

ENTROPY VISCOSITY FOR THE EULER EQUATIONS AND QUESTIONS REGARDING PARABOLIC REGULARIZATION

JEAN-LUC GUERMOND AND BOJAN POPOV

Department of Mathematics
Texas A&M University
College Station, TX 77843-3368, USA

ABSTRACT. This note describes a general class of regularizations for the compressible Euler equations. A unique regularization is identified that is compatible with all the generalized entropies à la [8] and satisfies the minimum entropy principle. All the results announced herein will be reported in detail in [5].

1. Introduction. A new numerical method for approximating nonlinear conservation laws using an artificial viscosity based on entropy production has been described in [4, 6, 16]. This so-called entropy viscosity method uses finite elements, either continuous or discontinuous, and consists of augmenting the numerical discretization at hand with a parabolic regularization where the nonlinear viscosity is based on the local size of a discrete entropy production. The idea of using the entropy to design numerical methods for nonlinear conservation equations is not new. For instance it is shown in [11] that the entropy production can be used as an a posteriori error indicator and therefore is useful for adaptive strategies. The main originality of the entropy viscosity method is that one directly uses the entropy production to construct an artificial viscosity. This strategy makes an automatic distinction between shocks and contact discontinuities. This method is simple to program and does not use any flux or slope limiters. The method can be reasonably justified for scalar conservation equations. For instance it is now well established that the solution of the parabolic regularization of a scalar conservation equation converges to the entropy solution as the regularization parameter goes to zero. This fundamental fact is the key justification for constructing approximation techniques based on artificial viscosity. Stability results have been established in [10] and [1] for fully discrete versions of the entropy viscosity method for scalar conservation equations using simple entropies. The extension of this strategy to hyperbolic systems is not so clear, since the question of how parabolic regularizations should be constructed for hyperbolic systems is still an open problem. In particular, our experience is that the Navier-Stokes system is not a robust regularization of the Euler system,

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one key reason being that there is no mechanism therein to help the density to stay positive, another one being that the Navier-Stokes regularization is known to violate the minimum entropy principle if the thermal diffusivity is zero.

The objective of this note is to investigate a nonstandard family of regularization of the Euler system that can serve as a reasonable starting point for an entropy viscosity technique. We identify a single family that preserves positivity of the density, satisfies a minimum entropy principle (see [14]), and is compatible with the largest class of generalized entropies inequalities of [8].

2. Statement of the problem. Consider the compressible Euler equations in conservative form in \mathbb{R}^d ,

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0, \quad \mathbf{U}(\mathbf{x}, 0) = (\rho_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x}), E(\mathbf{x}))^T, \quad (1)$$

where $\mathbf{U} = (\rho, \mathbf{m}, E)^T$, $\mathbf{F}(\mathbf{U}) = (\mathbf{m}, \mathbf{u} \otimes \mathbf{m} + p\mathbb{I}, \mathbf{u}(E + p))^T$. The dependent variables are the density, ρ , the momentum, \mathbf{m} and the total energy, E . We adopt the usual convention that for any vectors \mathbf{a}, \mathbf{b} , with entries $\{a_i\}_{i=1,\dots,d}$, $\{b_i\}_{i=1,\dots,d}$, the following holds: $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$ and $\nabla \cdot \mathbf{a} = \partial_{x_j} a_j$, $(\nabla \mathbf{a})_{ij} = \partial_{x_i} a_j$. Moreover, for any order 2 tensors \mathbf{g}, \mathbf{h} , with entries $\{g_{ij}\}_{i,j=1,\dots,d}$, $\{h_{ij}\}_{i,j=1,\dots,d}$, we define $(\nabla \cdot \mathbf{g})_j = \partial_{x_i} g_{ij}$, $\mathbf{a} \cdot \nabla = a_i \partial_{x_i}$, $(\mathbf{g} \cdot \mathbf{a})_i = g_{ij} a_j$, $\mathbf{g} : \mathbf{h} = g_{ij} h_{ij}$ where repeated indices are summed from 1 to d .

The equation of state is assumed to derive from a specific entropy, $s(\rho, e)$, through the thermodynamics identity: $T ds := de + p d\tau$, where $\tau := \rho^{-1}$, $e := \rho^{-1}E - \frac{1}{2}\mathbf{u}^2$ is the specific internal energy, $\mathbf{u} := \rho^{-1}\mathbf{m}$ is the velocity of the fluid particles. For instance it is usual to take $s = \log(e^{\frac{1}{\gamma-1}}\rho^{-1})$ for a polytropic ideal gas. Using the notation $s_e := \frac{\partial s}{\partial e}$ and $s_\rho := \frac{\partial s}{\partial \rho}$, this definition implies that $s_e := T^{-1}$, $s_\rho := -pT^{-1}\rho^{-2}$. The equation of state takes the form $p := -\rho^2 s_e s_\rho^{-1}$. The key structural assumption is that $-s$ is strictly convex with respect to $\tau := \rho^{-1}$ and e . Upon introducing $\sigma(\tau, e) := s(\rho, e)$, the convexity hypothesis is equivalent to assuming that $\sigma_{\tau\tau} \leq 0$, $\sigma_{ee} \leq 0$, and $\sigma_{\tau\tau}\sigma_{ee} - \sigma_{\tau e}^2 \leq 0$. This in turn implies that $\partial_\rho(\rho^2 s_\rho) < 0$, $s_{ee} < 0$, $0 < \partial_\rho(\rho^2 s_\rho)s_{ee} - \rho^2 s_{\rho e}^2$, or equivalently that the following matrix

$$\Sigma := \begin{pmatrix} \rho^{-1}\partial_\rho(\rho^2 s_\rho) & \rho s_{\rho e} \\ \rho s_{\rho e} & \rho s_{ee} \end{pmatrix}, \quad (2)$$

is negative definite. In the rest of the note we assume that the entropy is strictly convex and the temperature is positive, i.e., $0 < s_e$.

A physical way to regularize the Euler system (1) consists of considering this system as the limit of the Navier-Stokes equations. We claim that the Navier-Stokes regularization is not appropriate for numerical purposes. The first problem that we identify is that the minimum entropy principle cannot be satisfied for general initial data if the thermal dissipation is not zero. More precisely, assuming that the thermal diffusivity is nonzero, for any $r \in \mathbb{R}$, there exist initial data so that the set $\{s \geq r\}$ is not positively invariant, where s is the specific entropy, see e.g., [12, Thm 8.2.3]. Another argument often invoked against the presence of thermal dissipation is that it is incompatible with symmetrization of the Navier-Stokes system when using the generalized entropies of [7] for polytropic ideal gases. The function $\rho f(s)$ is said to be a generalized entropy if $f'(\gamma-1)\gamma^{-1} - f'' > 0$, $f' > 0$ and $f \in C^2(\mathbb{R}; \mathbb{R})$. It is proved in [9] that the only generalized entropy that symmetrizes the Navier-Stokes system is the trivial one ρs when the thermal diffusivity is nonzero, see also [15, (2.11) and Remark 2, page 460]. Although symmetrization of the viscous fluxes is not necessary

to establish entropy dissipation (see e.g., [13, §1.1]), it is nevertheless true that the Navier-Stokes system violates generalized entropy inequalities if $f''(s) \neq 0$.

The objective of this note is to introduce a regularization of (1) that is compatible with thermodynamics and can be used for numerical approximations.

3. General regularization. We investigate in this section the properties of following general regularization for the Euler system:

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \nabla \cdot \mathbf{T}, \quad \mathbf{U}(\mathbf{x}, 0) = (\rho_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x}), E(\mathbf{x}))^T, \quad (3)$$

where $\mathbf{T} = (\mathbf{f}, \mathbf{g}, \mathbf{h} + \mathbf{g} \cdot \mathbf{u})^T$ and the fluxes \mathbf{f} , \mathbf{g} , and \mathbf{h} are as general as possible. A regularization theory for general nonlinear hyperbolic system has been developed in [13] and [12, Chap 6]. Our objective in this note is more restrictive. We want to construct the fluxes \mathbf{f} , \mathbf{g} , and \mathbf{h} so that (3) gives a positive density, gives a minimum principle on the specific entropy, and is compatible with a large class of entropies. It is assumed in the rest of the note that (3) has a smooth solution.

3.1. Positivity of the density. Modulo mild regularity assumptions on the velocity, the theory of second-order elliptic equations implies that $\mathbf{f} = a(\rho, e)\nabla\rho$ is appropriate to guaranty the positivity of the density, where $a(\rho, e)$ is a smooth positive function. The following is established in [5]

Lemma 3.1 (Positive Density Principle). *Let $\mathbf{f} = a(\rho, e)\nabla\rho$ in (3), with $a \in L^\infty(\mathbb{R}^2; \mathbb{R})$ and $\inf_{(\xi, \eta) \in \mathbb{R}^2} a(\xi, \eta) > 0$. Assume that \mathbf{u} and $\nabla \cdot \mathbf{u} \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Assume also that there are constant states at infinity ρ^∞ , \mathbf{u}^∞ , so that the supports of $\rho(\cdot, \cdot) - \rho^\infty$ and $\mathbf{u}(\cdot, \cdot) - \mathbf{u}^\infty$ are compact in $\mathbb{R}^d \times (0, t)$, for any $t > 0$. Assume finally that $\rho_0 - \rho^\infty \in L^2(\mathbb{R}^d; \mathbb{R})$. Then the solution of (3) is such that*

$$\text{ess inf}_{\mathbf{x} \in \mathbb{R}^d} \rho(\mathbf{x}, t) \geq 0, \quad \forall t \geq 0. \quad (4)$$

3.2. Minimum entropy principle. Since physically admissible weak solutions of the Euler equations satisfy the following inequality $\partial_t s + \mathbf{u} \cdot \nabla s \geq 0$, they also satisfy a minimum entropy principle, i.e., the set $\{s \geq s_0\}$, where s_0 is the infimum of the specific entropy of the initial data, is positively invariant. The importance of the minimum entropy principle has been established by [14].

Requesting that the triple \mathbf{f} , \mathbf{h} and \mathbb{G} be such that the solution of (3) satisfies a minimum entropy principle narrows down the choices that can be made for the viscous fluxes. It is shown in [5] that the following structure is sufficient for this purpose:

$$\mathbf{f} = a(\rho, e)\nabla\rho \quad a(\rho, e) \geq 0, \quad (5)$$

$$\mathbf{g} = \mathbb{G}(\nabla^s \mathbf{u}) + \mathbf{f} \otimes \mathbf{u}, \quad \mathbb{G}(\nabla^s \mathbf{u}) : \nabla \mathbf{u} \geq 0, \quad (6)$$

$$\mathbf{h} = \mathbf{l} - \frac{1}{2}\mathbf{u}^2 \mathbf{f}, \quad \mathbf{l} = (a - d)(p\rho^{-1} + e)\nabla\rho + d\nabla(\rho e) \quad d(\rho, e) \geq 0. \quad (7)$$

Theorem 3.2 (Minimum Entropy Principle). *Assume that ρ_0 and e_0 are constant outside some compact set. Assume also that (5)-(6)-(7) hold. Assume that the solution to (3) is smooth, then the minimum entropy principle holds,*

$$\text{ess inf}_{\mathbf{x} \in \mathbb{R}^d} s(\mathbf{x}, t) \geq \text{ess inf}_{\mathbf{x} \in \mathbb{R}^d} s_0(\mathbf{x}), \quad \forall t \geq 0.$$

3.3. Generalized entropies. We investigate in this section whether the regularization of the Euler equations (3) is compatible with some or all generalized entropy inequalities identified in [8]. A function $\rho f(s)$ is called a generalized entropy if f is twice differentiable and

$$f'(s) > 0, \quad f'(s)c_p^{-1} - f''(s) > 0, \quad \forall(\rho, e) \in \mathbb{R}_+^2, \quad (8)$$

where $c_p(\rho, e) = T\partial_T s(p, T)$ is the specific heat at constant pressure. It is shown in [8] that $-\rho f(s)$ is strictly convex with respect to ρ^{-1} and e if and only if (8) holds, i.e., (8) characterizes the maximal set of admissible entropies for the compressible Euler equations that are of the form $\rho f(s)$. The following result is proved in [5]:

Theorem 3.3 (Entropy Inequalities). *Assume that (6)-(5)-(7) hold. Any weak solution to the regularized system (3) satisfies the entropy inequality*

$$\partial_t(\rho f(s)) + \nabla \cdot (\mathbf{u} \rho f(s) - d\rho \nabla f(s) - af(s)\nabla \rho) \geq 0, \quad (9)$$

for all generalized entropies $\rho f(s)$ if and only if $a = d$.

Corollary 1. *Any weak solution to the regularized system (3) satisfies the entropy inequality (9) for the physical entropy ρs (i.e., $f(s) = s$) if $2\Gamma - 2\Delta^{\frac{1}{2}} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac{1}{2}}$ where $\Gamma = \det(\Sigma)\rho^2 s_e^{-2} p_e^{-2}$ and $\Delta = \Gamma(1 + \Gamma)$.*

In the case of a polytropic ideal gases, i.e., $s = \log(e^{\frac{1}{\gamma-1}} \rho^{-1})$ with $\gamma > 1$, we have $c_p = \gamma(\gamma-1)^{-1}$, $\det(\Sigma) = (\gamma-1)^{-1}e^{-2}$, $\mathbf{f} = a\nabla \rho$, and $\mathbf{l} = \gamma de(\frac{a}{d} - 1 + \frac{1}{\gamma})\nabla \rho + d\rho \nabla e$. The range for the ratio ad^{-1} for Corollary 1 to hold is

$$\frac{2}{\gamma-1}(1 - \sqrt{\gamma}) < 1 - \frac{a}{d} < \frac{2}{\gamma-1}(1 + \sqrt{\gamma}). \quad (10)$$

In particular the choice $1 - \frac{a}{d} = \frac{1}{\gamma}$ is clearly in the admissible range. For this choice $\mathbf{l} = d\rho \nabla e$ and $\mathbf{f} = d\frac{\gamma-1}{\gamma}\nabla \rho$, i.e., \mathbf{l} does not involve any mass dissipation.

4. Conclusions. We show in this section that the regularization proposed above reconciles the Navier-Stokes and the parabolic regularization points of view.

4.1. Parabolic regularization. One natural question that comes to mind is how different is the general regularization (3) from the simple parabolic regularization:

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \epsilon \Delta \mathbf{U}, \quad \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}), \quad (11)$$

where $\mathbf{U} = (\rho, \mathbf{m}, E)^T$, $\mathbf{F}(\mathbf{U}) = (\mathbf{m}, \mathbf{u} \otimes \mathbf{m} + p\mathbb{I}, \mathbf{u}(E + p))^T$. The answer is given by the following, somewhat a priori frustrating result:

Proposition 1 (Parabolic regularization). *The parabolic regularization (11) is identical to (3) with (6)-(7) where $a = d = \epsilon$, $\mathbb{G} = \epsilon \rho \nabla \mathbf{u}$.*

Even when $a = d$, one important interest of the class of regularization (3), when compared to the monolithic parabolic regularization (11), is that it decouples the regularization on the velocity from that on the density and internal energy. In particular the regularization on the velocity can be made rotation invariant by making the tensor \mathbb{G} a function of the symmetric gradient $\nabla^s \mathbf{u}$. This decoupling was not a priori evident when looking at (11).

4.2. Connection with phenomenological models. Using the assumptions (6)–(7) in the balance equation (3) we obtain the following system:

$$\partial_t \rho + \nabla \cdot \mathbf{m} - \nabla \cdot \mathbf{f} = 0, \quad (12)$$

$$\partial_t \mathbf{m} + \nabla \cdot (\mathbf{u} \otimes \mathbf{m}) + \nabla p - \nabla \cdot (\mathbb{G}(\nabla^s \mathbf{u}) + \mathbf{f} \otimes \mathbf{u}) = 0, \quad (13)$$

$$\partial_t E + \nabla \cdot (\mathbf{u}(E + p)) - \nabla \cdot (\mathbf{l} + \frac{1}{2} \mathbf{u}^2 \mathbf{f} + \mathbb{G}(\nabla^s \mathbf{u}) \cdot \mathbf{u}) = 0, \quad (14)$$

When looking at (12)–(14) it is not immediately clear how this system can be reconciled either with the Navier-Stokes regularization or with any phenomenological modeling of dissipation. It is remarkable that this exercise can actually been done by introducing the quantity $\mathbf{u}_m = \mathbf{u} - \rho^{-1} \mathbf{f}$. The above conservation equations then become

$$\partial_t \rho + \nabla \cdot (\mathbf{u}_m \rho) = 0, \quad (15)$$

$$\partial_t \mathbf{m} + \nabla \cdot (\mathbf{u}_m \otimes \mathbf{m}) + \nabla p - \nabla \cdot (\mathbb{G}(\nabla^s \mathbf{u})) = 0, \quad (16)$$

$$\partial_t E + \nabla \cdot (\mathbf{u}_m E) - \nabla \cdot (\mathbf{l} - e \mathbf{f}) + \nabla \cdot ((p \mathbb{I} - \mathbb{G}(\nabla^s \mathbf{u})) \cdot \mathbf{u}) = 0. \quad (17)$$

This system resembles the Navier-Stokes regularization with two velocities. If one sets $a = d$, the term $\mathbf{l} - e \mathbf{f}$ becomes $d \rho \nabla e$, which upon assuming $de = c_v dt$, reduces to $d(\rho, e) \rho c_v \nabla T$, i.e., one obtains Fourier's law: $\mathbf{l} - e \mathbf{f} = d(\rho, e) \rho c_v \nabla T$.

The system (15)–(17) resembles, at least formally, a model of fluid dynamics of [2] (see e.g., equations (1) to (5) in [2]). The author has derived the above system of conservation equations (up to some non-essential disagreement on the term $\mathbf{l} - e \mathbf{f}$) by invoking phenomenological considerations. The mathematical properties of this system have been investigated by [3]. Brenner has been defending for years the idea that it makes phenomenological sense to distinguish the so-called mass velocity, \mathbf{u}_m , from the so-called volume velocity, \mathbf{u} . This idea seems to be supported by our mathematical derivation which did not invoke any had oc phenomenological assumption. Recall that our primal motivation in this project is to find a regularization of the compressible Euler equations that can serve as a good numerical device, and by being good we mean that the model must give positive density, positive internal energy, a minimum entropy principle and be compatible with a large class of entropy inequalities.

4.3. Concluding remarks. Let us finally rephrase our findings. In its most general form, the regularized system (15)–(17) can be re-written as follows:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}_m \rho) = 0, \quad (18)$$

$$\partial_t \mathbf{m} + \nabla \cdot (\mathbf{u}_m \otimes \mathbf{m}) + \nabla p - \nabla \cdot (G(\nabla^s \mathbf{u})) = 0, \quad (19)$$

$$\partial_t E + \nabla \cdot (\mathbf{u}_m E) - \nabla \cdot \mathbf{q} + \nabla \cdot ((p \mathbb{I} - G(\nabla^s \mathbf{u})) \cdot \mathbf{u}) = 0 \quad (20)$$

$$\mathbf{u}_m = \mathbf{u} - a(\rho, e) \nabla \log \rho \quad (21)$$

$$\mathbf{q} = (a - d) p \nabla \log \rho + d \rho \nabla e, \quad a(\rho, e) \geq 0, \quad d(\rho, e) \geq 0. \quad (22)$$

It is established in Lemma 3.1 that the definition of $\mathbf{f} = a(\rho, e) \nabla \rho$ is compatible with the positive density principle. The particular form of \mathbf{q} in (22) results from the definition of \mathbf{l} , see (7), which is required for the minimum entropy principle to hold, as established in Theorem 3.2. It is finally proved in Theorem 3.3 that the most robust regularization, i.e., that which is compatible with all the generalized entropy à la [8], corresponds to the choice $a = d$. A relaxation of the constraint $a = d$ is described in Corollary 1. As observed in §4.1, the parabolic regularization

can be put into the form (18)–(22) with the particular choice $\mathbb{G} = a\nabla\mathbf{u}$, which is not rotation invariant and uses the same viscosity coefficient for all fields.

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E-mail address: guermond@math.tamu.edu

E-mail address: popov@math.tamu.edu