so that $\lim_{N\to\infty} X/Y \to 0$, $X \sim Y$, indicates that X and Y are "of the same order of magnitude," namely, $X \lesssim Y \lesssim X$, and $X \approx Y$ indicates that $\lim_{N\to\infty} |X - Y| = 0$.

2. Detection of edges—concentration kernels. Consider an f which is *piecewise smooth* in the sense that it is sufficiently smooth except for finitely many jump discontinuities, say, at $z_1 < z_2, \ldots < z_m$, where

$$[f](z_j) := f(z_j+) - f(z_j-) \neq 0, \quad j = 1, 2, \ldots m.$$

Given the Fourier coefficients $\{\widehat{f}(k)\}_{k=-N}^{N}$, we are interested in detecting the edges of the underlying piecewise smooth f, namely, to detect their location $z_1, \ldots, z_m$ and their amplitudes $[f](z_1), \ldots, [f](z_m)$.

We utilize edge detection based on *concentration kernels*

$$\mathcal{K}_N^\sigma(y) := -\sum_{k=1}^{N} \sigma_N\left(\frac{k}{N}\right) \sin ky, \qquad \frac{\sigma_N(\xi)}{\xi} \in C^1[0,1]. \tag{2.1a}$$

We shall need the kernel $-\mathcal{K}_N^\sigma$ to have (approximately) unit mass

$$\int_0^\pi \mathcal{K}_N^\sigma(y)dy \approx -1.$$

To this end, we require that $\sigma(\xi) \equiv \sigma_N(\xi)$ be a properly normalized concentration factor so that $\sigma(\xi)$ satisfies

$$\int_0^1 \frac{\sigma(\xi)}{\xi} d\xi = 1. \tag{2.1b}$$

Indeed, the rectangular quadrature rule yields

$$\begin{aligned}\int_0^\pi \mathcal{K}_N^\sigma(y)dy &= \sum_{k=1}^{N} \sigma\left(\frac{k}{N}\right) \frac{(-1)^k - 1}{k} \\ &= -\sum_{k \text{ odd}\geq 1}^{N} \frac{\sigma(\xi_k)}{\xi_k}\frac{2}{N} = -\int_0^1 \frac{\sigma(\xi)}{\xi} d\xi + \delta_N, \quad \xi_k := \frac{k}{N},\end{aligned}$$

with an error estimate, e.g., [6, 1],

$$\left|\int_0^\pi \mathcal{K}_N^\sigma(y)dy + 1\right| \lesssim \delta_N, \qquad \delta_N := \frac{1}{N}\left\|\frac{\sigma_N(\xi)}{\xi}\right\|_{BV}. \tag{2.2}$$

We set[2]

$$K_N^\sigma f(x) := \mathcal{K}_N^\sigma * f(x) = \pi i \sum_{|k|\leq N} \operatorname{sgn}(k)\sigma_N\left(\frac{|k|}{N}\right)\widehat{f}(k)e^{ikx}.$$

Our purpose is to choose the *concentration factors* $\sigma_N(|k|/N)$ such that $K_N^\sigma f(x) \approx [f](x)$. Thus, $K_N^\sigma f(x)$ will detect the edges $[f](z_j)$, $j = 1, \ldots, m$, by concentrating near these $\mathcal{O}(1)$-edges, which are to be separated from a much smaller scale of order $K_N^\sigma f(x) \approx 0$ in regions of smoothness. In the following theorem we present a rather

[2]We observe that $K_N^\sigma f$ is the *operator* associated with, but otherwise different from, the concentration kernel $\mathcal{K}_N^\sigma(x)$.

general framework for edge detectors based on concentration factors. In particular, we track the precise dependence of the scale separation on the behavior of σ.

THEOREM 2.1 (concentration kernels). *Assume that $f(\cdot)$ is piecewise smooth such that the first variation of f is of locally bounded variation (BV)*

$$\omega_x(y) \equiv \omega_f(y,x) := \frac{f(x+y)-f(x-y)-[f](x)}{y} \in BV[-\pi,\pi]. \tag{2.3}$$

Let $\mathcal{K}_N^\sigma(x)$ be an admissible concentration kernel (2.1) *such that*

$$\delta_N := \frac{1}{N}\left\|\frac{\sigma_N(\xi)}{\xi}\right\|_{BV} \ll 1, \tag{2.4a}$$

$$\varepsilon_1(N) := \frac{1}{N}|\sigma_N(1)| \ll 1, \tag{2.4b}$$

$$\varepsilon_2(N) := \frac{1}{N}\int_{\xi\sim\frac{1}{N}}^1 \frac{|\sigma_N'(\xi)|}{\xi}d\xi \ll 1, \tag{2.4c}$$

$$\varepsilon_3(N) := \left|\sigma_N\left(\frac{1}{N}\right)\right| \ll 1. \tag{2.4d}$$

Set

$$\varepsilon_N := \varepsilon_1(N)+\varepsilon_2(N)+\varepsilon_3(N). \tag{2.5}$$

Then, the conjugate sum

$$K_N^\sigma f(x) = \pi i \sum_{|k|\le N} \operatorname{sgn}(k)\sigma_N\left(\frac{|k|}{N}\right)\widehat{f}(k)e^{ikx} \tag{2.6}$$

satisfies the concentration property

$$|K_N^\sigma f(x) - [f](x)| \lesssim \delta_N + \varepsilon_N\|\omega_x(\cdot)\|_{BV}. \tag{2.7}$$

Proof. We simplify the proof in [4, 8]. The key to the proof is to observe that $\mathcal{K}_N^\sigma$ is an appropriately normalized *derivative* of the delta function; in particular, since $\mathcal{K}_N^\sigma(\cdot)$ is odd

$$\begin{aligned}\mathcal{K}_N^\sigma * f(x) &= -\int_0^\pi K_N^\sigma(y)\Big(f(x+y)-f(x-y)\Big)dy \\ &= -\int_0^\pi \mathcal{K}_N^\sigma(y)\Big(f(x+y)-f(x-y)-[f](x)\Big)dy - [f](x)\times\int_0^\pi \mathcal{K}_N^\sigma(y)dy.\end{aligned}$$

Since σ is assumed normalized, the error estimate (2.2) tells us[3] that

$$\int_0^\pi \mathcal{K}_N^\sigma(y)dy = -1+\mathcal{O}(\delta_N), \qquad \delta_N = \frac{1}{N}\left\|\frac{\sigma_N(\xi)}{\xi}\right\|_{BV},$$

and we end up with the error estimate

$$\big|K_N^\sigma f(x) - [f](x)\big| \lesssim \left|\int_0^\pi \mathcal{K}_N^\sigma(y)y\omega_x(y)dy\right| + \mathcal{O}(\delta_N). \tag{2.8}$$

[3] In our case, the errors for the trapezoidal and rectangular rules provided, respectively, in [1] and [6] coincide modulo the $\mathcal{O}(\varepsilon_1(N)+\varepsilon_3(N))$ terms.

To upper bound the expression on the right, we use the following identity, which is derived by a straightforward summation by parts:

$$2\sin(y/2)\mathcal{K}_N^\sigma(y) \equiv \overbrace{\sigma_N(1)\cos\left(N+\frac{1}{2}\right)y}^{I_1(y)} + \sum_{k=1}^{N-1}\overbrace{\Big(\sigma_N(\xi_k)-\sigma_N(\xi_{k+1})\Big)\cos\left(k+\frac{1}{2}\right)y}^{I_{2k}(y)} - \overbrace{\sigma_N(\xi_1)\cos\left(\frac{y}{2}\right)}^{I_3(y)}.$$

The usual cancellation estimate $|\int_0^\pi \cos((k+\frac{1}{2})y)w(y)dy| \lesssim \|w\|_{BV}/|k+\frac{1}{2}|$ implies that

$$\left|\int_0^\pi I_1(y)\frac{y/2}{\sin(y/2)}\omega_x(y)dy\right| \lesssim |\sigma_N(1)|\cdot\frac{1}{N}\|\omega_x(\cdot)\|_{BV} \le \varepsilon_1(N)\|\omega_x(\cdot)\|_{BV},$$

$$\left|\int_0^\pi I_{2k}(y)\frac{y/2}{\sin(y/2)}\omega_x(y)dy\right| \lesssim |\sigma(\xi_{k+1})-\sigma(\xi_k)|\cdot\frac{1}{k+\frac{1}{2}}\|\omega_x(\cdot)\|_{BV},$$

$$\left|\int_0^\pi I_3(y)\frac{y/2}{\sin(y/2)}\omega_x(y)dy\right| \lesssim \varepsilon_3(N)\cdot\|\omega_x(\cdot)\|_{L^\infty}.$$

We conclude that

$$\begin{aligned}\left|\int_0^\pi \mathcal{K}_N^\sigma(y)y\omega_x(y)dy\right| &= \left|\int_0^\pi\left(I_1(y)+\sum_k I_{2k}(y)+I_3(y)\right)\frac{y/2}{\sin(y/2)}\omega_x(y)dy\right| \\ &\lesssim \left(\varepsilon_1(N)+\sum_{k=1}^{N-1}\frac{|\sigma(\xi_{k+1})-\sigma(\xi_k)|}{\xi_{k+\frac{1}{2}}}\frac{1}{N}+\varepsilon_3(N)\right)\left\|\frac{y/2}{\sin(y/2)}\omega_x(y)\right\|_{BV} \\ &\lesssim \Big(\varepsilon_1(N)+\varepsilon_2(N)+\varepsilon_3(N)\Big)\|\omega_x(\cdot)\|_{BV}.\end{aligned} \tag{2.9}$$

The result (2.7) follows from (2.8) and (2.9). ∎

As an example, we consider the noise-free case with a concentration factor for which

$$\frac{\sigma_N(\xi)}{\xi} \in C^1. \tag{2.10}$$

Clearly $\delta_N \lesssim 1/N$, and, since σ is bounded, $\varepsilon_1(N) \lesssim 1/N$. Moreover, since $\sigma_N'(\xi)$ is bounded,

$$\varepsilon_2(N) \lesssim \frac{1}{N}\int_{\xi\sim\frac{1}{N}}^1 \frac{1}{\xi}d\xi \lesssim \frac{\log(N)}{N}.$$

Finally, since $|\sigma_N(\xi)| \lesssim \xi$, then $\varepsilon_3(N) \lesssim 1/N$. Thus, Theorem 2.1 applies with $\delta_N = 1/N$ and $\varepsilon_N = \log(N)/N$, which we summarize in the following corollary.

Corollary 2.2 (see [8, Theorem 4.1]). *Assume that $f(\cdot)$ is a piecewise smooth function whose first variation is BV bounded, so that (2.3) holds. Let $\mathcal{K}_N^\sigma(x)$ be a* *normalized concentration kernel* (2.1) *with* $\frac{\sigma(\xi)}{\xi} \in C^1$. *Then* $K_N^\sigma f(x)$ *satisfies the concentration property*

$$|K_N^\sigma f(x) - [f](x)| \lesssim \delta_N + \frac{\log N}{N}\|\omega_x(\cdot)\|_{BV}, \qquad \omega_x(\cdot) \equiv \omega_f(\cdot, x). \tag{2.11}$$

Corollary 2.2 tells us that if x is sufficiently close to z_j, then $K_N^\sigma f(x)$ is close to $[f](z_j)$; but how close should x be? This is determined by the behavior of $\omega_f(\cdot, x)$, which in turn depends on the behavior of $f(\cdot)$ in the neighborhood of x. Our next theorem converts (2.11) into a precise error statement which depends solely on the position of x relative to the jump discontinuities but otherwise *is independent of the behavior of f near x*.

THEOREM 2.3 (concentration kernels revisited). *Assume that $f(\cdot) \in BV[-\pi, \pi]$ is a piecewise C^2-smooth function so that*

$$M_2 := \|f\|_{C^2\left(\cup_j (z_j, z_{j+1})\right)} < \infty.$$

Let $\mathcal{K}_N^\sigma(x)$ be a normalized concentration kernel (2.1) *with $\sigma(\xi)/\xi \in C^1$. Then, there exist constants depending on M_2 but otherwise independent of N such that $K_N^\sigma f(x)$ satisfies the concentration property*

$$K_N^\sigma f(x) = \begin{cases} [f](z_j) + \mathcal{O}\left(\dfrac{\log(N)}{N}\right) & \textit{if } |x - z_j| \lesssim \dfrac{\log(N)}{N}, \quad j = 1, \ldots, m, \\[2ex] \mathcal{O}\left(\dfrac{\log(N)}{N d_x}\right) & \textit{if } d_x := \textit{dist}\Big\{x, \{z_1, \ldots, z_m\}\Big\} \gg \dfrac{1}{N}. \end{cases} \tag{2.12}$$

Remark 2.4. Theorem 2.3 tells us that admissible kernels $\mathcal{K}_N^\sigma(x)$ tend to concentrate in the immediate $\mathcal{O}(1/N)$-neighborhood of the jump discontinuities z_j's, where they approach the amplitude of these jumps $[f](z_j)$, while they decay to zero away from these jumps, where $\log(N)/Nd_x \sim 0$ as $d_x \gg 1/N$. We note in passing that for the concentration property $K_N^\sigma f(x) \to [f](x)$ to hold, it suffices to require the integrability $\omega_f(\cdot, x) \in L^1$ $\forall x$, e.g., [3, 7, 5], and the references therein. The BV regularity of $\omega_f(\cdot, x)$ enables us to derive the first-order concentration rate stated in (2.11), and, when f satisfies the stronger assumption of piecewise C^2 smoothness, we have the improved first-order rate (2.12). In practice, the decay rate may be even *faster* than first order, depending on the fine properties of σ; consult [4, p. 1396].

The proof of Theorem 2.3 is based on the following sharp upper-bound:

$$\|\omega_x(\cdot)\|_{BV} \lesssim d_x \|f\|_{C^2(I_x)} + \frac{1}{d_x}\|f\|_{BV}, \tag{2.13}$$

here and below, I_x denotes the largest *punctured* interval of smoothness enclosing x, and d_x denotes the distance to the nearest jump, that is,[4]

$$\begin{aligned} I_x &= (x - d_x, x) \cup (x, x + d_x), \\ d_x &:= \begin{cases} \min\limits_{0 \le j \le m+1} |x - z_j| > 0 & \text{if } x \notin \{z_1, \ldots, z_m\}, \\ \min\limits_{\pm}\{|z_j - z_{j\pm1}|\} > 0 & \text{if } x = z_j, \quad j = 1, 2, \ldots m. \end{cases} \end{aligned} \tag{2.14}$$

[4]By periodicity, $z_0 := z_m - 2\pi$ and $z_{m+1} := z_1 + 2\pi$.

We postpone the proof of (2.13) to the end of this section (consult Lemma 2.5 below), and we turn to the proof of Theorem 2.3, where we distinguish between three cases.

• *Case* (i). If x is a discontinuity point, say, $x = z_j$, then (2.13) with $d_k := d_{z_k}$ states that

$$\|\omega_{z_j}(\cdot)\|_{BV} \lesssim C_{11} := \max_k \left\{ d_k \|f\|_{C^2(I_{z_k})} + \frac{1}{d_k}\|f\|_{BV} \right\}, \qquad j = 1, 2, \ldots, m,$$

and Corollary 2.2 implies that

$$|K_N^\sigma f(z_j) - [f](z_j)| \leq C_{11}\frac{\log N}{N}, \qquad j = 1, 2, \ldots, m. \tag{2.15a}$$

• *Case* (ii). Assume that x is near a discontinuity $|x - z_j| \lesssim \log(N)/N$. We note that $\mathcal{K}_N^\sigma$ is uniformly bounded:

$$\|\mathcal{K}_N^\sigma\|_{L^\infty} \lesssim \sum_{|k|\leq N} \sigma\left(\frac{|k|}{N}\right)|\widehat{f}(k)| \lesssim \sum_{k=1}^{N} \frac{\sigma(\xi_k)}{\xi_k}\frac{1}{N}\|f\|_{BV} \lesssim C_{12}, \qquad C_{12} = (1+\delta_N)\|f\|_{BV},$$

which in turn implies the Lipchitz continuity of $K_N^\sigma f(x)$:

$$|K_N^\sigma f(x_2) - K_N^\sigma f(x_1)| \leq \|\mathcal{K}_N^\sigma\|_{L^\infty} \int |f(x_2 - y) - f(x_1 - y)|dy \lesssim C_{12}|x_2 - x_1| \cdot \|f\|_{BV}. \tag{2.15b}$$

Combining (2.15a) and (2.15b) yields the first half of (2.12)

$$\begin{aligned} |K_N^\sigma f(x) - [f](z_j)| &\leq |K_N^\sigma f(x) - K_N^\sigma f(z_j)| + |K_N^\sigma f(z_j) - [f](z_j)| \\ &\leq C_{12}|x - z_j| \cdot \|f\|_{BV} + C_{11}\frac{\log N}{N} \lesssim \frac{\log N}{N}, \qquad |x - z_j| \lesssim \frac{\log(N)}{N}. \end{aligned}$$

• *Case* (iii). Finally, we assume that x is bounded away from the discontinuities $d_x \gg 1/N$. Then (2.13) and (2.11) yield the second half of (2.12)

$$|K_N^\sigma f(x)| \lesssim \frac{\log(N)}{N}\left(d_x\|f\|_{C^2(I_x)} + \frac{1}{d_x}\|f\|_{BV}\right) \lesssim \frac{\log(N)}{N d_x}, \qquad d_x \gg \frac{\log(N)}{N}.$$

We close this section with the following BV bound, which is the heart of the matter.

LEMMA 2.5 (BV bound). *Assume that* $f(\cdot)$ *is piecewise* C^2*-smooth, and let* $I_x = (x - d_x) \cup (x, x + d_x)$ *in* (2.14) *be the largest punctured interval of smoothness enclosing* x. *Then*

$$\|\omega_x(\cdot)\|_{BV} \lesssim d_x\|f\|_{C^2(I_x)} + \frac{1}{d_x}\|f\|_{BV}.$$

Proof. We decompose $\omega_x(y)$ into two symmetric parts:

$$\omega_x(y) = \frac{f(x + y) - f(x+)}{y} - \frac{f(x-) - f(x - y)}{y} =: \omega_+(y) - \omega_-(y).$$

To upper bound the variation of $\omega_+(\cdot)$ we compute

$$\begin{aligned} \omega_+'(y) &= -\frac{f(x + y) - f(x+) - yf'(x + y)}{y^2} \\ &= \frac{-1}{y^2}\left(\int_{x+}^{x+y} f'(\xi)d\xi - yf'(x + y)\right) \\ &= \frac{1}{y^2}\int_{x+}^{x+y} (\xi - x)f''(\xi)d\xi, \qquad x < y < x + d_x. \end{aligned}$$

The variation of ω_+ to the right of x is therefore upper bounded by

$$\begin{aligned}\|\omega_+(\cdot)\|_{BV(x+,\pi)} &\leq \int_{y=0}^{d_x}\left|\int_{\xi=x+}^{x+y}\frac{(\xi-x)}{y^2}f''(\xi)d\xi\right|dy \\ &\quad+\int_{y=d_x}^{\pi}\left|\frac{f'(x+y)}{y}\right|+\left|\frac{f(x+y)-f(x+)}{y^2}\right|dy \\ &\leq \frac{1}{2}\int_{y=0}^{d_x}\max_{\xi\in(x,x+d_x)}|f''(\xi)|dy \\ &\quad+\frac{1}{d_x}\int_{y=d_x}^{\pi}|f'(x+y)|+\left|\frac{f(x+y)-f(x+)}{y}\right|dy \\ &\leq \frac{d_x}{2}\|f\|_{C^2(x,x+d_x)}+\frac{2}{d_x}\|f\|_{BV}.\end{aligned}$$

Similarly, the variation of ω_+ to the left of x does not exceed $\|\omega_+(\cdot)\|_{BV(-\pi,x-)} \leq \frac{d_x}{2}\|f\|_{C^2(x-d_x,x)} + \frac{2}{d_x}\|f\|_{BV}$. We then have

$$\|\omega_+(\cdot)\|_{BV} \leq [f](x) + d_x\|f\|_{C^2(I_x)} + \frac{4}{d_x}\|f\|_{BV} \lesssim d_x\|f\|_{C^2(I_x)} + \frac{1}{d_x}\|f\|_{BV}.$$

A similar bound holds for $\|\omega_-(\cdot)\|_{BV}$, and the lemma follows. ☐

3. Noisy data—a new perspective based on constrained minimization. Assume that f experiences a single jump discontinuity at location z of height $[f](z)$. This dictates a first-order decay of the Fourier coefficients

$$\widehat{f}(k) = [f](z)\frac{e^{-ikz}}{2\pi ik} + \widehat{s}(k) + \widehat{n}(k). \tag{3.1}$$

Here, $\widehat{s}(k)$ are associated with the regular part of f after extracting the jumps $[f](z_j)$; their decay is of order $\sim |k|^{-2}$ or faster, depending on the smoothness of the regular part $s(\cdot)$. The new aspect of the problem enters through the $\widehat{n}(k)$'s, which are the Fourier coefficients of the noisy part corrupting the smooth part of the data. We assume $n(\cdot)$ to be white noise whose mean-square power at each frequency is $E(|\widehat{n}(k)|^2) = \eta$. With (3.1), the conjugate sum (2.6) becomes

$$\begin{aligned}K_N^\sigma f(x) = [f](z)\times 2\pi i\sum_{k=1}^{N}\frac{\sigma_N\left(\frac{k}{N}\right)}{k}\cos k(x-z) - 2\pi\sum_{k=1}^{N}\sigma\left(\frac{k}{N}\right)\widehat{s}(k)\sin kx \\ - 2\pi\sum_{k=1}^{N}\sigma\left(\frac{k}{N}\right)\widehat{n}(k)\sin kx.\end{aligned}$$

We quantify the "energy" of each of the three sums on the right. E_J and E_R are associated with the discontinuous and regular parts of f:

$$\sum_{k=1}^{N}\left(\frac{\sigma\left(\frac{k}{N}\right)}{k}\right)^2 \approx \frac{1}{N}\int_0^1\left(\frac{\sigma(\xi)}{\xi}\right)^2 d\xi =: E_J(\sigma), \tag{3.2a}$$

$$\sum_{k=1}^{N}\sigma^2\left(\frac{k}{N}\right)|\widehat{s}(k)|^2 \sim \frac{1}{N^3}\int_0^1\frac{\sigma^2(\xi)}{\xi^4}d\xi =: E_R(\sigma), \tag{3.2b}$$

and $E_\eta(\sigma)$ is associated with the noisy part of f, which was assumed to have variance η:

$$\sum_{k=1}^{N}\sigma^2\left(\frac{k}{N}\right)E(|\widehat{n}(k)|^2)\approx \eta N\int_0^1 \sigma^2(\xi)d\xi =: E_\eta(\sigma). \tag{3.2c}$$

Our perspective for construction of edge detectors for such noisy data is to treat the problem as a *constrained minimization.* We seek a function $\sigma(\xi)$ which minimizes the total energy, thus making the conjugate sum $K_N^\sigma f$ as localized as possible, subject to prescribed normalization constraint (2.1b),

$$\min\left\{a_J E_J(\sigma)+a_\eta E_\eta(\sigma)+a_R E_R(\sigma)\;\middle|\;\int_0^1\frac{\sigma(\xi)}{\xi}d\xi=1\right\}. \tag{3.3}$$

This yields

$$\sigma(\xi)=C_\sigma\frac{\frac{1}{\xi}}{a_J\frac{1}{N\xi^2}+a_\eta\eta N+a_R\frac{1}{N^3\xi^4}}=\frac{C_\sigma N^3\xi^3}{a_J N^2\xi^2+a_\eta\eta N^4\xi^4+a_R}.$$

We ignore the relatively negligible contribution of the regular part which becomes even smaller as $s(\cdot)$ becomes smoother. Setting $a_R=0$ we end up with concentration factors of the form

$$\sigma(\xi)=\frac{C_\sigma}{a_J}\cdot\frac{N\xi}{1+\eta\beta^2N^2\xi^2},\quad \beta:=\sqrt{\frac{a_\eta}{a_J}}. \tag{3.4}$$

The corresponding concentration kernel depends only on the *relative* size of the amplitudes $\beta^2=a_\eta/a_J$. Indeed, the normalization of $\sigma(\xi)/\xi$ (2.1b) causes the constant C_σ to satisfy

$$\int_0^1\frac{\sigma(\xi)}{\xi}d\xi=\frac{C_\sigma}{a_J\sqrt{\eta}\beta}\tan^{-1}\left(\sqrt{\eta}\beta N\right)=1,$$

and we end up with the *normalized* concentration factor

$$\sigma(\xi)\equiv\sigma_\eta(\xi)=\frac{1}{\tan^{-1}\left(\sqrt{\eta}\beta N\right)}\cdot\frac{\sqrt{\eta}\beta N\xi}{1+\eta\beta^2N^2\xi^2},\qquad \beta=\sqrt{\frac{a_\eta}{a_J}}. \tag{3.5}$$

The corresponding edge detector then takes the form

$$K_N^{\sigma_\eta}f(x)=\frac{\pi\sqrt{\eta}\beta}{\tan^{-1}\left(\sqrt{\eta}\beta N\right)}\sum_{|k|\le N}\frac{ik}{1+\eta\beta^2k^2}\widehat{f}(k)e^{ikx}. \tag{3.6}$$

The concentration factor $\sigma=\sigma_\eta$ now involves three factors: the ratio β, the noise variance η, and the number of modes N. The concentration kernel (3.6) tends to de-emphasize both the low frequencies which are "corrupted" by the jump discontinuity(ies) and the high frequencies which are corrupted by the noise. Different procedures yield different policies for the choice of $\beta=\beta(\eta)$; one will be discussed in the next subsection. It is worth noting the essential dependence of $\sigma_\eta(\xi)$ on the variance of the noise η. There are three scales involved—the small "smoothness" of order $\sim 1/N$, the noise scale of order $\sim\sqrt{\eta}$, and the $\mathcal{O}(1)$-scale of jump discontinuities. We distinguish between two cases. If $\sqrt{\eta}\ll 1/N$ so that $\sqrt{\eta}\beta N\ll 1$, then the noise can be considered part of the smooth variation of f and $\sigma_\eta(\xi) \approx \xi$ recovers the usual concentration factor for noise-free data. Indeed, $\sigma_\eta(\xi) = \xi$ at the limit of $\eta \downarrow 0$. Otherwise, when the $\mathcal{O}(1/N)$-smoothness scale is dominated by the $\mathcal{O}(\sqrt{\eta})$-noise scale in the sense that $\sqrt{\eta}\beta \gtrsim 1/N$ in which we assume the noise to be still well below the $\mathcal{O}(1)$-scale of the jumps,

$$\frac{1}{N} \lesssim \sqrt{\eta}\beta \ll \mathcal{O}(1).$$

In this case, we can ignore the bounded factor $1/\tan^{-1}(\sqrt{\eta}\beta N)$, and we compute the small scale dictated by Theorem 2.1. Setting $\zeta(\xi) := \sqrt{\eta}\beta N\xi$ we find

$$\begin{aligned}
\delta_N &= \frac{1}{N}\left\|\frac{\sigma_\eta(\xi)}{\xi}\right\|_{BV} \lesssim \frac{\sqrt{\eta}\beta N}{N}\int \frac{1}{(1+\zeta^2)^2}d\zeta \lesssim \sqrt{\eta}\beta,\\
\varepsilon_1(N) &= \frac{1}{N}|\sigma_\eta(1)| \lesssim \frac{1}{\sqrt{\eta}\beta N^2},\\
\varepsilon_2(N) &= \frac{1}{N}\int_{1/N}^{1} \frac{1-\zeta^2}{\xi(1+\zeta^2(\xi))^2}d\xi \lesssim \sqrt{\eta}\beta\log\big(\sqrt{\eta}\beta\big),\\
\varepsilon_3(N) &= \left|\sigma_\eta\left(\frac{1}{N}\right)\right| \lesssim \sqrt{\eta}\beta.
\end{aligned}$$

Hence (2.4) holds with

$$\varepsilon_N \equiv \varepsilon_\eta := \sqrt{\eta}\beta\log\big(\sqrt{\eta}\beta\big).$$

It is remarkable to see how the small scale of smoothness in the noiseless case $\mathcal{O}(\log(N)/N)$ is now replaced by the small scale of noise $\varepsilon_\eta = \mathcal{O}(\sqrt{\eta}\beta\log(\sqrt{\eta}\beta))$. We now appeal to (2.7): since $\sqrt{\eta} \ll 1$, Theorem 2.1 implies that $K_N^{\sigma_\eta} f$ separates the $\mathcal{O}(1)$-scale of the edges from the noise scale of order $\varepsilon_\eta \ll 1$.

THEOREM 3.1 (edge detection in noisy data). *Assume that $f(\cdot)$ is piecewise smooth in the sense that* (2.3) *holds. Assume that its spectral data contain white noise with variance $\eta \ll 1$. Let $K_N^{\sigma_\eta} f(x)$ be a normalized concentration kernel* (3.6)

$$K_N^{\sigma_\eta} f(x) = \frac{\pi\sqrt{\eta}\beta}{\tan^{-1}(\sqrt{\eta}\beta N)}\sum_{|k|\le N}\frac{ik}{1+\eta\beta^2k^2}\widehat{f}(k)e^{ikx}$$

associated with the concentration factor

$$\sigma_\eta(\xi) = \frac{1}{\tan^{-1}(\sqrt{\eta}\beta N)}\cdot\frac{\sqrt{\eta}\beta N\xi}{1+\eta\beta^2N^2\xi^2}.$$

We distinguish between two cases:

(i) *if $\sqrt{\eta}\beta N \ll 1$, we set the small scale $\varepsilon = \varepsilon_N := \log(N)/N$;*

(ii) *if $1/N \lesssim \sqrt{\eta}\beta \ll 1$, we set the small scale $\varepsilon = \varepsilon_\eta := \sqrt{\eta}\beta\log\big(\sqrt{\eta}\beta\big)$. Then, $K_N^{\sigma_\eta} f(x)$ satisfies the following concentration property:*

$$K_N^{\sigma_\eta} f(x) = \begin{cases} [f](z_j) + \mathcal{O}\big(\varepsilon\big), & |x - z_j| \lesssim \varepsilon, \quad j = 1,\dots,m,\\[2mm] \mathcal{O}\bigg(\dfrac{\varepsilon}{d_x}\bigg), & d_x = dist\Big\{x, \{z_1,\dots,z_m\}\Big\} \gg \varepsilon. \end{cases} \tag{3.7}$$

Remark 3.2. We conclude that the concentration factor advocated in Theorem 3.1 is a rescaling of

$$\sigma(\xi) = \frac{\xi}{1+\xi^2}.$$

Thus, $\sigma_\eta(\xi) = C_\sigma \sigma(\sqrt{\eta}\beta N \xi)$, where C_σ is the proper normalization factor.

4. Numerical results. I.

4.1. Noiseless data. To illustrate the results of the previous section, we present two sets of numerical results. We begin with the case of *noiseless data* depicted in Figure 4.1—a standard periodic sawtooth function with a single jump continuity at $x = 1$. We still use the concentration factors advocated in Theorem 3.1, $C_\sigma \sigma(\sqrt{\eta}\beta N \xi)$. We set $\beta = 1$, corresponding to equal weights for the errors due to the noise and the discontinuous parts of the signal. The edge detector concentrates near the unit jump discontinuity at $x = 1$, and it decays to zero away from it. Since the data are noiseless, η is treated as an extra parameter. As η gets smaller, $K_N^{\sigma_\eta} f(x)$ tends to zero faster for x away from 1. The results of Figure 4.1 show that, by decreasing the value of η by a factor of 10, the decay of $K_N^{\sigma_\eta} f(x)$ away from $x = 1$ is accelerated by a factor of $0.00365/0.00137 = 2.66 \approx \sqrt{10}$; thus, the decay ratio in the smooth regime is of order $\sqrt{\eta}$, as it should be.

4.2. Noisy data—balancing the different types of errors. We now turn to the case of *noisy data*. In order to choose the free parameter β, it is important to know how β influences the error at the output of our edge detector (3.6). Let us

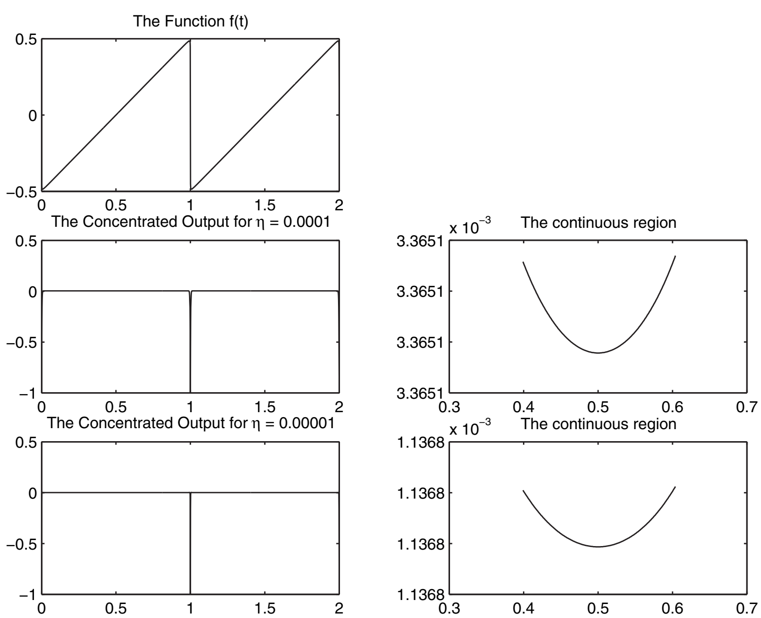

FIG. 4.1. *The output for various values of η when there is actually no noise at the input.*

consider E_η. We have seen that

$$\begin{aligned} E_\eta &= \eta N \int_0^1 \sigma^2(\xi)\, d\xi \\ &= \frac{\eta N}{(\tan^{-1}(\sqrt{\eta}\beta N))^2} \int_0^1 \frac{\eta \beta^2 N^2 \xi^2}{(1+\eta\beta^2 N^2 \xi^2)^2}\, d\xi \\ &= \frac{\sqrt{\eta}}{\beta(\tan^{-1}(\sqrt{\eta}\beta N))^2} \int_0^{\sqrt{\eta}\beta N} \frac{\zeta^2}{(1+\zeta^2)^2}\, d\zeta \overset{\sqrt{\eta}\beta N \gtrsim 1}{\sim} \frac{\sqrt{\eta}}{\beta}. \end{aligned}$$

E_η is approximately the variance of the contribution due to noise to the edge detector. If we want to consider the *size* of the noise, we should consider the standard deviation of this contribution. It is customary to bound the noise by some constant multiple of the standard deviation—we will use the factor 2. We define the *effective size* of the noise's contribution to be

$$E_{\eta,eff} \equiv 2\sqrt{E_\eta} \sim \frac{\eta^{1/4}}{\sqrt{\beta}}.$$

By (3.2a), the effective contribution from the jump is of order

$$E_{J,eff} = E_J(\sigma_\eta) \sim \eta^{1/2}\beta.$$

Minimizing $E_{\eta,eff} + E_{J,eff}$ with respect to β yields $\beta \sim \eta^{-1/6}$. Figure 4.2 demonstrates the edge detected in noisy data using the concentration kernel (3.6) with the advocated $\beta = \pi\eta^{-1/6}$.

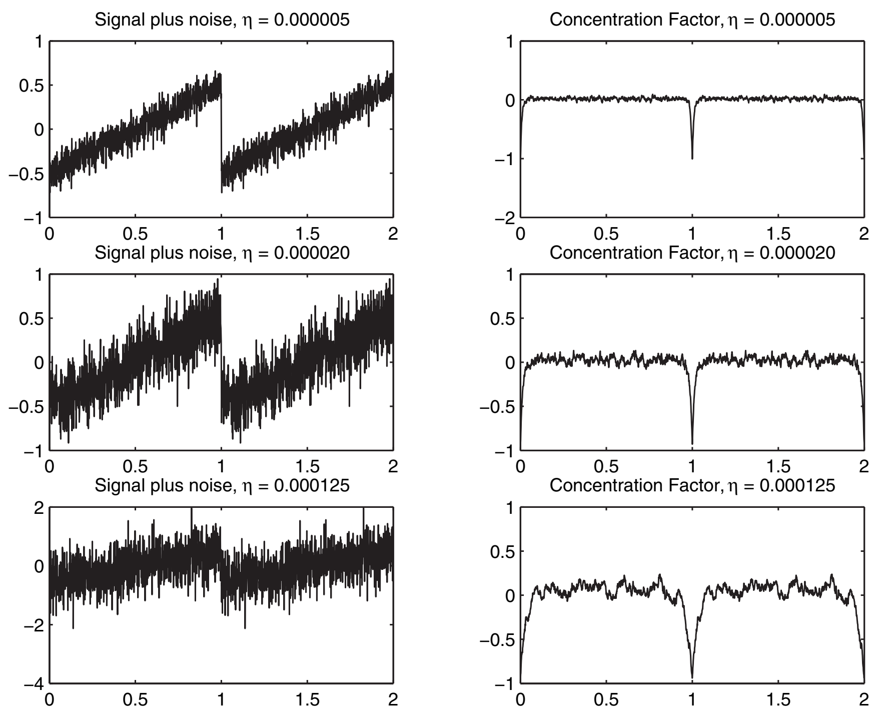

FIG. 4.2. *Detection of edges in a noisy sawtooth function corrupted with various values of* η *using the concentration kernel* (3.6) *with* $\beta = \pi\eta^{-1/6}$.

5. Noisy data and smoothness—concentration kernels revisited. As an alternative approach to the L^2-minimization offered in section 3, we now replace the L^2-"averaged" effect of the regular part taken in (3.2b) by the BV-like quantity

$$E_R(\sigma) \approx \sum_{k=1}^{N} \left| \sigma\left(\frac{k}{N}\right) \right| \cdot |\widehat{s}(k)|, \tag{5.1a}$$

where the regular part is sufficiently smooth that

$$|\hat{s}(k)| \sim \frac{1}{k^2}. \tag{5.1b}$$

As in (3.3), we consider the constrained minimization

$$\min \left\{ J(\sigma) \;\middle|\; \int_0^1 \frac{\sigma(\xi)}{\xi}\, d\xi = 1 \right\}, \tag{5.2a}$$

with $J(\sigma) = a_J E_J(\sigma) + a_\eta E_\eta(\sigma) + a_R E_R(\sigma)$, where $E_J(\sigma)$ and $E_\eta(\sigma)$ are given by (3.2a) and (3.2c), respectively, but with an alternative expression for the "energy" of the regular part motivated by (5.1): $E_R(\sigma) := \int_0^1 |\sigma(\xi)|\xi^{-2} d\xi/N$. We find that

$$J(\sigma) := \frac{a_J}{N} \int_0^1 \frac{\sigma^2(\xi)}{\xi^2}\, d\xi + a_\eta \eta N \int_0^1 \sigma^2(\xi)\, d\xi + \frac{a_R}{N} \int_0^1 \frac{|\sigma(\xi)|}{\xi^2}\, d\xi. \tag{5.2b}$$

Proceeding formally, the solution for the first variation of (5.2) leads to

$$\frac{2a_J \sigma(\xi)}{N\xi^2} + 2\eta a_\eta N \sigma(\xi) + \frac{a_R \cdot \operatorname{sgn}(\sigma(\xi))}{N\xi^2} = \frac{\lambda}{\xi}, \qquad \operatorname{sgn}(\sigma) = \begin{cases} 1, & \sigma > 0, \\ 0, & \sigma = 0, \\ -1, & \sigma < 0. \end{cases}$$

We will show that the resulting optimal concentration factor is given by

$$\sigma(\xi) = C_\sigma \frac{(N\xi - k_0)_+}{1 + \eta\beta^2 N^2 \xi^2}, \qquad k_0 := \frac{a_R}{\lambda}, \quad \beta = \sqrt{\frac{a_\eta}{a_J}}, \quad C_\sigma = \frac{\lambda}{2a_J}. \tag{5.3}$$

Indeed, to justify the passage to (5.3), one may consider a *regularized* version of the variational statement (5.2) $\min J_\varepsilon(\sigma)$, where

$$J_\varepsilon(\sigma) := \frac{a_J}{N} \int_0^1 \frac{\sigma^2(\xi)}{\xi^2}\, d\xi + a_\eta \eta N \int_0^1 \sigma^2(\xi)\, d\xi + \frac{a_R}{N} \int_0^1 \frac{|\sigma(\xi)|_\varepsilon}{\xi^2}\, d\xi$$

involves a *mollified* absolute value function

$$|\sigma|_\varepsilon := \begin{cases} |\sigma|, & |\sigma| \geq \epsilon, \\ \dfrac{\sigma^2}{2\epsilon} + \dfrac{\epsilon}{2}, & |\sigma| \leq \epsilon. \end{cases}$$

The solution of the corresponding regularized first variation yields the minimizer

$$\sigma(\xi) = C_\sigma \frac{N\xi - k_0 \cdot \operatorname{sgn}_\varepsilon(\sigma(\xi))}{\eta\beta^2 N^2 \xi^2 + 1}, \qquad \operatorname{sgn}_\varepsilon(\sigma) := \begin{cases} 1, & \sigma > \varepsilon, \\ \sigma/\varepsilon, & |\sigma| \leq \varepsilon, \\ -1, & \sigma < -\varepsilon. \end{cases}$$

Thus, we end up with the optimal concentration factor $\sigma(\xi) = \sigma_\varepsilon(\xi)$,

$$\sigma_\varepsilon(\xi) = \begin{cases} C_\sigma \dfrac{N\xi - k_0}{1 + \eta\beta^2 N^2 \xi^2}, & N\xi - k_0 > \varepsilon, \\ C_\sigma \dfrac{\varepsilon N\xi}{\varepsilon(1 + \eta\beta^2 N^2 \xi^2) + C_\sigma k_0}, & N\xi - k_0 \le \varepsilon, \end{cases}$$

and (5.3) is recovered by letting $\varepsilon \downarrow 0$. Clearly, the resulting optimal concentration factor is nonnegative.

It remains to calculate the normalization factor C_σ, for which

$$C_\sigma \int_{k_0/N}^{1} \frac{1}{\xi} \frac{N\xi - k_0}{\eta\beta^2 N^2 \xi^2 + 1}\, d\xi = 1.$$

The integral on the left is found to be

$$\int_{k_0/N}^{1} \left(\frac{-k_0}{\xi} + \frac{k_0\eta\beta^2 N^2 \xi + N}{\eta\beta^2 N^2 \xi^2 + 1} \right) d\xi$$
$$= k_0 \log\left(\frac{k_0}{N}\right) + \frac{k_0}{2} \log\left(\frac{\eta\beta^2 N^2 + 1}{\eta\beta^2 k_0^2 + 1}\right) + \frac{1}{\sqrt{\eta}\beta}\Big(\tan^{-1}(\sqrt{\eta}\beta N) - \tan^{-1}(\sqrt{\eta}\beta k_0)\Big).$$

We focus our attention on the "noisy" case when $1/N \lesssim \sqrt{\eta}\beta \ll 1$ so that the fourth term on the right is negligible while the second term on the right is approximated by

$$\frac{k_0}{2} \log\left(\frac{\eta\beta^2 N^2 + 1}{\eta\beta^2 k_0^2 + 1}\right) \approx k_0 \log(\sqrt{\eta}\beta N).$$

We end up with an approximated integral

$$\int_{k_0/N}^{1} \Big(\dots \Big) d\xi \approx k_0 \log(\sqrt{\eta}\beta N) + \frac{1}{\sqrt{\eta}\beta} \tan^{-1}(\sqrt{\eta}\beta N).$$

The balance between these two terms depends on the specific policy for β and the detailed balance between $\sqrt{\eta}\beta$ and N. Our normalized concentration factor takes the form

$$\sigma_\eta(\xi) = \frac{1}{\sqrt{\eta}\beta k_0 \log(\sqrt{\eta}\beta N) + \tan^{-1}(\sqrt{\eta}\beta N)} \cdot \frac{\sqrt{\eta}\beta(N\xi - k_0)_+}{1 + \eta\beta^2 N^2 \xi^2}. \tag{5.4a}$$

We can simplify this concentration factor in several ways; we mention two here.

(i) When N is large enough, we have $\tan^{-1}(\sqrt{\eta}\beta N) \approx \pi/2$ yielding

$$\sigma_\eta(\xi) = \frac{1}{k_0 \log(\sqrt{\eta}\beta N) + \pi/(2\sqrt{\eta}\beta)} \cdot \frac{(N\xi - k_0)_+}{1 + \eta\beta^2 N^2 \xi^2}. \tag{5.4b}$$

(ii) Observe that $\sigma_\eta(\xi)$ is rapidly decreasing at $\xi \approx 1$, with

$$\sigma_\eta(\xi) \sim \frac{1}{\eta\beta^2 N \log N}, \qquad \xi \approx 1,$$

so $\sigma_\eta(\xi)$ can be set to zero for $\xi \approx 1$ when N is large enough. In order to properly normalize the resulting concentration factor, N must be replaced by N_0. This leads us to the following:

$$\text{(5.4c)} \qquad \sigma_\eta\left(\frac{k}{N}\right) = \begin{cases} 0, & k < k_0, \\ \dfrac{1}{k_0 \log(\sqrt{\eta}\beta N_0) + \pi/(2\sqrt{\eta}\beta)} \cdot \dfrac{(k-k_0)_+}{1+\eta\beta^2 k^2}, & k_0 < k < N_0, \\ 0, & N_0 < k < N. \end{cases}$$

6. Numerical results. II. We consider two examples depicted in Figure 6.1. In the first case, we have a noise of variance $\eta = 2\times 10^{-5}$ to be detected out of the first $N \gg 1000$ modes. The use of the edge detector (5.4) depends on three parameters, namely, β, k_0, and N_0. For β we use, as before, $\beta = \pi\eta^{-1/6} \sim 15$. The value of k_0 was chosen as $k_0 = 8\pi$: observe that, according to (5.3), $k_0 C_\sigma$ equals the ratio $a_R/2a_J$, and we found optimal results when the ratio of these amplitudes is set to be of order 10, yielding $k_0 = 10/0.3984 \sim 8\pi$. Finally, N_0 was taken as $N_0 = 1000$ since $\sigma_\eta(k/N) \leq 0.1$ for $N \geq N_0$. With these parameters we find

$$\sigma_\eta\left(\frac{k}{N}\right) = \begin{cases} 0, & k = 1, \ldots, 24, 1001, 1002, \ldots, \\ 0.3985 \cdot \dfrac{(k-8\pi)_+}{1+\eta(15k)^2}, & 25 \leq k \leq 1000. \end{cases}$$

In the second case, the noise variance is $\eta = 4.5 \times 10^{-5}$. This led us to the choice of $\beta = \pi\eta^{-1/6} \sim 13$. Setting $k_0 = 6\pi$ (as before, $k_0 \sim 10/C_\sigma = 10/0.5070$) and $N_0 = 1000$, we have

$$\sigma_\eta\left(\frac{k}{N}\right) = \begin{cases} 0, & k = 1, \ldots, 19, 1001, 1002, \ldots, \\ 0.5070 \cdot \dfrac{(k-6\pi)_+}{1+\eta(13k)^2}, & 20 \leq k \leq 1000. \end{cases}$$

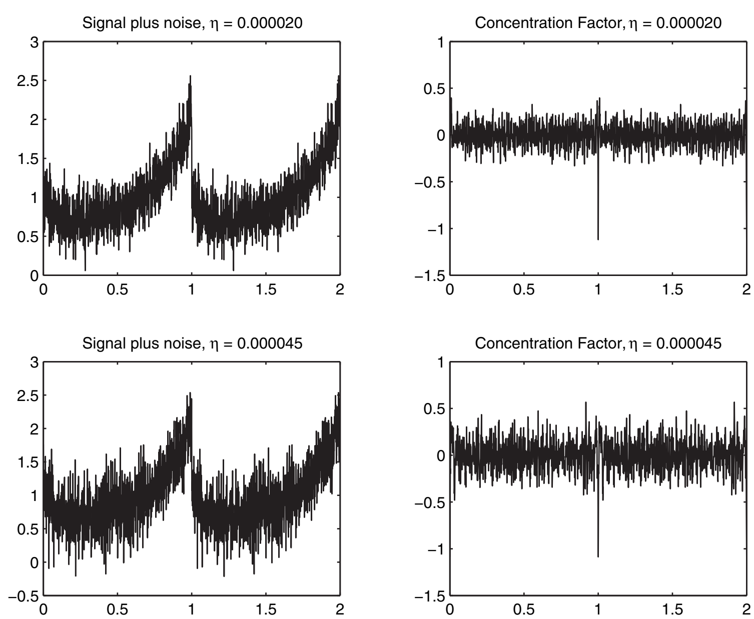

FIG. 6.1. *The output for various values of η when the input consists of a piecewise smooth function and noise (of the specified power spectral density).*

Note that in calculating the constants we made use of the *exact* normalization factor C_σ in (5.4b). $N_0 = 1000$ is not large enough to make the approximate normalization factor C_σ given in (5.4c) useful.

Note that, even with a large amount of white noise and a smooth signal, the location of the jump discontinuity is still clear. When considering jumps "corrupted" by low frequency data, we avoid low frequency signals by not using low frequency data. This helps keep the smooth signal from corrupting our results. On the other hand, because the jump discontinuity has most of its energy at low frequencies as well, our technique will increase the noise's effect. Comparing Figures 4.2 and 6.1, we find that the latter is not as clean as the former in the subfigure where the strength of the noise is the same.

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