Edge Detection Using Fourier Coefficients

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1. INTRODUCTION. Edge detection and the detection of discontinuities are important in many fields. In image processing, for example, one often needs to determine the boundaries of the items of which a picture is composed. (For more information about edge detection in image processing, see [**10**].) We consider the problem of detecting the edges present in a function when given the Fourier coefficients of the function.

There are numerical methods that estimate the Fourier coefficients of a function of interest rather than directly estimating the solution. The spectral viscosity method, a numerical method used to solve nonlinear partial differential equations (PDEs), is an example of such a method [**12**]. The method approximates the Fourier coefficients of the solution of a PDE. The Fourier coefficients are then used to calculate an approximation to the solution. The accurate reconstruction of the solution requires that the positions of the discontinuities of the solution be known [**5**]. In this paper we discuss techniques for using a function's Fourier coefficients to determine the locations and sizes of the jump discontinuities of the function.

At first glance the spectral representation of the signal—the Fourier series or transform associated with the signal—does not seem to be the ideal place to look for information about discontinuities in the signal. When a signal is discontinuous the convergence of the Fourier series or transform associated with the signal is not uniform; in such cases the Gibbs phenomenon [**11**] appears and truncating the series after any finite number of terms always leads to $O(1)$ oscillations in the reconstructed signal. (For a nice, detailed treatment of the Gibbs phenomenon, see [**6**].)

Considering the question again, however, one realizes that if a discontinuity is characterized by a "phenomenon," then the existence of the discontinuity is indeed encoded in the coefficients. The question becomes how to effectively "decode" the discontinuity. One does not do this by directly summing the series—one uses the spectral representation in a somewhat different way to "concentrate" the function about the discontinuity. In what follows, we explain how this is done. We restrict ourselves to periodic (or compactly supported) functions and only consider Fourier series. (Those interested in seeing a more general theory of concentration factors are referred to [**3, 4**].)

Much of the information in this article is well known [**3, 4**]. The use of the Euler-Mascheroni constant to improve the performance of the concentration factor in Section 4 is, to the best of our knowledge, new.

In the next section we give some of the background necessary for our study. In the following sections we present the classical method of finding the discontinuities, we explain its shortcomings, and we present a better method and analyze its properties.

2. SOME BACKGROUND.

2.1. Piecewise Continuity and Piecewise Differentiability. A function is called piecewise continuous if it has only a finite number of discontinuities in any interval, and its left and right limits exist (but are not equal) at each discontinuity. A function is said to be piecewise differentiable if the function and its derivative are both piecewise continuous. (A piecewise differentiable function need not be continuous.)

In signal processing one is often interested in piecewise differentiable functions—functions like those in Figure 1. Piecewise differentiable functions are composed of

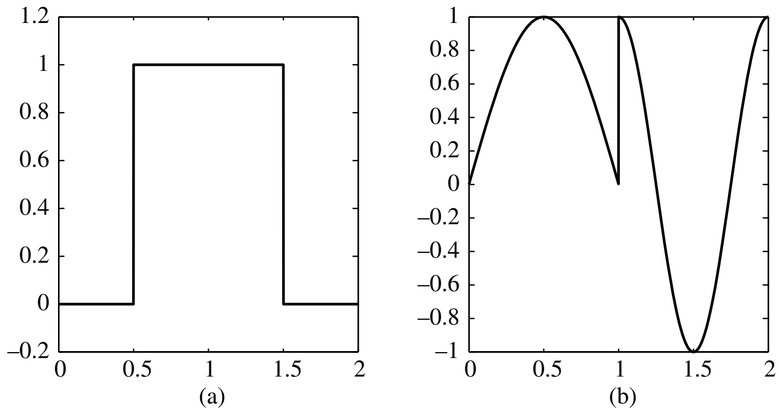


Figure 1. Two piecewise differentiable functions.

differentiable functions that have been "pasted" together like the constant values in Figure 1(a) and the sine waves in Figure 1(b).

If one acquires a signal from a piece of equipment and if changes in the equipment's operating mode are signaled by discontinuous changes in the signal, then locating the discontinuities in the signal is important. If one is processing a picture, the transition from a region that depicts a face to a region that depicts a wall is generally discontinuous. If one would like to determine the edges of the face, it is important to be able to determine the curves along which the function that represents the image is discontinuous.

Because of the role that piecewise differentiable functions play in signal processing, we characterize piecewise differentiable functions carefully. As the discontinuities of piecewise differentiable functions are of particular interest to us, we develop techniques for examining such discontinuities.

2.2. The Convergence of the Fourier Series. Let $f(t)$ be a periodic function with period T. The Fourier coefficients of $f(t)$ are

$$c_n = \frac{1}{T}\int_x^{x+T} e^{-in\omega t} f(t)\,dt,$$

where $\omega = 2\pi/T$, x is an arbitrary point, and T is the period of the function $f(t)$.

In many cases, it is possible to reconstruct a function from its Fourier coefficients. We consider three different senses in which a function is represented by its Fourier series. First, consider a piecewise differentiable periodic function, $f(t)$. At all points at which $f(t)$ is continuous we have

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}.$$

At points of discontinuity, the convergence is to the mean of the values to which the function tends from the left and the right of the discontinuity [**2**]. That is, for piecewise differentiable functions the convergence of the Fourier series to the function is pointwise wherever the function is continuous and is to the average value of the function at the jumps in the function's value. This shows that if a piecewise differentiable function $f(t)$ is continuous, then the Fourier series converges pointwise to $f(t)$. If $f(t)$ is discontinuous, then so is the function to which the Fourier series converges.

Now suppose that $f(t)$ is periodic with period T and is square integrable in each period—that $f(t) \in L^2[0, T]$. Then (as described in [**8**]) the Fourier series converges to the function in $L^2[0, T]$ (and this is a weaker form of convergence than *uniform* convergence). Additionally, for square integrable functions Parseval's equation states that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T} \int_x^{x+T} |f(t)|^2 \, dt.$$

Parseval's equation says that if a function is square integrable, then the function's Fourier coefficients are square summable. (Because any piecewise continuous function is bounded on any finite interval, every periodic piecewise continuous function is square integrable.)

Finally, suppose that the Fourier coefficients of $f(t)$ are absolutely summable; that is,

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

As $|e^{in\omega t}| = 1$, the absolute summability of the Fourier coefficients establishes the uniform convergence of the Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}.$$

As the functions $e^{in\omega t}$ are continuous and we know that the uniform limit of continuous functions is a continuous function, we find that if the Fourier coefficients are absolutely summable, then the Fourier series converges to a continuous function. As we have already seen that the Fourier series of a piecewise differentiable function tends to a continuous function if and only if the function is actually continuous, we find that *if the Fourier coefficients of a piecewise differentiable function are absolutely summable, then the function is continuous*.

2.3. Smoothness and Convergence. In what follows two properties of the Fourier coefficients are important. One property concerns the relation between the smoothness of $f(t)$ and convergence of the Fourier series associated with $f(t)$. We treat this question here. The other property concerns the effect that shifting a function has on the function's Fourier coefficients and is treated in Section 2.4.

Let us consider the connection between the smoothness of $f(t)$ and the summability of the Fourier coefficients. If a function is continuous and piecewise differentiable, then the Fourier coefficients of the derivative of the function are square summable (as the derivative is piecewise continuous). Let $t_0, t_1, \ldots, t_{M-1}$ be the points in the interval $[0, T)$ at which the derivative of $f(t)$ changes in a discontinuous fashion—at which there are jumps in the derivative's values—and let a_n be the Fourier coefficients that correspond to $f'(t)$. Because $f(t)$ is continuous, it is clear that

$$\begin{aligned} T a_n &= \int_0^T e^{-in\omega t} f'(t) \, dt \\ &= \int_0^{t_0} e^{-in\omega t} f'(t) \, dt + \cdots + \int_{t_{M-1}}^T e^{-in\omega t} f'(t) \, dt \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{by parts}}{=} e^{-in\omega t} f(t)\Big|_0^{t_0} + \cdots + e^{-in\omega t} f(t)\Big|_{t_{M-1}}^{T} \\
&\quad + in\omega \int_0^{t_0} e^{-in\omega t} f(t)\,dt + \cdots + in\omega \int_{t_{M-1}}^{T} e^{-in\omega t} f(t)\,dt \\
&\overset{\text{continuity}}{=} in\omega \int_0^{t_0} e^{-in\omega t} f(t)\,dt + \cdots + in\omega \int_{t_{M-1}}^{T} e^{-in\omega t} f(t)\,dt \\
&= in\omega T c_n.
\end{aligned}
$$

Making use of the Cauchy-Schwarz inequality, we see that

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_n| &= \frac{1}{\omega}\left(\sum_{n=-\infty}^{-1} \frac{1}{n}|in\omega c_n| + \omega|c_0| + \sum_{n=1}^{\infty} \frac{1}{n}|in\omega c_n|\right) \\
&\leq \frac{1}{\omega}\left(\sqrt{\sum_{n=-\infty}^{-1} \frac{1}{n^2}}\sqrt{\sum_{n=-\infty}^{-1} |a_n|^2} + \omega|c_0| + \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty} |a_n|^2}\right) \\
&< \infty.
\end{aligned}
$$

That is, the Fourier coefficients of a function that is both continuous and piecewise differentiable are absolutely summable. Combining this result with the final result of Section 2.2, we find that a piecewise differentiable function is continuous if and only if its Fourier coefficients are absolutely summable. *Thus, if a piecewise differentiable function $f(t)$ can be written*

$$f(t) = f_d(t) + f_c(t),$$

where $f_d(t)$ and $f_c(t)$ are piecewise differentiable, $f_d(t)$ is discontinuous, and $f_c(t)$ is continuous, then (because of the linearity of the Fourier coefficients) the Fourier coefficients of $f(t)$ can be written as a sum of two parts. The part that corresponds to $f_d(t)$ cannot have absolutely summable Fourier coefficients. The part that corresponds to $f_c(t)$ must have absolutely summable Fourier coefficients.

2.4. Shifts of a Function. The second property we are interested in concerns the effect that shifting a function has on the function's Fourier coefficients. Let the Fourier coefficients of $f(t)$ be denoted by c_n. What are the Fourier coefficients, a_n, of $f(t-\tau)$? A simple calculation shows that

$$
\begin{aligned}
a_n &= \frac{1}{T}\int_x^{x+T} e^{-in\omega t} f(t-\tau)\,dt \\
&\overset{u=t-\tau}{=} \frac{1}{T}\int_{x-\tau}^{x-\tau+T} e^{-in\omega(u+\tau)} f(u)\,du \\
&= e^{-in\omega\tau} c_n.
\end{aligned}
$$

2.5. An Important Example. Without loss of generality, in the rest of this exposition we consider functions that are periodic with period 1. Consider $k(t)$ defined by

$$k(t) \equiv t - \frac{1}{2}$$

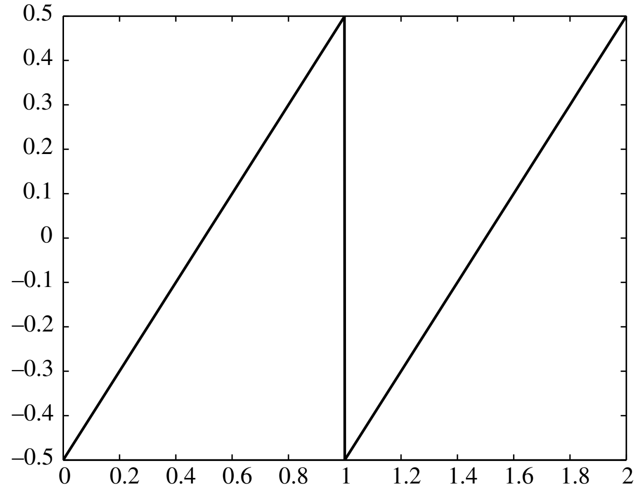


Figure 2. The function $k(t)$.

in the interval $t \in [0, 1)$ and defined elsewhere by periodically extending the function. (See Figure 2.) The function as defined has a jump of height 1 at every integer. (The height of the jump in the value of a function $k(t)$ at a point t_0 is defined as $\lim_{t\to t_0^-} k(t) - \lim_{t\to t_0^+} k(t)$.)

The Fourier coefficients of $k(t)$ are

$$c_n = \int_0^1 \left(t - \frac{1}{2}\right) e^{-i2\pi nt}\, dt.$$

For $n = 0$, it is clear that $c_0 = 0$. For $n \neq 0$, we see that

$$\begin{aligned} c_n &= \int_0^1 \left(t - \frac{1}{2}\right) e^{-i2\pi nt}\, dt \\ &\overset{\text{by parts}}{=} \left(t - \frac{1}{2}\right) \frac{e^{-i2\pi nt}}{-in2\pi}\bigg|_{t=0}^1 + \frac{1}{i2\pi n}\int_0^1 e^{-i2\pi nt}\, dt \\ &= \frac{i}{2\pi n}. \end{aligned}$$

The coefficients are square summable—as they must be—but they are not summable. The Fourier series that corresponds to $k(t)$ is

$$\sum_{n=-\infty}^{-1} \frac{i}{2\pi n} e^{i2\pi nt} + \sum_{n=1}^{\infty} \frac{i}{2\pi n} e^{i2\pi nt}. \tag{1}$$

2.6. An Interesting Sum. Using Parseval's equation for $k(t) = t - 1/2$ and its Fourier coefficients, we have

$$\begin{aligned} \int_0^1 k^2(t)\, dt &= \int_0^1 \left(t - \frac{1}{2}\right)^2 dt \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned}$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{4\pi^2n^2}$$
$$= \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

As it is easy to see that

$$\int_0^1\left(t-\frac{1}{2}\right)^2 dt = \frac{1}{12},$$

it is simple to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \tag{2}$$

This sum is used in Section 3.2.

2.7. Decomposing a Function. In the sections to come, we will need to split a piecewise differentiable function into its continuous and discontinuous parts. We now consider one way to perform this decomposition. Suppose that one has a piecewise differentiable function $f(t)$ with m jumps at the locations $t_1, \ldots, t_m$ with the heights $h_1, \ldots, h_m$, respectively. The function

$$w(t) = f(t) - \sum_{j=1}^{m} h_j k(t - t_j)$$

has no jumps and is continuous and piecewise differentiable. We find that

$$f(t) = w(t) + \sum_{j=1}^{m} h_j k(t - t_j),$$

where $w(t)$ is continuous and piecewise differentiable and the sum $\sum_{j=1}^{m} h_j k(t - t_j)$ is discontinuous and piecewise differentiable. The Fourier coefficients of $w(t)$, which we denote by b_n, are absolutely summable. Making use of the linearity of the Fourier coefficients, we find that the Fourier coefficients of the sum, denoted by a_n, are

$$a_n = \begin{cases} \sum_{j=1}^{m} h_j \dfrac{ie^{-i2\pi n t_j}}{2\pi n} & n \neq 0, \\ 0 & n = 0. \end{cases}$$

Clearly these coefficients are not summable. By the linearity of the Fourier coefficients, we find that the coefficients of $f(t)$, denoted by c_n, are

$$c_n = a_n + b_n.$$

3. THE CLASSICAL APPROACH—THE HILBERT TRANSFORM. In order to determine where the edges of the data are in a "minimally invasive way," we want to find a transformation of the Fourier coefficients that changes the Fourier coefficients as little as possible, but that causes the partial sums of the Fourier series of a discontinuous function to grow at the discontinuities but not elsewhere. Note that the reason that the (symmetric) partial sums corresponding to the Fourier series (1)

$$\sum_{n=-N}^{-1} \frac{i}{2\pi n} e^{i2\pi nt} + \sum_{n=1}^{N} \frac{i}{2\pi n} e^{i2\pi nt}$$

do not diverge at $t = 0$ is that at $t = 0$ the terms corresponding to $\pm n$ cancel one another.

Consider c_n, the Fourier coefficients of a function that is piecewise continuous but not continuous. Making use of the decomposition of Section 2.7, decompose the function into its discontinuous and continuous parts, and denote their Fourier coefficients by a_n and b_n respectively. Then $c_n = a_n + b_n$.

Now consider a transformation of the sequence c_n. Define r_n, the transformed sequence, by the equation

$$r_n = \begin{cases} -ic_n = -ia_n - ib_n & n \geq 1, \\ 0 & n = 0, \\ ic_n = ia_n + ib_n & n \leq -1. \end{cases}$$

This transformation is known as the *Hilbert transform* [**9**]. Clearly the transformed version of the b_n is still absolutely summable, while the transformed version of the a_n is not absolutely summable. Thus, the continuous part of the function is transformed into a continuous function by the Hilbert transform while the Hilbert transform of the discontinuous part is still—at the very least—discontinuous. (If the coefficient of the constant term of the original function is zero, then the Hilbert transform is an l^1 isometry. If the coefficient of the constant term is nonzero, the Hilbert transform is an l^1 contraction.)

3.1. The Effect of the Transformation on $k(t - \tau)$. Let us consider the function that one recovers from the Fourier series using the transformed coefficients of the discontinuous function $k(t - \tau)$. The function one recovers is

$$g(t) = \sum_{n=-\infty}^{-1} \frac{-1}{2\pi n} e^{i2\pi n(t-\tau)} + \sum_{n=1}^{\infty} \frac{1}{2\pi n} e^{i2\pi n(t-\tau)} = \sum_{n=1}^{\infty} \frac{\cos(2\pi n(t-\tau))}{n\pi}.$$

Note that at $t = \tau + m$ this series is the harmonic series and $g(\tau + m)$ diverges (and at $t = \tau + m + 1/2$ the series is the alternating harmonic series and converges conditionally to $\ln(2)/\pi$).

To proceed with our analysis we must analyze the partial sums

$$g_N(\xi) = \sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi}, \qquad \xi = t - \tau$$

more carefully. To this end, we consider the properties of the *Dirichlet kernel* [**14**] defined by

$$D_N(\xi) \equiv \sum_{n=-N}^{N} e^{i2\pi n\xi} = 1 + 2\sum_{n=1}^{N} \cos(2\pi n\xi). \tag{3}$$

This is a finite geometric series whose sum is

$$\begin{aligned} D_N(\xi) &= e^{-i2\pi N\xi} \sum_{n=0}^{2N} e^{i2\pi n\xi} \\ &= e^{-i2\pi N\xi} \frac{1 - e^{i(2N+1)2\pi\xi}}{1 - e^{i2\pi\xi}} \\ &= \frac{\sin((2N+1)\pi\xi)}{\sin(\pi\xi)}. \end{aligned}$$

It follows that

$$|D_N(\xi)| \le \frac{1}{|\sin(\pi\xi)|}.$$

We now consider the partial sum

$$g_N(\xi) = \sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi}.$$

From (3) it follows that this sum can be written as

$$g_N(\xi) = \frac{\cos(2\pi\xi)}{\pi} + \sum_{n=2}^{N} \frac{D_n(\xi) - D_{n-1}(\xi)}{2n\pi}.$$

Rewriting this, we find that

$$g_N(\xi) = \frac{\cos(2\pi\xi)}{\pi} + \frac{D_N(\xi)}{2N\pi} - \frac{1}{2\pi} \sum_{n=2}^{N} D_{n-1}(\xi) \left(\frac{1}{n} - \frac{1}{n-1}\right) - \frac{D_1(\xi)}{2\pi}$$

which, from the fact that $D_1(\xi) = 1 + 2\cos(2\pi\xi)$, can be further simplified to

$$g_N(\xi) = \frac{D_N(\xi)}{2N\pi} + \frac{1}{2\pi} \sum_{n=2}^{N} \frac{1}{n(n-1)} D_{n-1}(\xi) - \frac{1}{2\pi}.$$

Considering our previous bound on $|D_n(\xi)|$, we have

$$|g_N(\xi)| = \left| \sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi} \right| \le \frac{1}{|\sin(\pi\xi)|} \left(\frac{1}{2N\pi} + \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \right) + \frac{1}{2\pi}.$$

Note that

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Making use of (2) it follows that

$$|g_N(\xi)| = \left| \sum_{n=1}^{N} \frac{\cos(2\pi n\xi)}{n\pi} \right| \le \frac{1}{|\sin(\pi\xi)|} \left(\frac{1}{2N\pi} + \frac{1}{2\pi} \frac{\pi^2}{6} \right) + \frac{1}{2\pi}.$$

This shows that as long as ξ is not an integer—as long as $t \neq \tau + m$—$g_N(\xi)$ is bounded, and the dependence of the bound on ξ is known.

This leaves us in the position of knowing that the partial sums diverge like the harmonic series at $t = \tau + m$ and are bounded elsewhere. We take advantage of this fact by dividing the partial sum by the (approximate) value of the partial sum of the (divergent) harmonic series. This causes the partial sum to tend to 1 at the point at which the discontinuity occurred and to tend to zero elsewhere.

When $\xi = m$ (or, equivalently, $t = \tau + m$) the partial sum is

$$g_N(\xi) = \sum_{n=1}^{N} \frac{1}{n\pi} = \frac{1}{\pi} \sum_{n=1}^{N} \frac{1}{n}.$$

We would like to develop a closed form estimate of this sum. We note that

$$\int_{n}^{n+1} \frac{1}{x}\, dx \leq \frac{1}{n} \leq \int_{n-1}^{n} \frac{1}{x}\, dx.$$

Thus

$$1 + \int_{2}^{N+1} \frac{1}{x}\, dx \leq \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \int_{1}^{N} \frac{1}{x}\, dx.$$

It follows that

$$1 + \ln(N+1) - \ln(2) \leq \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \ln(N).$$

Dividing all three terms by $\ln(N)$ and taking the limit as $N \to \infty$, we find that

$$\frac{\sum_{n=1}^{N} \frac{1}{n}}{\ln(N)} \to 1.$$

Combining our results, we find that

$$\lim_{N\to\infty} \frac{\pi}{\ln(N)} g_N(\xi) = \begin{cases} 0 & \xi \notin \mathbb{Z}, \\ 1 & \xi \in \mathbb{Z}. \end{cases}$$

3.2. Edge Detection in Piecewise Differentiable Functions. Now let us consider the effects of the transformation on a generic piecewise differentiable function $f(t)$. If the Fourier coefficients of $f(t)$ are c_n, then the sum that we consider is

$$\text{edge}_1(t; N) \equiv \frac{\pi}{\ln(N)} \left(\sum_{n=-N}^{-1} i c_n e^{i2\pi nt} + \sum_{n=1}^{N} -i c_n e^{i2\pi nt} \right).$$

The function $\text{edge}_1(t; N)$ is the function that corresponds to the Fourier coefficients of $f(t)$ after they have been Hilbert transformed. This sum is our first edge detector, and it has two important properties. As $N \to \infty$ the value of the sum tends to the height of the jump in the original function at any point at which a jump occurs. At all other points, the sum tends to zero.

To prove these claims, we make use of the decomposition of Section 2.7. By linearity we can consider the effect of the operation on each set of Fourier coefficients. As the the sum of the b_n converges absolutely, it is clear that

$$\frac{\pi}{\ln(N)}\left(\sum_{n=-N}^{-1} ib_n e^{i2\pi nt} + \sum_{n=1}^{N} -ib_n e^{i2\pi nt}\right) = O\left(\frac{1}{\ln(N)}\right).$$

For any fixed n the coefficient a_n is just the coefficient that corresponds to

$$\sum_{j=1}^{m} h_j k(t - t_j).$$

By the linearity of $\text{edge}_1(t; N)$ relative to the coefficients input to it and from the results of Section 3.1 we see that as $N \to \infty$ the function $\text{edge}_1(t; N)$ converges to h_j at t_j and to zero, like $O(1/\ln(N))$, elsewhere.

4. SHORTCOMINGS OF THE TECHNIQUE. Let us consider the function $k(t) = t - 1/2$, and let us see how well our edge detector $\text{edge}_1(t; N)$ works. In Figure 3(a), $k(t)$ is approximated using the Fourier series with $N = 1000$, and in Figure 3(b) we see the output of $\text{edge}_1(t; 1000)$.

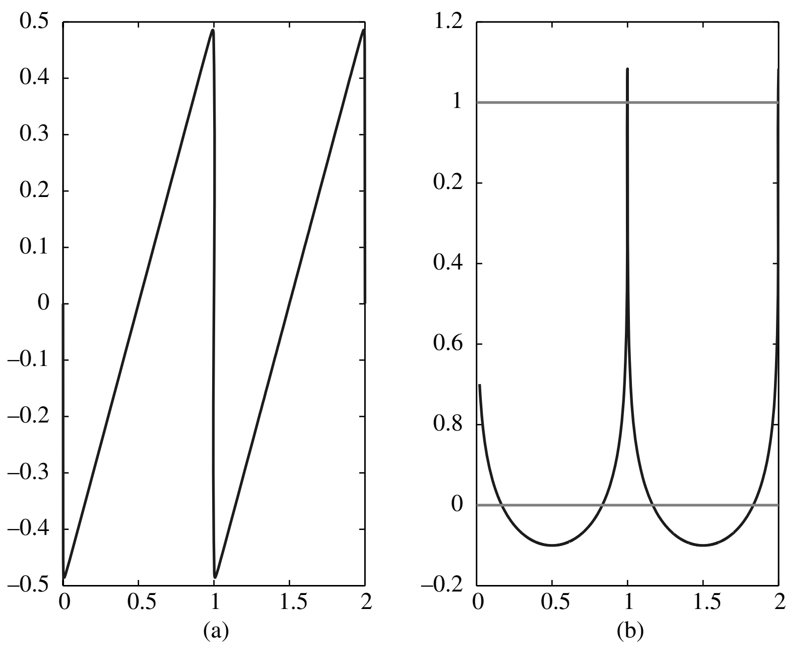


Figure 3. The function $k(t)$ and the output of $\text{edge}_1(t; 1000)$.

Upon looking at Figure 3(b), two points are immediately obvious. First of all, the measured value of the jump, which should be exactly 1—is about 1.08. Second of all, even though N is rather large, the points away from the jump are not particularly close to zero.

The second point is *the* fundamental problem with this method. Because we divide a finite number by $\ln(N)$, and because $\ln(N)$ does not increase quickly, we need a very large value of N in order to force the points away from the jumps to zero.

The first problem, however, is curable. Consider the partial sums that correspond to the harmonic series again. We substituted $\ln(N)$ for the partial sum. It is well known that

$$\lim_{N\to\infty}\left(\left(\sum_{k=1}^{N}\frac{1}{k}\right)-\ln(N)\right)\equiv\gamma=0.577215\ldots.$$

Furthermore, it has been shown [**13**] that

$$\frac{1}{2(N+1)}<\left(\sum_{k=1}^{N}\frac{1}{k}\right)-\ln(N)-\gamma<\frac{1}{2N}.$$

The constant γ is known as the Euler-Mascheroni constant. Rather than dividing the sum by $\ln(N)$, we divide it by $\ln(N)+\gamma$. This defines a second, improved, edge detector, $\text{edge}_2(t;N)$:

$$\begin{aligned}\text{edge}_2(t;N)&\equiv\frac{1}{\ln(N)+\gamma}\left(\sum_{n=-N}^{-1}ic_ne^{i2\pi nt}+\sum_{n=1}^{N}-ic_ne^{i2\pi nt}\right)\\&=\frac{\ln(N)}{\ln(N)+\gamma}\text{edge}_1(t;N).\end{aligned}$$

The improved edge detector returns Figure 4. Here the jump is indeed measured as one unit, but the convergence away from the jumps is still very slow.

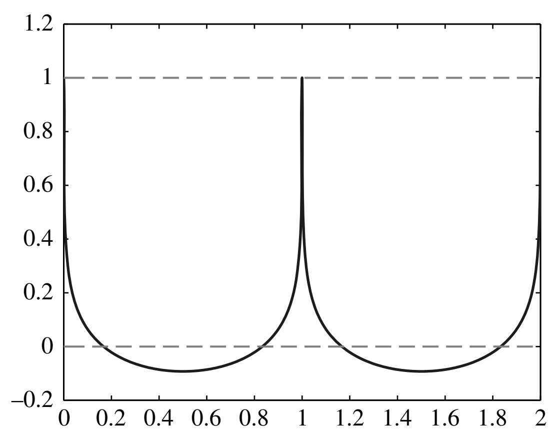


Figure 4. The output of $\text{edge}_2(t;1000)$.

5. A BETTER AND SIMPLER TECHNIQUE. If one's goal is to determine the locations of the discontinuities of a function, there is no reason to require that the processing of the Fourier coefficients only minimally affect the coefficients. The problem with the previous method was that we were dividing a bounded function that we wanted to force to zero by $\ln(N)$. This caused the decrease towards zero away from the jumps to be very slow. It would be better to divide the bounded part by something larger, if possible.

Let us consider the following method of transforming the Fourier coefficients of our data. If c_n are the coefficients of the function, let the transformed coefficients s_n be

$$s_n=-inc_n.$$

This transformation emphasizes the high frequency components of the original function (and gives, up to a constant factor, the Fourier coefficients of the derivative of the original function). As jump discontinuities have a large high frequency component, it is reasonable that such a transformation can help us locate jump discontinuities.

Let us consider the function that one recovers when one starts with the coefficients that correspond to the shifted sawtooth wave, $k(t-\tau)$:

$$c_n = \begin{cases} \dfrac{ie^{-i2\pi n\tau}}{2\pi n} & n \neq 0, \\ 0 & n = 0. \end{cases}$$

We find that the partial sums using the transformed coefficients are

$$g_N(t) = \sum_{n=-N}^{N} s_n e^{i2\pi nt} = \frac{1}{\pi}\sum_{n=1}^{N}\cos(2\pi n(t-\tau)).$$

Considering the definition and the properties of the Dirichlet kernel (as developed in Section 3.1), one finds that

$$g_N(t) = \frac{1}{\pi}\frac{D_N(\xi)-1}{2} = \frac{\sin((2N+1)\pi\xi)}{2\pi\sin(\pi\xi)} - \frac{1}{2\pi}.$$

We see that the partial sums are bounded as long as ξ is not an integer. When ξ is an integer, the sums equals N/π.

Note that the transformation performed on the coefficients causes the series associated with the discontinuous part to diverge like N/π. The coefficients of the continuous part, on the other hand, will not diverge as quickly. In fact, using arguments similar to those of Section 2 it is easy to show that for a sufficiently smooth continuous part the sum will be absolutely and uniformly convergent.

Therefore, we can produce an effective edge detector by considering

$$\text{edge}_3(t; N) \equiv \frac{\pi}{N}\sum_{n=-N}^{N} s_n e^{i2\pi nt} = -\frac{\pi}{N}\sum_{n=-N}^{N} inc_n e^{i2\pi nt}. \tag{4}$$

The discontinuous piece contributes a component that converges to the height of the jump at the location of the jump and tends to zero like $1/N$ away from the jump. The continuous piece, if it is smooth enough, will decay as $1/N$ as well. This technique is superior to the preceding one (except insofar as it is invasive—it requires that the l^1 norm of the coefficients be greatly altered). In Figure 5(b) we see the output of $\text{edge}_3(t; 100)$ for $k(t) = t - 1/2$, and in Figure 5(d) we see the output of $\text{edge}_3(t; 1000)$ for the same input. For the sawtooth wave, the latter detector performs just as one would hope, and even the former gives reasonable results.

In Figure 6 we consider the output of $\text{edge}_3(t; 1000)$ when the input has two jumps (of heights 2 and -2) in close proximity to one another. We see that $\text{edge}_3(t; 1000)$ detects all the jumps quite cleanly.

In Figure 7 we see the input to and the output of $\text{edge}_3(t; N)$ when the input to the detector is a large continuous waveform with small jumps (of height 1.5) embedded in it. The output of $\text{edge}_3(t; 1000)$ is much cleaner than the output of $\text{edge}_3(t; 100)$. As explained above, as N increases the influence of any smooth pieces decreases while the influence of the jumps remains the same.

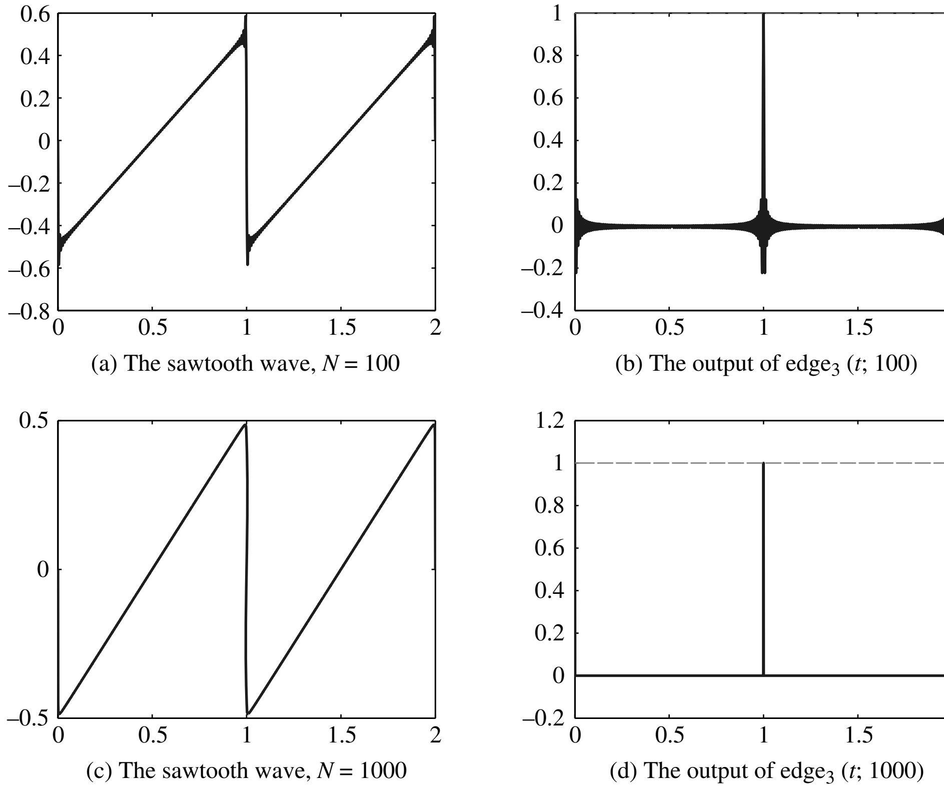


Figure 5. The edges of $k(t)$ as detected by $\text{edge}_3(t;\, N)$.

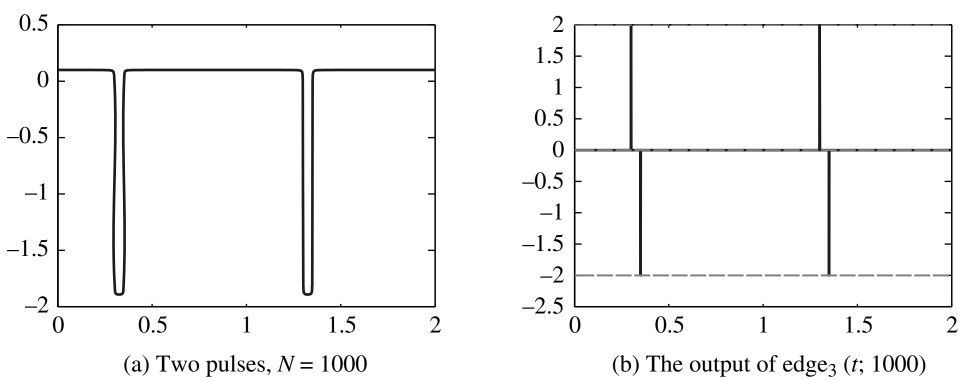
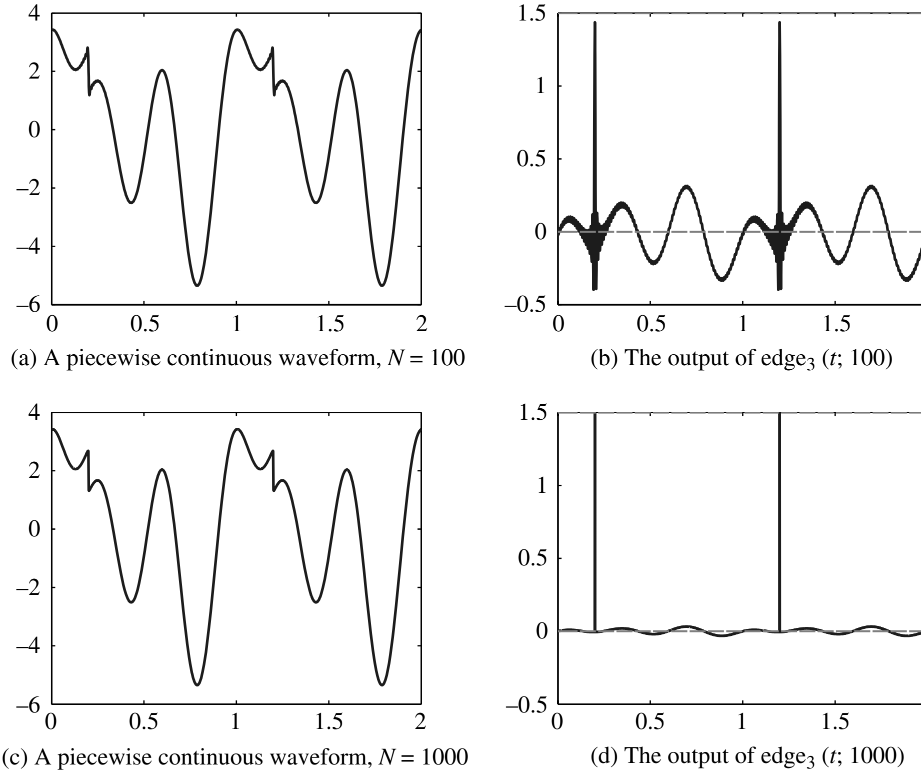


Figure 6. The input to and output of $\text{edge}_3(t;\, 1000)$ when the data has two jumps (of heights 2 and -2) in close proximity to one another.

In the case at hand, there are only a few nonzero Fourier coefficients in the Fourier series expansion of the smooth part of the data of Figure 7. Thus, for all sufficiently large N the sum in (4) is fixed. The number that multiplies the sum in (4), however, decreases linearly with N. Thus, the influence of the smooth part decreases linearly with increasing N while the influence of the jumps is unchanged by increasing N. This explains more precisely why the output of $\text{edge}_3(t;\, 1000)$ is much cleaner than that of $\text{edge}_3(t;\, 100)$.



(a) A piecewise continuous waveform, $N = 100$ (b) The output of edge_3 (t; 100)

(c) A piecewise continuous waveform, $N = 1000$ (d) The output of edge_3 (t; 1000)

Figure 7. The input to and output of $\text{edge}_3(t;\, N)$ when the data has a large continuous part.

6. CONCLUSIONS. We have developed three edge detectors that work by using spectral data about a function to "concentrate" the function about its discontinuities. The first two detectors are based on the Hilbert transform and are minimally invasive; unfortunately they are not very effective. The third method is both invasive and effective. All three methods have been carefully analyzed using techniques that are elegant and elementary.

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Mathematics Is . . .

"Mathematics is an activity governed by the same rules imposed upon the symphonies of Beethoven, the paintings of DaVinci, and the poetry of Homer."

Edward Kasner and James Newman, *Mathematics and the Imagination*, Simon & Schuster, New York, 1940, p. 362.

"Mathematics is like music, freely exploring the possibilities of form."

George Santayana, *The Realm of Truth: Book Third of Realms of Being*, Charles Scribner's Sons, New York, 1938, p. 2.

—Submitted by Carl C. Gaither, Killeen, TX