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VISCOUS REGULARIZATION OF THE EULER EQUATIONS AND ENTROPY PRINCIPLES*

JEAN-LUC GUERMOND† AND BOJAN POPOV†

Abstract. This paper investigates a general class of viscous regularizations of the compressible Euler equations. A unique regularization is identified that is compatible with all the generalized entropies, à la [Harten et al., *SIAM J. Numer. Anal.*, 35 (1998), pp. 2117–2127], and satisfies the minimum entropy principle. A connection with a recently proposed phenomenological model by [H. Brenner, *Phys. A*, 370 (2006), pp. 190–224] is made.

Key words. conservation equations, hyperbolic systems, parabolic regularization, entropy, viscosity solutions

AMS subject classifications. 76N15, 35L65, 65M12

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1. Introduction.

Proving positivity of the density and internal energy and proving a minimum principle on the specific entropy of numerical approximations of the compressible Euler equations is a challenging task that has so far been achieved for very few numerical schemes on arbitrary meshes in two and higher space dimensions. The Godunov scheme [6] and some variants of the Lax[1] scheme [12] are known to satisfy all these properties (see [2] for the Godunov scheme, [14, appendix] for the explicit Lax algorithm, and [19] for the implicit version of the Lax algorithm). The argumentation for the Godunov scheme relies on the fact that Riemann problems are solved exactly at each time step, and averaging Riemann solutions preserves the above-mentioned properties. None of the above arguments can be readily extended to central high-order schemes or more generally to schemes that are based on Galerkin approximations. One way to address this issue consists of using the standard parabolic regularization of the Euler equations to construct a scheme for which the vanishing viscosity is proportional to the mesh size. The problem with this approach is that the regularization acts on the conserved variables, which are the density, momentum, and total energy. Since the momentum and total energy are not Galilean invariant, a change of reference frame by translation and/or rotation changes the regularization. A way out of this dilemma consists of considering the Navier–Stokes regularization as a starting point for constructing a numerical method. However, one then encounters two serious difficulties. The first is that the Navier–Stokes equations do not include any regularization in the continuity equation, which is inconsistent with most numerical discretizations. The second is that, whereas it is known that the Euler equations satisfy a minimum entropy principle on the specific entropy (see, e.g., [17]), it is also known that the Navier–Stokes equations violate this minimum principle if the thermal

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†Department of Mathematics, Texas A&M University, College Station, TX 77843-3368 (guermond@math.tamu.edu, popov@math.tamu.edu). The work of the first author was performed while on leave from CNRS, France.

[1]The Lax scheme is often called the Lax–Friedrichs scheme in the literature.

diffusivity is nonzero; see, e.g., [15, Theorem 8.2.3]. These two observations make the Navier–Stokes regularization inconvenient for numerical purposes. One is then led to ponder the following question: Is it possible to find a regularization of the Euler equations that is Galilean invariant, ensures positivity of the density and internal energy, satisfies a minimum entropy principle, and is compatible with a large class of entropy inequalities? The objective of this paper is to propose answers to this question.

The paper is organized as follows. The parabolic and the Navier–Stokes regularizations and their apparent shortcomings mentioned above are discussed in section 2. A general family of regularizations is introduced and investigated in sections 3 and 4. The minimum entropy principle is investigated in section 3, and the compatibility with entropy inequalities is studied in section 4. The key result of this paper is Theorem 4.1: Only one regularization technique satisfies the minimum entropy principle and is compatible with all the generalized entropies of [9]. This formulation is compared in section 5 with a reformulation of the Navier–Stokes equations proposed by [1] that is based on heuristic arguments. A striking observation is that, by distinguishing the so-called mass and volume velocities, it is possible to rewrite the proposed regularization into a form similar to that of the Navier–Stokes equations with rotation invariant viscous fluxes. This way of looking at the regularization reconciles the parabolic and Navier–Stokes regularizations and shows that these are two faces of the same coin. The key results of the paper are summarized in subsection 5.3 and illustrated in subsection 5.4. Standard identities and inequalities from thermodynamics that are used in this paper are collected in Appendix A.

2. Standard regularizations.

We review in this section some well-known regularization techniques and discuss the pros and cons thereof.

2.1. Statement of the problem.

Consider the compressible Euler equations in conservative form in $\mathbb{R}^d$,

$$\partial_t\rho + \nabla\cdot\boldsymbol{m} = 0, \tag{2.1}$$

$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p = 0, \tag{2.2}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) = 0, \tag{2.3}$$

$$\rho(\boldsymbol{x},0) = \rho_0, \qquad \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0, \qquad E(\boldsymbol{x},0) = E_0, \tag{2.4}$$

where the dependent variables are the density, ρ, the momentum, $\boldsymbol{m}$, and the total energy, E. We adopt the usual convention that for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, with entries $\{a_i\}_{i=1,\dots,d}$, $\{b_i\}_{i=1,\dots,d}$, the following holds: $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} = a_i b_j$ and $\nabla\cdot\boldsymbol{a} = \partial_{x_j} a_j$, $(\nabla\boldsymbol{a})_{ij} = \partial_{x_i} a_j$. Moreover, for any order 2 tensors $\mathbb{g}$, $\mathbb{h}$, with entries $\{g_{ij}\}_{i,j=1,\dots,d}$, $\{h_{ij}\}_{i,j=1,\dots,d}$, we define $(\nabla\cdot\mathbb{g})_j = \partial_{x_i} g_{ij}$, $\boldsymbol{a}\cdot\nabla = a_i\partial_{x_i}$, $(\mathbb{g}\cdot\boldsymbol{a})_i = g_{ij}a_j$, and $\mathbb{g}{:}\mathbb{h} = g_{ij}\mathbb{h}_{ij}$, where repeated indices are summed from 1 to d.

The pressure, p, is given by the equation of state, which we assume to derive from a specific entropy, $s(\rho,e)$, defined through the thermodynamics identity:

$$T\,\mathrm{d}s := \mathrm{d}e + p\,\mathrm{d}\tau, \tag{2.5}$$

where $\tau := \rho^{-1}$, $e := \rho^{-1}E - \frac{1}{2}\boldsymbol{u}^2$ is the specific internal energy, and $\boldsymbol{u} := \rho^{-1}\boldsymbol{m}$ is the velocity of the fluid particles. For instance, it is common to take $s = \log(e^{\frac{1}{\gamma-1}}\rho^{-1})$ for a polytropic ideal gas. Using the notation $s_e := \frac{\partial s}{\partial e}$ and $s_\rho := \frac{\partial s}{\partial \rho}$, the identity (2.5) implies that

$$s_e := T^{-1}, \qquad s_\rho := -pT^{-1}\rho^{-2}. \tag{2.6}$$

The equation of state takes the form $p := -\rho^2 s_\rho s_e^{-1}$ or

$$ps_e + \rho^2 s_\rho = 0. \tag{2.7}$$

The key structural assumption is that $-s$ is strictly convex with respect to $\tau := \rho^{-1}$ and e. Upon introducing $\sigma(\tau, e) := s(\rho, e)$, the convexity hypothesis is equivalent to assuming that $\sigma_{\tau\tau} \le 0$, $\sigma_{ee} \le 0$, and $\sigma_{\tau\tau}\sigma_{ee} - \sigma_{\tau e}^2 \le 0$ (see, e.g., [5]). This in turn implies that

$$\partial_\rho(\rho^2 s_\rho) < 0, \qquad s_{ee} < 0, \qquad 0 < \partial_\rho(\rho^2 s_\rho)s_{ee} - \rho^2 s_{\rho e}^2, \tag{2.8}$$

or equivalently that the matrix

$$\mathbb{\Sigma} := \begin{pmatrix} \rho^{-1}\partial_\rho(\rho^2 s_\rho) & \rho s_{\rho e} \\ \rho s_{\rho e} & \rho s_{ee} \end{pmatrix} \tag{2.9}$$

is negative definite. In the rest of the paper we assume that (2.8) holds and that the temperature is positive,

$$0 < s_e. \tag{2.10}$$

Remark 2.1. Note in passing that, contrary to what is sometimes done in the literature, we do not assume that the pressure is positive, which requires $s_\rho < 0$ (see, e.g., [5, p. 99], [9, (2.3)]). For instance, the assumptions (2.8) and (2.10) hold for stiffened gases, but the quantity s_ρ can change sign. It is shown in the appendix (see Remark A.1) that the convexity assumption (2.8) and the positivity of the temperature (2.10) are sufficient to prove that the Euler system is hyperbolic. This fact was first established by [7] in one dimension. It was established again in [4] and [9].

The objective of the present paper is to introduce a viscous regularization of (2.1)–(2.4) that is compatible with thermodynamics and that can serve as a reasonable starting point for numerical approximation.

2.2. Monolithic parabolic regularization. A common regularization of (2.1) for theoretical and numerical purposes consists of the following monolithic parabolic regularization:

$$\partial_t \rho + \nabla\cdot\boldsymbol{m} = \epsilon\Delta\rho, \tag{2.11}$$

$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p = \epsilon\Delta m, \tag{2.12}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) = \epsilon\Delta E, \tag{2.13}$$

$$\rho(\boldsymbol{x},0) = \rho_0, \qquad \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0, \qquad E(\boldsymbol{x},0) = E_0, \tag{2.14}$$

where ϵ is a small parameter. We call this regularization monolithic since no distinction is made between the conserved quantities; i.e., the operator $\epsilon\Delta$ is applied blindly to all the conserved quantities.

It can be shown that the Lax–Friedrichs scheme and its parabolic analogue introduced in [14] are approximations of (2.11). For instance, considering a nonlinear conservation equation $\partial_t \boldsymbol{U} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U}) = 0$, where $\boldsymbol{U}$ is the dependent vector-valued variable in $\mathbb{R}^m$, the scheme introduced in [12, p. 163] in one space dimension consists of considering

$$\begin{aligned} \boldsymbol{U}_i^{n+1} &= \frac{1}{2}(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n) - \frac{1}{2}\lambda(\boldsymbol{F}(\boldsymbol{U}_{i+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{i-1}^n)) \\ &= \boldsymbol{U}_i^n - \frac{1}{2}\lambda(\boldsymbol{F}(\boldsymbol{U}_{i+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{i-1}^n)) + \tau\frac{1}{2}h^2\tau^{-1}\frac{(\boldsymbol{U}_{i+1}^n - 2\boldsymbol{U}_i^n + \boldsymbol{U}_{i-1}^n)}{h^2}, \end{aligned} \tag{2.15}$$

where h is the mesh size, τ is the time step, and $\lambda := \tau h^{-1}$. Assuming the flux $\boldsymbol{F}$ to be uniformly Lipschitz, to simplify, and upon introducing the maximum wave speed $\beta := \|\boldsymbol{F}'\|_{L^\infty(\mathbb{R}^m;\mathbb{R}^m\times\mathbb{R}^m)}$ and the CFL number cfl $:= \beta\tau h^{-1}$, (2.15) is the centered second-order approximation of the following parabolic regularization of the conservation equation, $\partial_t\boldsymbol{U} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U}) - \epsilon\Delta\boldsymbol{U} = 0$, with the artificial viscosity $\epsilon := \frac{1}{2}h\lambda^{-1} = \frac{1}{\text{cfl}}\frac{1}{2}\beta h$. In other words, the Lax–Friedrichs scheme is a centered second-order approximation of (2.11)–(2.14) with the numerical viscosity $\epsilon = \frac{1}{\text{cfl}}\frac{1}{2}h\|\|\boldsymbol{u}\| + c\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}_+)}$, where c is the speed of sound. That the CFL number appears at the denominator of the artificial viscosity makes this scheme over-dissipative. It is often more appropriate to consider the following alternative:

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{1}{2}\lambda(\boldsymbol{F}(\boldsymbol{U}_{i+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{i-1}^n)) + \frac{1}{2}\lambda|\beta|h^2\frac{(\boldsymbol{U}_{i+1}^n - 2\boldsymbol{U}_i^n + \boldsymbol{U}_{i-1}^n)}{h^2},$$

which is also a centered second-order approximation of the parabolic regularization $\partial_t\boldsymbol{U} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U}) - \epsilon\Delta\boldsymbol{U} = 0$ with the viscosity $\frac{1}{2}\beta h$, which is more traditionally associated with up-winding. This algorithm is often abusively referred to as the Lax–Friedrichs scheme. Both the above numerical schemes have interesting positivity and entropy properties; see, e.g., [11, 17, 18, 14].

Despite its appealing mathematical properties, the above regularization is often criticized by physicists since it seemingly violates the Galilean and rotational invariance. It also dissipates the density, the momentum, and the total energy, which seemingly are again aberrations from the physical point of view. When looking at (2.11)–(2.14), it is indeed difficult to see how this set of equations can be reconciled with the Navier–Stokes equations, which are usually viewed by physicists to be the acceptable regularization of the Euler equations.

2.3. Navier–Stokes regularization. As mentioned above, a common "physical" way to regularize the Euler system (2.1)–(2.4) consists of considering this system as the limit of the Navier–Stokes equations

$$\partial_t\rho + \nabla\cdot\boldsymbol{m} = 0, \tag{2.16}$$

$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p - \nabla\cdot\mathbb{g} = 0, \tag{2.17}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) - \nabla\cdot(\boldsymbol{h} + \mathbb{g}\cdot\boldsymbol{u}) = 0, \tag{2.18}$$

$$\rho(\boldsymbol{x},0) = \rho_0, \qquad \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0, \qquad E(\boldsymbol{x},0) = E_0, \tag{2.19}$$

where $\mathbb{g}$ and $\boldsymbol{h}$ are the viscous and thermal fluxes. The most elementary model compatible with Galilean invariance consists of assuming that

$$\mathbb{g} = 2\mu\nabla^s\boldsymbol{u} + \lambda\nabla\cdot\boldsymbol{u}\mathbb{I}, \qquad \boldsymbol{h} = \kappa\nabla T, \tag{2.20}$$

where $\nabla^s\boldsymbol{u} := \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$, $\mathbb{I}$ is the identity matrix in $\mathbb{R}^d$, and T is the temperature, $T := s_e^{-1}$. The viscosity μ and the thermal diffusivity κ are required to be nonnegative by the Clausius–Duhem inequality, although these two parameters may depend on the state (ρ, e).

We claim that (2.16)–(2.20) is not appropriate for numerical purposes, and we identify at least two obstructions. The first problem is that the minimum entropy principle cannot be satisfied for general initial data if the thermal dissipation is not zero. More precisely, assuming $\kappa \neq 0$, for any $r \in \mathbb{R}$ there exist initial data so that the set $\{s \geq r\}$ is not positively invariant. Let us recall a simple proof of this statement borrowed from [15, Theorem 8.2.3]. The specific entropy for the Navier–Stokes system satisfies

$$\partial_t s + \boldsymbol{u}\cdot\nabla s = \frac{1}{\rho T}\left(\mathbb{g}{:}\nabla^s \boldsymbol{u} + \nabla\cdot(\kappa\nabla T)\right). \tag{2.21}$$

Assume that $\boldsymbol{u}_0 := \boldsymbol{m}_0\rho_0^{-1}$ is constant. Assume also that the equation of state of the fluid is such that $p_e \neq 0$. Then one can use T and s as independent state variables, since $\rho^2\det\left(\frac{D(T,s)}{D(\rho,e)}\right) = \frac{\rho^2}{s_e^2}(s_\rho s_{ee} - s_e s_{\rho e}) = p_e \neq 0$ (see (A.6)). One can then choose s_0 with global minimum at 0 and T_0 so that $\Delta T_0(0) < 0$ and $\nabla T_0(0) = 0$. Without loss of generality, we assume that $\kappa > 0$ in a neighborhood of 0. Then $\partial_t s(0,0) = \kappa\rho_0^{-1}(0)T_0(0)^{-1}\Delta T_0(0) < 0$, thereby proving that $\{s \geq r\}$ is not positively invariant for the regularized system (2.16)–(2.20).

Another argument often invoked against the presence of thermal dissipation is that it is incompatible with symmetrization of the Navier–Stokes system when using the generalized entropies of Harten for polytropic ideal gases. The function $\rho f(s)$ is said to be a generalized entropy if $f'(\gamma-1)\gamma^{-1} - f'' > 0$, $f' > 0$, and $f \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$; see [8]. (Note that the above inequality slightly differs from that in [8] since Harten's definition of the specific entropy is $s = \log(e\rho^{1-\gamma})$.) It is proved in [10] that the only generalized entropy that symmetrizes the Navier–Stokes system (2.16)–(2.20) is the trivial one ρs when $\kappa \neq 0$; see also [18, equation (2.11) and Remark 2, p. 460]. Note, though, that symmetrization of the viscous fluxes is not a necessary condition for proving entropy dissipation; see, e.g., [16, section 1.1]. It is nevertheless true that the Navier–Stokes system with $\kappa \neq 0$ does not admit a generalized entropy inequality if $f''(s) \neq 0$, and this fact is a simple consequence of the following quadratic form not being nonnegative: $f'(s)X^2 - f''(s)XY$, $(X,Y) \in \mathbb{R}^2$.

The above two arguments seem to imply that one should take $\kappa = 0$ if one wants to use the Navier–Stokes system as a numerical device that regularizes the Euler equations, satisfies the minimum entropy principle, and satisfies entropy inequalities. In that case, one then faces a serious obstruction when solving for contact waves. For instance, assuming that the initial data, ρ_0, $\boldsymbol{m}_0$, $\boldsymbol{E}_0$ are such that the exact velocity is constant in time and space, say $\boldsymbol{u} = \beta\boldsymbol{e}_x$, the problem (2.16)–(2.19) reduces to solving two linear transport equations:

$$\partial_t\rho + \beta\partial_x\rho = 0, \quad \rho(\cdot,0) = \rho_0, \tag{2.22}$$

$$\partial_t E + \beta\partial_x E = 0, \quad E(\cdot,0) = E_0. \tag{2.23}$$

Note that $\boldsymbol{u}$ being constant implies that the pressure gradient is zero. The exact solution is $\rho(\boldsymbol{x},t) = \rho_0(\boldsymbol{x} - \beta t\boldsymbol{e}_x)$. To make things a little bit more interesting, assume that ρ_0 is piecewise constant, say $\rho_0(x) = 1$ if $x < 0$ and $\rho_0(x) = 2$ if $x > 0$. In the absence of some sort of regularization, the above two linear transport equations are difficult to solve numerically. Except for the method of characteristics and Lagrangian-based techniques, we are not aware of any numerical methods that can solve these equations without resorting to some kind of viscous regularization.

In conclusion, if positivity of the density, the minimum entropy principle, and a reasonable approximation of contact discontinuities is desired, the Navier–Stokes regularization does not seem to be appropriate to regularize (2.1)–(2.4), whether κ is zero or not.

3. General regularization. We investigate in this section the properties of a class of regularizations that we expect to be as general as possible. More precisely, let us consider the following general regularization for the Euler system:

$$\partial_t \rho + \nabla\cdot \boldsymbol{m} - \nabla\cdot \boldsymbol{f} = 0, \tag{3.1}$$

$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p - \nabla\cdot \mathbb{g} = 0, \tag{3.2}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) - \nabla\cdot(\boldsymbol{h} + \mathbb{g}\cdot\boldsymbol{u}) = 0, \tag{3.3}$$

where for the time being we let the fluxes $\boldsymbol{f}$, $\mathbb{g}$, and $\boldsymbol{h}$ be as general as possible. A theory of viscous regularization for general nonlinear hyperbolic system has been developed in [16] and [15, Chapter 6]. This theory identifies classes of entropy-dissipative viscous regularizations and establishes short-term existence results. Our objective in this paper is more restrictive. We want to construct the fluxes $\boldsymbol{f}$, $\mathbb{g}$, and $\boldsymbol{h}$ so that (3.1)–(3.3) gives a positive density, gives a minimum principle on the specific entropy, and is compatible with a large class of entropies. (Note in passing that the positivity of the internal energy will be a consequence of the positivity of the density and the minimum entropy principle.) In the rest of the paper, we are going to work under the assumption that (3.1)–(3.3) has a smooth solution.

3.1. Positivity of the density. Let us now choose the flux $\boldsymbol{f}$ so that it regularizes the mass conservation equation. From the theory of second-order elliptic equations we conjecture that $a(\rho,e)\nabla\rho$ should be appropriate, where $a(\rho,e)$ is a smooth positive function of ρ and e. In particular, it is reasonable to expect that the following choice implies positivity of the density:

$$a(\rho,e) = \chi(\rho,e)\varphi'(\rho), \tag{3.4}$$

where χ is a smooth positive function of ρ and e and φ is a strictly increasing function. This definition gives $\boldsymbol{f} = \chi(\rho,e)\nabla\varphi(\rho)$. This regularization is at least compatible with the nonnegative density principle as stated in the following lemmas.

Lemma 3.1 (nonnegative density principle). *Let* $\boldsymbol{f} = a(\rho,e)\nabla\rho$ *in* (3.1), *with* $a \in L^\infty(\mathbb{R}^2;\mathbb{R})$ *and* $\inf_{(\xi,\eta)\in\mathbb{R}^2} a(\xi,\eta) > 0$. *Assume that the smooth solution of* (3.1) *satisfies* (i) $\boldsymbol{u}$ *and* $\nabla\cdot\boldsymbol{u} \in L^1(\mathbb{R}^d\times\mathbb{R}_+;\mathbb{R})$*;* (ii) $a\nabla\rho \in L^1((0,\infty);L^1(\mathbb{R}^d))$*; and* (iii) $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) \in L^1((0,\infty);L^1(\mathbb{R}^d))$. *Then the solution of* (3.1) *is such that*

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathbb{R}^d} \rho(\boldsymbol{x},t) \geq 0 \qquad \forall t \geq 0. \tag{3.5}$$

Proof. Let $\epsilon > 0$, and let $h_\epsilon(x)$ be a smooth concave function that approximates $\min(x,0)$ uniformly over $\mathbb{R}$; say there is $c > 0$ so that $\sup_{s\in\mathbb{R}} |h_\epsilon(s) - \min(s,0)| + |h_\epsilon(s) - sh'_\epsilon(s)| < c\epsilon$ and $h''_\epsilon \leq 0$. Let $t > 0$ be some fixed time. (In the remainder of the paper c is a generic constant whose value may change at each occurrence.) Let $B(0,R)$ be the ball centered at 0 of radius R such that

$$\int_0^t \int_{\mathbb{R}^d\setminus B(0,R)} |a\nabla\rho| + |\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u})| \,\mathrm{d}\boldsymbol{x} < \epsilon.$$

Let χ be a regularized characteristic function with the following properties: $\chi|_{B(0,R)} = 1$ and $\chi|_{\mathbb{R}^d\setminus B(0,R+1)} = 0$. Multiplying the weak form of (3.1) by the legitimate test function $\chi h'_\epsilon(\rho)$, we obtain

$$\int_{\mathbb{R}^d} \Big(h'_\epsilon(\rho)\chi(\boldsymbol{x})(\partial_t\rho + \nabla\cdot(\boldsymbol{u}\rho)) + a\nabla\rho\nabla(\chi h'_\epsilon(\rho)) \Big) \,\mathrm{d}\boldsymbol{x} = 0.$$

Using that $h''_\epsilon \leq 0$, we infer that

$$\int_{\mathbb{R}^d} \left(\chi h'_\epsilon(\rho)(\partial_t \rho + \nabla\cdot(\boldsymbol{u}\rho)) + a h'_\epsilon(\rho)\nabla\rho\cdot\nabla\chi\right) \mathrm{d}\boldsymbol{x} \geq 0.$$

This in turn implies that

$$\begin{aligned}
\int_{\mathbb{R}^d} & h'_\epsilon(\rho)(\partial_t\rho + \nabla\cdot(\boldsymbol{u}\rho)) \,\mathrm{d}\boldsymbol{x} \\
&\geq \int_{\mathbb{R}^d} (1-\chi) h'_\epsilon(\rho)(\partial_t\rho + \nabla\cdot(\boldsymbol{u}\rho)) \,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^d} a h'_\epsilon(\rho)\nabla\rho\cdot\nabla\chi \,\mathrm{d}\boldsymbol{x} \\
&\geq - \int_{\mathbb{R}^d\setminus B(0,R)} |h'_\epsilon(\rho)(\partial_t\rho + \nabla\cdot(\boldsymbol{u}\rho))| \,\mathrm{d}\boldsymbol{x} - c\int_{\mathbb{R}^d\setminus B(0,R)} |a\nabla\rho| \,\mathrm{d}\boldsymbol{x} \\
&\geq -c \int_{\mathbb{R}^d\setminus B(0,R)} (|a\nabla\rho| + |\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u})|) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Observing that $h'_\epsilon(\rho)(\partial_t\rho + \nabla\cdot(\boldsymbol{u}\rho)) = \partial_t h_\epsilon(\rho) + \nabla\cdot(h_\epsilon(\rho)\boldsymbol{u}) + (\rho h'_\epsilon(\rho) - h_\epsilon(\rho))\nabla\cdot\boldsymbol{u}$, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\partial_t h_\epsilon(\rho) + \nabla\cdot(h_\epsilon(\rho)\boldsymbol{u})\right) \mathrm{d}\boldsymbol{x} &\geq - \int_{\mathbb{R}^d} |(\rho h'_\epsilon(\rho) - h_\epsilon(\rho))\nabla\cdot\boldsymbol{u}| \,\mathrm{d}\boldsymbol{x} \\
&\quad - c\int_{\mathbb{R}^d\setminus B(0,R)} (|a\nabla\rho| + |\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u})|) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Now, we proceed as in [17]. We introduce the cone $\mathcal{C} := \{(\boldsymbol{x},\tau) \mid \|\boldsymbol{x}\| \leq R + (t-\tau)\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty},\ 0 \leq \tau \leq t\}$ and, using the above estimate together with the assumptions regarding the behavior of $\boldsymbol{u}$, ρ, and a, we infer that

$$\int_{\mathcal{C}} \left(\partial_t h_\epsilon(\rho) + \nabla\cdot(h_\epsilon(\rho)\boldsymbol{u})\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \geq \int_0^t\int_{\mathbb{R}^d} \left(\partial_t h_\epsilon(\rho) + \nabla\cdot(h_\epsilon(\rho)\boldsymbol{u})\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t - c\epsilon \geq -c\epsilon.$$

Denoting by $\boldsymbol{n} = (\boldsymbol{n_x}, n_t)$ the unit exterior normal to the mantle, $\mathcal{M}$, of the cone $\mathcal{C}$, we have $\boldsymbol{n_x} = (1 + \|\boldsymbol{u}\|^2_{\boldsymbol{L}^\infty})^{-\frac{1}{2}} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ and $n_t = (1 + \|\boldsymbol{u}\|^2_{\boldsymbol{L}^\infty})^{-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}$, and we observe that $(n_t + \boldsymbol{u}\cdot\boldsymbol{n_x}) \geq 0$. Then,

$$\begin{aligned}
\int_{\mathcal{C}} \left(\partial_t h_\epsilon(\rho) + \nabla\cdot(h_\epsilon(\rho)\boldsymbol{u})\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \int_{\|\boldsymbol{x}\|\leq R} h_\epsilon(\rho(\boldsymbol{x},t)) \,\mathrm{d}\boldsymbol{x} - \int_{\|\boldsymbol{x}\|\leq R+t\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}} h_\epsilon(\rho_0(\boldsymbol{x})) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\mathcal{M}} h_\epsilon(\rho(\boldsymbol{x},t))(n_t + \boldsymbol{u}\cdot\boldsymbol{n_x}) \,\mathrm{d}\mathcal{M} \\
&\leq \int_{\|\boldsymbol{x}\|\leq R} h_\epsilon(\rho(\boldsymbol{x},t)) \,\mathrm{d}\boldsymbol{x} - \int_{\|\boldsymbol{x}\|\leq R+t\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}} h_\epsilon(\rho_0(\boldsymbol{x})) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

In conclusion, we obtain $\int_{\mathbb{R}^d} h_\epsilon(\rho(\boldsymbol{x},t)) \,\mathrm{d}\boldsymbol{x} \geq -c\,\epsilon + \int_{\mathbb{R}^d} h_\epsilon(\rho_0(\boldsymbol{x}))$. We can now pass to the limit on ϵ using the Lebesgue dominated convergence, and we obtain $\int_{\mathbb{R}^d} \min(\rho(\boldsymbol{x},t),0) \geq 0$. The result follows readily. ∎

By proceeding as in [3], it is possible to improve the above nonnegativity principle and to establish positivity of the density.

LEMMA 3.2 (positivity density principle). *In addition to the assumptions of Lemma* 3.1, *we further assume that there exists* $\rho_m > 0$ *such that for all* t *there is* R_t

such that $\inf_{\|\boldsymbol{x}\|\geq R_t} \rho(\boldsymbol{x},\tau) \geq \rho_m$ *for all* $\tau \in (0,t)$*, and* $(\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t))_+ \in L^1(0,t;L^\infty(\mathbb{R}^d))$*. Then,*

$$\text{meas}(\{\boldsymbol{x} \in \mathbb{R}^d \mid \rho(\boldsymbol{x},t) = 0\}) = 0 \quad \forall t \geq 0,$$

meaning that the vacuum zones (if any) have zero Lebesgue measure.

Proof. Let ϕ be defined as follows: $\phi(z) = \ln z$ if $0 < z \leq 1$, and $\phi(z) = 0$ if $z > 1$. Note that ϕ is nonpositive, monotonically increasing, and concave and satisfies $|z\phi'(z)| \leq 1$ for all $z > 0$. As in Lemma 3.1, we take a regularization ϕ_ϵ of ϕ which preserves these properties and approximates ϕ uniformly on $\mathbb{R}_+$; say there is $c > 0$ so that $\sup_{z\in\mathbb{R}_+} |\phi_\epsilon(z) - \phi(z)| < c\epsilon$, $|z\phi_\epsilon'(z)| \leq 1$ and $\phi_\epsilon\ \phi_\epsilon'' \leq 0$.

Let $B(0,R)$ be the ball centered at 0 of radius R such that

$$\int_0^t \int_{\mathbb{R}^d\setminus B(0,R)} |a\nabla\rho| + |\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u})| \,\mathrm{d}\boldsymbol{x} < \epsilon$$

and $\inf_{\|\boldsymbol{x}\|\geq R} \rho(\boldsymbol{x},\tau) \geq \rho_m > 0$ for all $\tau \in (0,t)$. We introduce the cone $\mathcal{C} := \{(\boldsymbol{x},\tau) \mid \|\boldsymbol{x}\| \leq R + (t-\tau)\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty},\ 0 \leq \tau \leq t\}$ and the regularized characteristic function χ such that $\chi|_{\{\|\boldsymbol{x}\|+\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}(\tau-t)\leq R\}} = 1$ and $\chi|_{\{\|\boldsymbol{x}\|+\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}(\tau-t)\leq R+1\}} = 0$.

Multiplying the weak form of (3.1) by the test function $\chi\phi_\epsilon'(\rho)$, we obtain

$$\int_{\mathbb{R}^d} \Big((\partial_t\phi_\epsilon(\rho) + \boldsymbol{u}\nabla\phi_\epsilon(\rho) + \rho\phi_\epsilon'(\rho)\nabla\cdot\boldsymbol{u})\chi + a\phi_\epsilon''\chi|\nabla\rho|^2 + a\phi_\epsilon'(\rho)\nabla\rho\nabla\chi\Big) \,\mathrm{d}\boldsymbol{x} = 0.$$

Using the assumptions of the lemma, this also gives

$$\begin{aligned}\int_{\mathbb{R}^d}\chi\Big(\partial_t\phi_\epsilon(\rho) + \nabla\cdot(\boldsymbol{u}\phi_\epsilon(\rho))\Big)\,\mathrm{d}\boldsymbol{x} &\geq \int_{\mathbb{R}^d}\Big(\chi(\phi_\epsilon(\rho) - \rho\phi_\epsilon'(\rho))\nabla\cdot\boldsymbol{u} + a\phi_\epsilon'(\rho)\nabla\rho\nabla\chi\Big)\,\mathrm{d}\boldsymbol{x} \\ &\geq -c\int_{\mathbb{R}^d}(|\nabla\cdot\boldsymbol{u}| + |a\nabla\rho|)\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d}\phi_\epsilon(\nabla\cdot\boldsymbol{u})_+\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

By proceeding as in the proof of the previous lemma, using the properties of χ and ϕ_ϵ, we infer that

$$\begin{aligned}\int_{\mathcal{C}}\Big(\partial_t\phi_\epsilon(\rho) + \nabla\cdot(\boldsymbol{u}\phi_\epsilon(\rho))\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &\geq \int_{\mathbb{R}^d}\chi\Big(\partial_t\phi_\epsilon(\rho) + \nabla\cdot(\boldsymbol{u}\phi_\epsilon(\rho))\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\quad - c\int_0^t\int_{\mathbb{R}^d\setminus B(0,R)}\Big(|\nabla\cdot\boldsymbol{u}| + |\partial_t\rho + \nabla\cdot(\boldsymbol{u}\rho)|\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\end{aligned}$$

which, owing to the definition of the cone $\mathcal{C}$, in turn implies that

$$\int_{\|\boldsymbol{x}\|\leq R}(-\phi_\epsilon(\rho))\,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^d}(-\phi_\epsilon(\rho_0))\,\mathrm{d}\boldsymbol{x} + c + \int_0^t\int_{\mathbb{R}^d}(-\phi_\epsilon)(\nabla\cdot\boldsymbol{u})_+\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

We now pass to the limit on $R \to +\infty$,

$$\int_{\mathbb{R}^d}(-\phi_\epsilon(\rho))\,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^d}(-\phi_\epsilon(\rho_0))\,\mathrm{d}\boldsymbol{x} + c + \int_0^t\left[\sup_{\boldsymbol{x}\in\mathbb{R}^d}(\nabla\cdot\boldsymbol{u})_+\right]\int_{\mathbb{R}^d}(-\phi_\epsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Applying Gronwall's inequality for $\int_{\mathbb{R}^d}(-\phi_\epsilon(\rho(\boldsymbol{x},t)))\,\mathrm{d}\boldsymbol{x}$ and passing to the limit on ϵ, we conclude that

$$\int_{\mathbb{R}^d}|\min(0,\ln(\rho(\boldsymbol{x},t)))|\,\mathrm{d}\boldsymbol{x} \leq c$$

for any $t > 0$, which implies that $\text{meas}\{x \in \mathbb{R}^d \mid \rho(\boldsymbol{x}, t) = 0\} = 0$. ∎

Remark 3.1. The assumptions of Lemma 3.2 are reasonable. They are fully justified in [3]. The existence of $\rho_m > 0$ such that for all t there is R_t such that $\inf_{\|\boldsymbol{x}\| \geq R_t} \rho(\boldsymbol{x}, \tau) \geq \rho_m$ for all $\tau \in (0, t)$ is related to the fact that ρ converges to an unperturbed state at infinity; i.e., finite speed of propagation plus exponential decay of the parabolic regularization do not perturb the state at infinity too much. The assumption $(\nabla\cdot\boldsymbol{u}(\boldsymbol{x}, t))_+ \in L^1(0, t; L^\infty(\mathbb{R}^d))$ can be justified by using techniques similar to those in [3].

3.2. Minimum entropy principle. We now investigate under which conditions on the fluxes $\boldsymbol{f}$, $\mathbb{g}$, and $\boldsymbol{h}$ a minimum principle on the specific entropy holds. To simplify the computations and to account for the impact of the viscous part in the mass conservation equation, we change the notation of the various viscous fluxes introduced in (3.1)–(3.3) and assume that the following structural properties hold:

$$\mathbb{g} = \mathbb{G}(\nabla^s \boldsymbol{u}) + \boldsymbol{f} \otimes \boldsymbol{u}, \qquad \boldsymbol{h} = \boldsymbol{l} - \tfrac{1}{2}\boldsymbol{u}^2 \boldsymbol{f}, \qquad \mathbb{G}(\nabla^s \boldsymbol{u}){:}\nabla\boldsymbol{u} \geq 0, \tag{3.6}$$

$$\boldsymbol{f} = a(\rho, e)\nabla\rho, \qquad a(\rho, e) \geq 0, \tag{3.7}$$

$$\boldsymbol{l} = s_e^{-1}(e s_e - \rho s_\rho)\boldsymbol{f} + d(\rho, e)\rho s_e^{-1}\nabla s, \qquad d(\rho, e) \geq 0. \tag{3.8}$$

LEMMA 3.3. *The specific entropy for the system* (3.1)–(3.3) *satisfies*

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) - \nabla\cdot(\rho d \nabla s) - \boldsymbol{f}\cdot\nabla(e s_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e - s_e \mathbb{G}{:}\nabla\boldsymbol{u} = 0. \tag{3.9}$$

Proof. We rewrite (3.1)–(3.3) in nonconservative form as follows:

$$\begin{aligned}
&\partial_t\rho + \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{f} = 0,\\
&\rho(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \boldsymbol{u}\nabla\cdot\boldsymbol{f} + \nabla p - \nabla\cdot\mathbb{g} = 0,\\
&\rho(\partial_t\mathcal{E} + \boldsymbol{u}\cdot\nabla\mathcal{E}) + \mathcal{E}\nabla\cdot\boldsymbol{f} + \nabla\cdot(\boldsymbol{u}p) - \nabla\cdot(\boldsymbol{h} + \mathbb{g}\cdot\boldsymbol{u}) = 0,
\end{aligned}$$

where we have defined $\mathcal{E} = \rho^{-1}E$. Then we obtain the equation controlling the internal energy, $e = \mathcal{E} - \frac{1}{2}\boldsymbol{u}^2$, by multiplying the momentum equation by $\boldsymbol{u}$ and subtracting the result from the total energy equation:

$$\rho(\partial_t e + \boldsymbol{u}\cdot\nabla e) + (e - \tfrac{1}{2}\boldsymbol{u}^2)\nabla\cdot\boldsymbol{f} + p\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{h} - \mathbb{g}{:}\nabla\boldsymbol{u} = 0.$$

The key to obtaining the equation that controls the entropy is to multiply the mass conservation by ρs_ρ, multiply the internal energy balance by s_e, and add the two resulting equations. This linear combination is motivated by the following observation, $\partial_\alpha s = s_\rho \partial_\alpha \rho + s_e \partial_\alpha e$, which holds for any independent variable $\alpha \in \{t, \boldsymbol{x}\}$. We then obtain

$$\begin{aligned}
\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + s_e(e - \tfrac{1}{2}\boldsymbol{u}^2)\nabla\cdot\boldsymbol{f} + (p s_e + \rho^2 s_\rho)\nabla\cdot\boldsymbol{u}&\\
- s_e(\nabla\cdot\boldsymbol{h} + \mathbb{g}{:}\nabla\boldsymbol{u}) - \rho s_\rho \nabla\cdot\boldsymbol{f} &= 0.
\end{aligned}$$

The definition of the pressure implies that the quantity $p s_e + \rho^2 s_\rho$ is zero; see (2.7). This simplification yields

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + (e s_e - \rho s_\rho)\nabla\cdot\boldsymbol{f} - s_e(\mathbb{g}{:}\nabla\boldsymbol{u}) - s_e\tfrac{1}{2}\boldsymbol{u}^2\nabla\cdot\boldsymbol{f} - s_e\nabla\cdot\boldsymbol{h} = 0.$$

We now regroup the terms,

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + (e s_e - \rho s_\rho)\nabla\cdot\boldsymbol{f} - s_e\nabla\cdot(\boldsymbol{h} + \tfrac{1}{2}\boldsymbol{u}^2\boldsymbol{f}) - s_e(\mathbb{g}{:}\nabla\boldsymbol{u} - (\boldsymbol{f}\otimes\boldsymbol{u}){:}\nabla\boldsymbol{u}) = 0,$$

and conclude by using the definitions $\mathbb{G}(\nabla^s \boldsymbol{u}) := \mathbb{g} - \boldsymbol{f}\otimes\boldsymbol{u}$ and $\boldsymbol{l} = \boldsymbol{h} + \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f}$ with $\boldsymbol{l} := s_e^{-1}(es_e - \rho s_\rho)\boldsymbol{f} + d(\rho,e)\rho s_e^{-1}\nabla s$. ∎

Remark 3.2. The conditions $\mathbb{G}(\nabla^s\boldsymbol{u}){:}\nabla\boldsymbol{u} \geq 0$, $a(\rho,e)\geq 0$, and $d(\rho,e)\geq 0$ are essential to establishing the minimum principle on the specific entropy and the entropy inequalities (see Theorems 3.5 and 4.1).

Remark 3.3. The structural assumption $\boldsymbol{l} = s_e^{-1}(es_e - \rho s_\rho)\boldsymbol{f} + d(\rho,e)\rho s_e^{-1}\nabla s$ is crucial; it implies that $\nabla\cdot((es_e - \rho s_\rho)\boldsymbol{f} - s_e\boldsymbol{l}) = -\nabla\cdot(d\rho\nabla s)$. The definition of $\boldsymbol{l}$ makes sense since thermodynamics requires that $s_e = T^{-1} > 0$ (see (2.10)). Note that, given (3.7), the following alternative expressions hold: $\boldsymbol{l} = (d-a)\rho s_\rho s_e^{-1}\nabla\rho + ae\nabla\rho + d\rho\nabla e$ or $\boldsymbol{l} = (a-d)(p\rho^{-1}+e)\nabla\rho + d\nabla(\rho e)$. Note in particular that $\boldsymbol{l} = d\nabla(\rho e)$ if one chooses $a = d$.

Let us define the quantity

$$J(\nabla\rho,\nabla e) := -\boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e + a\nabla\rho\cdot\nabla s, \tag{3.10}$$

which is a quadratic form with respect to $\nabla\rho$ and ∇e and whose coefficients depend on ρ, e, $a(\rho,e)$, $c(\rho,e)$, and $d(\rho,e)$.

Let $\mathbb{I}_d$ be the $d\times d$ identity matrix. For any symmetric 2×2 block matrix $\mathbb{N}$,

$$\mathbb{N} := \begin{pmatrix} n_{11}\mathbb{I}_d & n_{12}\mathbb{I}_d \\ n_{12}\mathbb{I}_d & n_{22}\mathbb{I}_d\end{pmatrix}, \quad \text{we denote} \quad \mathbb{N}_2 := \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22}\end{pmatrix}.$$

Given row vectors $\boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^d$, the quadratic form $(\boldsymbol{X},\boldsymbol{Y})\cdot\mathbb{N}\cdot(\boldsymbol{X},\boldsymbol{Y})^T$, generated by the 2×2 block matrix $\mathbb{N}$, is negative semidefinite if and only if $\mathbb{N}_2$ is negative semidefinite, i.e., $n_{22}\leq 0$ and $\det(\mathbb{N}_2)\leq 0$.

LEMMA 3.4. *Assume that* (3.7)–(3.8) *hold. The quadratic form* $J(\nabla\rho,\nabla e)$ *is negative semidefinite if and only if*

$$ad\det(\Sigma) - \frac{1}{4}(d-a)^2\rho^{-2}s_e^2p_e^2 \geq 0. \tag{3.11}$$

Moreover, let $\lambda\in\mathbb{R}$ *such that* $d(1+\lambda) = a$; *then*

$$J(\nabla\rho,\nabla e) + \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s \leq 0. \tag{3.12}$$

The inequality (3.12) *becomes strict if* $a>0$ *and* $d>0$.

Proof. Using the definition of $\boldsymbol{l}$, we rewrite J in the following form:

$$\begin{aligned} J = -as_e\nabla\rho\cdot\nabla e - ae\nabla\rho\cdot\nabla s_e + as_\rho|\nabla\rho|^2 + a\rho\nabla\rho\cdot\nabla s_\rho + ae\nabla\rho\cdot\nabla s_e - a\rho s_\rho\nabla\rho\cdot\nabla s_e \\ + d\rho s_e^{-1}\nabla s_e\cdot(s_\rho\nabla\rho + s_e\nabla e) + a\nabla\rho\cdot(s_\rho\nabla\rho + s_e\nabla e).\end{aligned}$$

This expression can be further simplified as follows:

$$\begin{aligned} J(\nabla\rho,\nabla e) &= 2as_\rho|\nabla\rho|^2 + a\rho\nabla\rho\cdot(s_{\rho\rho}\nabla\rho + s_{\rho e}\nabla e) \\ &\quad + (d-a)\rho s_\rho s_e^{-1}\nabla\rho\cdot(s_{\rho e}\nabla\rho + s_{ee}\nabla e) + d\rho\nabla e\cdot(s_{\rho e}\nabla\rho + s_{ee}\nabla e) \\ &= (\nabla\rho,\nabla e)^T\mathbb{N}(\nabla\rho,\nabla e),\end{aligned}$$

where the matrix $\mathbb{N}$ is defined by

$$\mathbb{N} = \begin{pmatrix} n_{11}\mathbb{I}_d & n_{12}\mathbb{I}_d \\ n_{12}\mathbb{I}_d & n_{22}\mathbb{I}_d\end{pmatrix}, \qquad \begin{aligned} n_{11} &= (d-a)\rho s_\rho s_e^{-1}s_{\rho e} + a\rho^{-1}\partial_\rho(\rho^2 s_\rho), \\ 2n_{12} &= (d-a)\rho s_\rho s_e^{-1}s_{ee} + (d+a)\rho s_{\rho e}, \\ n_{22} &= d\rho s_{ee}.\end{aligned}$$

Let us define the 2×2 block matrix $\mathbb{Q}$ obtained by setting $a = 0$ and $d = 1$ in $\mathbb{N}$:

$$q_{11} = \rho s_\rho s_e^{-1} s_{\rho e}, \qquad q_{12} = \rho s_\rho s_e^{-1} s_{ee} + \rho s_{\rho e}, \qquad q_{22} = \rho s_{ee}.$$

Notice that this definition implies that the quadratic form induced by $\mathbb{Q}$ is

$$(\nabla\rho, \nabla e)\cdot\mathbb{Q}\cdot(\nabla\rho, \nabla e)^T = \frac{\rho}{s_e}\nabla s_e\cdot\nabla s.$$

Now let us consider the 2×2 block matrix $\mathbb{M} = \mathbb{N} + \lambda d\mathbb{Q}$, where $\lambda \in \mathbb{R}$. Let us set $d' = d(1 + \lambda)$ and observe that

$$\begin{aligned} m_{11} &= (d' - a)\rho s_\rho s_e^{-1} s_{\rho e} + a\rho^{-1}\partial_\rho(\rho^2 s_\rho), \\ 2m_{12} &= (d' - a)\rho s_\rho s_e^{-1} s_{ee} + (d' + a)\rho s_{\rho e}, \\ m_{22} &= d'\rho s_{ee}. \end{aligned}$$

Observe finally that $J(\nabla\rho, \nabla e) + \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s = (\nabla\rho, \nabla e)\cdot\mathbb{M}\cdot(\nabla\rho, \nabla e)^T$. The matrix $\mathbb{M}$ is negative semidefinite if and only if $m_{22} = d'\rho s_{ee} \le 0$ and $\det(\mathbb{M}_2) \le 0$. Note that $m_{22} \le 0$ if and only if $0 \le d'$ since $s_{ee} < 0$, owing to the convexity assumption (2.8). We also need $\det(\mathbb{M}_2)$ to be nonnegative,

$$\begin{aligned} \det(\mathbb{M}_2) &= ((d' - a)\rho s_\rho s_e^{-1} s_{\rho e} + a\rho^{-1}\partial_\rho(\rho^2 s_\rho))d'\rho s_{ee} \\ &\quad - \tfrac{1}{4}((d' - a)\rho s_\rho s_e^{-1} s_{ee} + (d' + a)\rho s_{\rho e})^2 \\ &= ad'\left(\partial_\rho(\rho^2 s_\rho)s_{ee} - \rho^2 s_{\rho e}^2\right) - \tfrac{1}{4}(d' - a)^2\rho^2 s_e^{-2}(s_e s_{\rho e} - s_\rho s_{ee})^2. \end{aligned}$$

Now if we set λ so that $d' = d(1+\lambda) = a$, then $\det(\mathbb{M}_2)$ is nonnegative and $d' = a \ge 0$. Note in passing that upon setting $\lambda = 0$, this computation shows that $J(\nabla\rho, \nabla e) \le 0$ if and only if (3.11) holds. ∎

Remark 3.4. Note that we could avoid invoking the convexity of the entropy in the above argument by taking $a = 0$ and $\lambda = -1$. This would, however, defeat the purpose of our enterprise, whose primary goal is to find a nonzero viscous regularization of the mass conservation equation that ensures positivity of the density and is entropy-compatible.

Remark 3.5. Note that $J(\nabla\rho, \nabla e) < 0$ when $a = d$.

THEOREM 3.5 (minimum entropy principle). *Let the assumptions of Lemmas* 3.1 *and* 3.2 *hold. Assume that* ρ_0 *and* e_0 *are constant outside some compact set. Assume also that* (3.6)–(3.8) *hold. Assume that the solution to* (3.1)–(3.3) *is smooth; then the minimum entropy principle holds:*

$$\inf_{\boldsymbol{x}\in\mathbb{R}^d} s(\boldsymbol{x}, t) \ge \inf_{\boldsymbol{x}\in\mathbb{R}^d} s_0(\boldsymbol{x}) \qquad \forall t \ge 0.$$

Proof. Using definition (3.10), we have $-\boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e = J - a\nabla\rho\cdot\nabla s$, which implies that (3.9) can be rewritten as follows:

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) - \nabla\cdot(d\rho\nabla s) - a\nabla\rho\cdot\nabla s = -J + s_e\mathbb{G}{:}\nabla\boldsymbol{u} \ge 0. \tag{3.13}$$

Owing to Lemma 3.4, there is $\lambda = \frac{a}{d} - 1$ so that $J + \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s \le 0$. Finally we have proved that

$$\begin{aligned} &\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) - \nabla\cdot(d\rho\nabla s) - (a\nabla\rho + \lambda d\frac{\rho}{s_e}\nabla s_e)\cdot\nabla s \\ &\qquad = -J - \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s + s_e\mathbb{G}{:}\nabla\boldsymbol{u} \ge 0. \end{aligned} \tag{3.14}$$

By assumption, all the fields are smooth and $s \to s^\infty$ uniformly when $|\boldsymbol{x}| \to \infty$ because ρ and e approach the constant states ρ^∞ and e^∞ uniformly when $|\boldsymbol{x}| \to \infty$. For each time t, let us denote $s_{\min}(t) = \inf_{\boldsymbol{x}\in\mathbb{R}^d} \; s(\boldsymbol{x}, t)$. If $s_{\min}(t) = s^\infty$, we have nothing to prove, since $s^\infty \geq \inf_{\boldsymbol{x}\in\mathbb{R}^d} s_0(\boldsymbol{x})$. Otherwise let $x_{\min}(t)$ be one point where the minimum of s is reached; then $\nabla s(x_{\min}(t), t) = 0$ and $\Delta s(x_{\min}(t), t) \geq 0$. Equation (3.14) implies that

$$\rho\partial_t s((x_{\min}(t), t)) - d\rho\Delta s(x_{\min}(t), t) \geq 0,$$

which in turn implies that $\rho\partial_t s((x_{\min}(t), t)) \geq 0$, and using the positivity of density, we conclude that the minimum entropy principle holds. ∎

Remark 3.6. Note that the condition (3.11) is not required to hold in order for the minimum principle to hold.

Remark 3.7. The minimum entropy principle together with the nonnegativity of the density implies that the internal energy is nonnegative. More precisely, let $s_{\min}$ be the infimum of the density at time $t = 0$. Assume that the equation of state is such that $\inf_{\rho\in\mathbb{R}_+} e(s_{\min}, \frac{1}{\rho}) \geq 0$. Then the thermodynamics assumption $s_e > 0$ implies that $e_s > 0$, which in turn gives $e(s, \frac{1}{\rho}) \geq e(s_{\min}, \frac{1}{\rho}) \geq 0$ for all time t since ρ is nonnegative.

4. Entropy inequalities.

We investigate in this section whether the regularization of the Euler equations (3.1)–(3.3) is compatible with some or all generalized entropy inequalities.

4.1. Generalized entropies.

Let us consider all the generalized entropy identified in [9]. A function $\rho f(s)$ is called a generalized entropy if f is twice differentiable and

$$f'(s) > 0, \qquad f'(s)c_p^{-1} - f''(s) > 0 \qquad \forall(\rho, e) \in \mathbb{R}^2_+, \tag{4.1}$$

where $c_p(\rho, e) = T\partial_T s(p, T)$ is the specific heat at constant pressure. It is shown in [9] that $-\rho f(s)$ is strictly convex if and only if (4.1) holds; i.e., (4.1) characterizes the maximal set of admissible entropies for the compressible Euler equations that are of the form $\rho f(s)$.

THEOREM 4.1 (entropy inequalities). *Assume that* (3.6)–(3.8) *hold. Any smooth solution to the regularized system* (3.1)–(3.3) *satisfies the entropy inequality*

$$\partial_t(\rho f(s)) + \nabla\cdot\big(\boldsymbol{u}\rho f(s) - d\rho\nabla f(s) - af(s)\nabla\rho\big) \geq 0 \tag{4.2}$$

for all generalized entropies $\rho f(s)$ if and only if $a = d$.

Proof. Let us multiply (3.13) by $f'(s)$:

$$\begin{aligned}\rho(\partial_t f(s) + \boldsymbol{u}\cdot\nabla f(s)) - \nabla\cdot(d\rho\nabla f(s)) + d\rho f''(s)|\nabla s|^2 - af'(s)\nabla\rho\cdot\nabla s \\ + Jf'(s) = f'(s)s_e\mathbb{G}{:}\nabla\boldsymbol{u}.\end{aligned}$$

We now multiply the mass conservation equation (3.1) by $f(s)$, and we add the result to the above equation:

$$\begin{aligned}\partial_t(\rho f(s)) + \nabla\cdot(\boldsymbol{u}\rho f(s)) - \nabla\cdot(d\rho\nabla f(s) + af(s)\nabla\rho) \\ + d\rho f''(s)|\nabla(s)|^2 + Jf'(s) = f'(s)s_e\mathbb{G}{:}\nabla\boldsymbol{u}.\end{aligned}$$

We now investigate the sign of the quantity $d\rho f''(s)|\nabla s|^2 + Jf'(s)$. Owing to (4.1), we have

$$d\rho f''(s)|\nabla s|^2 + Jf'(s) < (d\rho c_p^{-1}|\nabla s|^2 + J)f'(s). \tag{4.3}$$

We now need to determine the sign of the quadratic form on the right-hand side of the above inequality:

$$\begin{aligned} d\rho c_p^{-1}|\nabla s|^2 + J &= d\rho c_P^{-1}|s_\rho \nabla\rho + s_e \nabla e|^2 + J \\ &= d\rho c_P^{-1}(s_\rho^2|\nabla\rho|^2 + 2s_\rho s_e \nabla\rho\cdot\nabla e + s_e^2|\nabla e|^2) + J = d\rho(\nabla\rho, \nabla e)\cdot\mathbb{S}\cdot(\nabla\rho, \nabla e)^T, \end{aligned}$$

where the coefficients of the 2×2 block matrix $\mathbb{S}$ are defined as

$$\begin{aligned} ds_{11} &= dc_P^{-1}s_\rho^2 + \left((d-a)s_\rho s_e^{-1}s_{\rho e} + a\rho^{-2}\partial_\rho(\rho^2 s_\rho)\right), \\ 2ds_{12} &= 2dc_P^{-1}s_\rho s_e + \left((d-a)s_\rho s_e^{-1}s_{ee} + (d+a)s_{\rho e}\right), \\ ds_{22} &= d(c_P^{-1}s_e^2 + s_{ee}), \end{aligned}$$

and can be rewritten into the following form:

$$\begin{aligned} ds_{11} &= d(c_P^{-1}s_\rho^2 + \rho^{-2}\partial_\rho(\rho^2 s_\rho)) + (d-a)s_e^{-1}\left(s_\rho s_{\rho e} - s_e\rho^{-2}\partial_\rho(\rho^2 s_\rho)\right), \\ 2ds_{12} &= 2d(c_P^{-1}s_\rho s_e + s_{\rho e}) + (d-a)s_e^{-1}\left(s_\rho s_{ee} - s_e s_{\rho e}\right), \\ ds_{22} &= d(c_P^{-1}s_e^2 + s_{ee}). \end{aligned}$$

Then upon setting $x = 1 - \frac{a}{d}$, we infer that

$$s_{11} = h_{11} + x\rho^{-2}s_e p_\rho, \qquad 2s_{12} = 2h_{12} + x\rho^{-2}s_e p_e, \qquad s_{22} = h_{22}, \tag{4.4}$$

where the 2×2 matrix $\mathbb{H}_2$ defined by

$$\mathbb{H}_2 = \begin{pmatrix} s_\rho^2 c_P^{-1} + \rho^{-2}\partial_\rho(\rho^2 s_\rho) & s_\rho s_e c_P^{-1} + s_{\rho e} \\ s_\rho s_e c_P^{-1} + s_{\rho e} & s_e^2 c_P^{-1} + s_{ee} \end{pmatrix}$$

is shown to be negative in Lemma A.3. In particular we have $s_{22} = h_{22} = s_e^2 c_P^{-1} + s_{ee} < 0$ owing to the inequality $c_p T_e > 1$ established in (A.12). As a result, the matrix $\mathbb{S}$ is negative semidefinite if and only if the determinant of $\mathbb{S}_2$ is nonnegative,

$$\begin{aligned} \det(\mathbb{S}_2) &= h_{11}h_{22} + xh_{22}\rho^{-2}s_e p_\rho - (h_{12} + \tfrac{1}{2}x\rho^{-2}s_e p_e)^2 \\ &= \det(\mathbb{H}_2) + x\rho^{-2}s_e(h_{22}p_\rho - h_{12}p_e) - \tfrac{1}{4}x^2\rho^{-4}s_e^2 p_e^2. \end{aligned}$$

According to Lemma A.3 we have $\det(\mathbb{H}_2)$ and $h_{22}p_\rho - h_{12}p_e = 0$. This proves that

$$\det(\mathbb{S}_2) = -\tfrac{1}{4}x^2\rho^{-4}s_e^2 p_e^2.$$

In conclusion, $\mathbb{S}$ is negative semidefinite if and only if $x = 0$, i.e., $a = d$.

The above argument shows that $d\rho f''(s)|\nabla s|^2 + Jf'(s) < 0$ if $a = d$. This proves that all the generalized entropy inequalities are satisfied if $a = d$.

If $a \neq d$, we consider generalized entropies such that $f''(s) = (1-\epsilon)f'(s)c_p(s,\rho)$, $\epsilon \in (0,1)$. (It is always possible to solve this ODE for any fixed value of ρ.) For this subclass of generalized entropies, we have

$$d\rho f''(s)|\nabla s|^2 + Jf'(s) = ((1-\epsilon)d\rho c_p^{-1}|\nabla s|^2 + J)f'(s). \tag{4.5}$$

From the proof of Theorem 4.1, we know that the quadratic form $d\rho c_p^{-1}|\nabla s|^2 + J = d\rho(\nabla\rho, \nabla e)\cdot\mathbb{S}(\rho, e)\cdot(\nabla\rho, \nabla e)^T$ is negative semidefinite if and and only $a = d$. Let $(\rho^*, e^*) \in \mathbb{R}^2_+$ be a pair of positive numbers so that $a(\rho^*, e^*) \neq d(\rho^*, e^*)$. Since the quadratic form generated by $\mathbb{S}(\rho^*, e^*)$ is not negative semidefinite, there exists a pair of row vectors $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^d$ so that $(\boldsymbol{X}, \boldsymbol{Y})\cdot\mathbb{S}(\rho^*, e^*)\cdot(\boldsymbol{X}, \boldsymbol{Y})^T > 0$. It is always possible to choose ϵ small enough so that

$$(\boldsymbol{X}, \boldsymbol{Y})\cdot\mathbb{S}(\rho^*, e^*)\cdot(\boldsymbol{X}, \boldsymbol{Y}) - \epsilon d^*\rho^*(c_p^*)^{-1}|s_\rho^*\boldsymbol{X} + s_e^*\boldsymbol{Y}|^2 f'(s^*) > 0.$$

Now we define an initial state so that in the neighborhood of the origin we have the following data: $\boldsymbol{m}_0 = 0$, $\rho_0(\boldsymbol{x}) = \rho^* + \boldsymbol{x}\cdot\boldsymbol{X}$, $e_0(\boldsymbol{x}) = e^* + \boldsymbol{x}\cdot\boldsymbol{Y}$. Notice that with this choice, $\nabla\boldsymbol{u}_0 = 0$, $\nabla\rho_0 = \boldsymbol{X}$, and $\nabla e_0 = \boldsymbol{Y}$; therefore $d\rho_0 f''(s_0)|\nabla(s_0)|^2 + J(\rho_0, e_0)f'(s_0) - f'(s_0)s_e(\rho_0, e_0)\mathbb{G}{:}\nabla\boldsymbol{u}_0 > 0$, which proves that the entropy inequality is violated at the origin close to the initial time. In conclusion, $a = d$ is a necessary condition for all the generalized entropy inequalities to be satisfied. ∎

Remark 4.1. Upon redefining the velocity $\widetilde{\boldsymbol{u}} = \boldsymbol{u} + (d - a)\nabla\log\rho$, the entropy inequality (4.2) can be rewritten into the following form:

$$\partial_t(\rho f(s)) + \nabla\cdot(\widetilde{\boldsymbol{u}}\rho f(s)) - \nabla\cdot(d\rho\nabla\rho f(s)) \geq 0. \tag{4.6}$$

Remark 4.2. Theorem 4.1 proves that the family of regularization such that $a = d$ is the most robust in the sense that it is the most dissipative. This result suggests that the choice $a = d$ may be a very good candidate for constructing a robust first-order numerical method for solving the compressible Euler equations.

Corollary 4.2. *Let α be a real number, $\alpha < 1$, and assume that* (3.6)–(3.8) *hold. Any smooth solution to the regularized system* (3.1)–(3.3) *satisfies the entropy inequality* (4.2) *for all the generalized entropies $\rho f(s)$ such that $f' > 0$ and $\alpha c_p^{-1} f' \geq f''$ if $2\Gamma - 2\Delta^{\frac12} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac12}$, where $\Gamma = (1-\alpha)det(\mathbb{\Sigma})\rho^2 s_e^{-2} p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$.*

Proof. We proceed as in the proof of Theorem 4.1, replacing $\mathbb{H}$ by $\mathbb{H}^\alpha$, where $\mathbb{H}^\alpha$ is obtained from $\mathbb{H}$ by substituting c_p^{-1} by αc_p^{-1}. Upon replacing c_p^{-1} by αc_p^{-1} in the proof of Lemma A.3, we infer that $\det(\mathbb{H}_2^\alpha) = (1-\alpha)\rho^{-2}\det(\mathbb{\Sigma})$ and $s_e(h_{22}^\alpha p_\rho - h_{21}^\alpha p_e) = (1-\alpha)\rho^{-2}\det(\mathbb{\Sigma})$. Then by defining $\mathbb{S}^\alpha$ as in (4.4), where $\mathbb{H}$ is substituted by $\mathbb{H}^\alpha$, we obtain

$$\begin{aligned}\det(\mathbb{S}_2^\alpha) &= (1-\alpha)\rho^{-2}\det(\mathbb{\Sigma}) + x\rho^{-2}(1-\alpha)\det(\mathbb{\Sigma}) - \tfrac14 x^2\rho^{-4}s_e^2p_e^2\\ &= \rho^{-2}((1-\alpha)\det(\mathbb{\Sigma})(1+x) - \tfrac14 x^2\rho^{-2}s_e^2p_e^2),\end{aligned}$$

where we defined $x = 1 - \frac{a}{d}$. Then upon setting $\Gamma = (1-\alpha)\det(\mathbb{\Sigma})\rho^2 s_e^{-2}p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$, we conclude that the matrix $\mathbb{S}^\alpha$ is negative definite if

$$2\Gamma - 2\Delta^{\frac12} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac12},$$

which ends the proof. ∎

Corollary 4.3. *Any weak solution to the regularized system* (3.1)–(3.3) *satisfies the entropy inequality* (4.2) *for the physical entropy ρs (i.e., $f(s) = s$) if $2\Gamma - 2\Delta^{\frac12} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac12}$, where $\Gamma = det(\mathbb{\Sigma})\rho^2 s_e^{-2} p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$.*

Proof. Take $\alpha = 0$ in Corollary 4.2 or use (3.11). ∎

4.2. Ideal gas. Let us illustrate the above theory in the case of ideal gases, i.e., $s = \log(e^{\frac{1}{\gamma-1}}\rho^{-1})$ with $\gamma > 1$. We have $c^2 = \gamma(\gamma-1)e$, $c_p = \gamma(\gamma-1)^{-1}$, $\det(\Sigma) = (\gamma-1)^{-1}e^{-2}$, $\boldsymbol{f} = a\nabla\rho$, and $\boldsymbol{l} = \gamma de(\frac{a}{d} - 1 + \frac{1}{\gamma})\nabla\rho + d\rho\nabla e$. The range for the ratio ad^{-1} for Corollary 4.3 to hold is

$$\frac{2}{\gamma-1}(1-\sqrt{\gamma}) < 1 - \frac{a}{d} < \frac{2}{\gamma-1}(1+\sqrt{\gamma}). \tag{4.7}$$

In particular, the choice $1 - \frac{a}{d} = \frac{1}{\gamma}$ is clearly in the admissible range for the physical entropy inequality. This particular choice is such that $\boldsymbol{l} = d\rho\nabla e$ and $\boldsymbol{f} = d\frac{\gamma-1}{\gamma}\nabla\rho$; i.e., $\boldsymbol{l}$ does not involve any mass dissipation. The choice $a = d$ implies $\boldsymbol{f} = a\nabla\rho$ and $\boldsymbol{l} = a\nabla(\rho e)$.

5. Discussion. We show in this section that the regularization proposed above is a bridge between the Navier–Stokes and parabolic regularizations of the Euler equations that reconciles the two points of view.

5.1. Parabolic regularization. The first natural question that comes to mind is how different is the general regularization (3.1)–(3.3) from other known regularizations? In particular, how does it differ from the parabolic regularization (2.11)–(2.14)? The answer is given by the following, somewhat a priori frustrating, result.

PROPOSITION 5.1. *The parabolic regularization* (2.11)–(2.13) *is identical to* (3.1)–(3.3) *with* (3.6)-(3.8)*, where* $a = d = \epsilon$*,* $\mathbb{G} = \epsilon\rho\nabla\boldsymbol{u}$*.*

Proof. The equality $a = \epsilon$ comes from the identification $\boldsymbol{f} = \epsilon\nabla\rho$ in the mass conservation equation in (2.11) and (3.1). The identity $\epsilon\nabla\boldsymbol{m} = \epsilon\nabla\rho\otimes\boldsymbol{u} + \epsilon\rho\nabla\boldsymbol{u}$ implies that, upon setting $\mathbb{g} = \boldsymbol{f}\otimes\boldsymbol{u} + \mathbb{G}$ with $\mathbb{G} = \epsilon\rho\nabla\boldsymbol{u}$, the momentum conservation equations in (2.12) and (3.2) are identical. Upon observing that

$$\mathbb{g}\cdot\boldsymbol{u} = \boldsymbol{u}^2\boldsymbol{f} + \mathbb{G}\cdot\boldsymbol{u} = \epsilon\boldsymbol{u}^2\nabla\rho + \frac{1}{2}\epsilon\rho\nabla\boldsymbol{u}^2 = \epsilon\nabla\frac{1}{2}\rho\boldsymbol{u}^2 + \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f},$$

we obtain that

$$\epsilon\nabla\boldsymbol{E} = \epsilon\nabla(\rho e) + \nabla\frac{1}{2}\epsilon\rho\boldsymbol{u}^2 = \epsilon\nabla(\rho e) - \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f} + \mathbb{g}\cdot\boldsymbol{u}.$$

As a result, the energy equations in (2.13) and (3.3) are identical if one sets $\boldsymbol{h} = \boldsymbol{l} - \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f}$, with $\boldsymbol{l} = \epsilon\nabla(\rho e)$, meaning $d = \epsilon$. ∎

Remark 5.1. Even when $a = d$, one important interest of the class of regularization (3.1)–(3.3), when compared to the monolithic parabolic regularization, is that it decouples the regularization on the velocity from that on the density and internal energy. In particular, the regularization on the velocity can be made rotation invariant by making the tensor $\mathbb{G}$ a function of the symmetric gradient $\nabla^s\boldsymbol{u}$. This decoupling was not a priori evident (at least to us) when looking at the monolithic parabolic regularization (2.11)–(2.13).

5.2. Connection with phenomenological models. When introducing the structural assumptions (3.6)–(3.8) into the balance equations (3.1)–(3.3), we obtain the following system:

$$\partial_t\rho + \nabla\cdot\boldsymbol{m} - \nabla\cdot\boldsymbol{f} = 0, \tag{5.1}$$

$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p - \nabla\cdot(\mathbb{G}(\nabla^s\boldsymbol{u}) + \boldsymbol{f}\otimes\boldsymbol{u}) = 0, \tag{5.2}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) - \nabla\cdot(\boldsymbol{l} + \tfrac{1}{2}\boldsymbol{u}^2\boldsymbol{f} + \mathbb{G}(\nabla^s\boldsymbol{u})\cdot\boldsymbol{u}) = 0. \tag{5.3}$$

When looking at (5.1)–(5.3), it is not immediately clear how this system can be reconciled either with the Navier–Stokes regularization or with any phenomenological modeling of dissipation.

It is remarkable that this exercise can actually be done by introducing the quantity $\boldsymbol{u}_m = \boldsymbol{u} - \rho^{-1}\boldsymbol{f}$. The above conservation equations can then be rewritten as follows:

$$\partial_t\rho + \nabla\cdot(\boldsymbol{u}_m\rho) = 0, \tag{5.4}$$

$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}_m \otimes \boldsymbol{m}) + \nabla p - \nabla\cdot(\mathbb{G}(\nabla^s\boldsymbol{u})) = 0, \tag{5.5}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}_m E) - \nabla\cdot(\boldsymbol{l} - e\boldsymbol{f}) + \nabla\cdot\big((p\mathbb{I} - \mathbb{G}(\nabla^s\boldsymbol{u}))\cdot\boldsymbol{u}\big) = 0, \tag{5.6}$$

with again $\boldsymbol{m} = \rho\boldsymbol{u}$ and $E = \rho e + \frac{1}{2}\rho\boldsymbol{u}^2$. It is surprising that this system involves two velocities. It is also somewhat surprising to observe that the above system resembles the Navier–Stokes regularization. In particular, if one sets $a = d$, the term $\boldsymbol{l} - e\boldsymbol{f}$ becomes $d\rho\nabla e$, which upon assuming $\mathrm{d}e = c_v\,\mathrm{d}T$, reduces to $d(\rho,e)\rho c_v\nabla T$; i.e., one recovers Fourier's law: $\boldsymbol{l} - e\boldsymbol{f} = d(\rho,e)\rho c_v\nabla T$.

During the preparation of this paper, it was brought to our attention that the regularization model that we propose above somewhat resembles, at least formally, a model of fluid dynamics of [1] (see, e.g., equations (1) to (5) in [1]). The author has derived the above system of conservation equations (up to some nonessential disagreement on the term $\boldsymbol{l} - e\boldsymbol{f}$) by invoking theoretical arguments from [13] and phenomenological considerations. The mathematical properties of this system have been investigated thoroughly by [3]. Brenner has been defending the idea that it makes phenomenological sense to distinguish the so-called mass velocity, $\boldsymbol{u}_m$, from the so-called volume velocity, $\boldsymbol{u}$, since 2004 (or so). We do not want to enter this debate, but this idea seems to be supported by our mathematical derivation of (5.4)–(5.6), which did not invoke any ad hoc phenomenological assumption. Recall that our primary motivation in this project is to find a regularization of the compressible Euler equations that can serve as a good numerical device, and by good we mean that the model must give positive density, positive internal energy, and a minimum entropy principle and be compatible with a large class of entropy inequalities.

5.3. Conclusions. Let us finally rephrase our findings. In its most general form, the regularized system (5.4)–(5.6) can be rewritten as follows:

$$\partial_t\rho + \nabla\cdot(\boldsymbol{u}_m\rho) = 0, \tag{5.7}$$

$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}_m \otimes \boldsymbol{m}) + \nabla p - \nabla\cdot(G(\nabla^s\boldsymbol{u})) = 0, \tag{5.8}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}_m E) - \nabla\cdot\boldsymbol{q} + \nabla\cdot\big((p\mathbb{I} - G(\nabla^s\boldsymbol{u}))\cdot\boldsymbol{u}\big) = 0, \tag{5.9}$$

$$\boldsymbol{u}_m = \boldsymbol{u} - a(\rho,e)\nabla\log\rho, \tag{5.10}$$

$$\boldsymbol{q} = (a-d)p\nabla\log\rho + d\rho\nabla e, \qquad a(\rho,e) \ge 0,\ d(\rho,e) \ge 0, \tag{5.11}$$

where a and d must satisfy the inequalities established in Corollary 4.3, i.e., $2\Gamma - 2\Delta^{\frac{1}{2}} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac{1}{2}}$, where $\Gamma = \det(\Sigma)\rho^2 s_e^{-2} p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$.

It is established in Lemma 3.1 that the definition of $\boldsymbol{f} = a(\rho,e)\nabla\rho$ is compatible with the positive density principle. The particular form of $\boldsymbol{q}$ in (5.11) results from the definition of $\boldsymbol{l}$ (see (3.8)), which is required for the minimum entropy principle to hold, as established in Theorem 3.5. It is finally proved in Theorem 4.1 that the most robust regularization, i.e., that which is compatible with all the generalized entropy à la [9], corresponds to the choice $a = d$. Various relaxations of the constraint $a = d$ are described in Corollaries 4.2 and 4.3. As observed in subsection 5.1, the parabolic regularization can be put into the form (5.7)–(5.11) with the particular choice $\mathbb{G} = a\nabla\boldsymbol{u}$, which is not rotation invariant and uses the same viscosity coefficient for all fields.

Since the case $a = d$ is the most robust, we finally rewrite (5.7)–(5.11) with this choice in a form that is more suitable for numerical implementation:

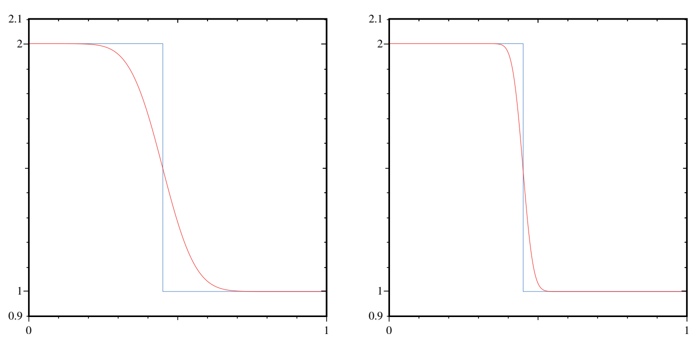
$$\partial_t\rho + \nabla{\cdot}(\boldsymbol{u}\rho) - \nabla{\cdot}(a\nabla\rho) = 0, \tag{5.12}$$

$$\partial_t\boldsymbol{m} + \nabla{\cdot}(\boldsymbol{u}\otimes\boldsymbol{m} + p\mathbb{I}) - \nabla{\cdot}(a\nabla\rho\otimes\boldsymbol{u} + G(\nabla^s\boldsymbol{u})) = 0, \tag{5.13}$$

$$\partial_t E + \nabla{\cdot}(\boldsymbol{u}(E+p)) - \nabla{\cdot}\big(a\nabla(\rho e) + \tfrac{1}{2}\boldsymbol{u}^2 a\nabla\rho + G(\nabla^s\boldsymbol{u}){\cdot}\boldsymbol{u}\big) = 0. \tag{5.14}$$

5.4. Numerical illustrations. We finish with some theoretical and numerical illustrations of the points made in the paper.

To illustrate that the Navier–Stokes regularization is not appropriate for solving the Euler equations, as claimed in subsection 2.3, let us consider a contact wave and a polytropic ideal gas. Let $\boldsymbol{u}_0$ be a uniform flow field and p_0 be a uniform pressure field. Let ρ_0 be some initial density field. Let $\rho(\boldsymbol{x},t)$ be the solution of $\partial_t\rho+\nabla{\cdot}(\boldsymbol{u}_0\rho)-\nabla{\cdot}(a\nabla\rho) = 0$ with $\rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x})$. We claim that $\rho(\boldsymbol{x},t)$, $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_0$ and $p(\boldsymbol{x},t) = p_0$ solve (5.12) and (5.13), respectively. We now verify that (5.14) is solved as well. Using that $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_0$ and $p(\boldsymbol{x},t) = p_0$, (5.14) gives $\partial_t(\rho e) + \frac{1}{2}\boldsymbol{u}_0^2\partial_t\rho + +\nabla{\cdot}(\boldsymbol{u}_0\rho e) + \frac{1}{2}\boldsymbol{u}_0^2\nabla{\cdot}(\boldsymbol{u}_0\rho) - \nabla{\cdot}(a\nabla(\rho e)) - \frac{1}{2}\boldsymbol{u}_0^2\nabla{\cdot}(a\nabla\rho) = 0$, which reduces to $\partial_t(\rho e)+\nabla{\cdot}(\boldsymbol{u}_0\rho e)-\nabla{\cdot}(a\nabla(\rho e)) = 0$ due to mass conservation (5.12). This equation is trivially satisfied for a polytropic ideal gas since $\rho e = (\gamma-1)p_0$ is a uniform constant over the domain. In conclusion, (5.12)–(5.14) is compatible with contact waves in polytropic ideal gases. The reader can verify that this is not the case for the Navier–Stokes regularization (2.16)–(2.18) unless $\kappa = 0$, which then does not leave any regularization of the density and the internal energy.

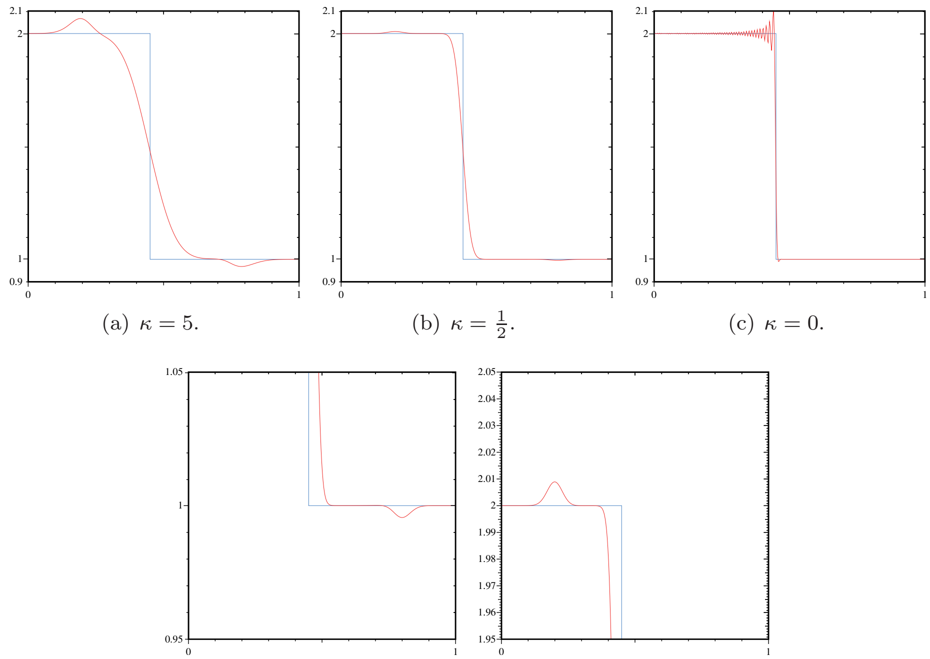


(a) Density, $t = 0.3$, $\kappa = 5$.

(b) Density, $t = 0.3$, $\kappa = \frac{1}{2}$.

FIG. 1. *Density field: New regularization (red) versus exact solution (blue).*

We further illustrate this point by solving the above contact wave problem numerically. We consider a one-dimensional contact wave problem with initial data $\boldsymbol{u}_0 = \boldsymbol{e}_x$, $p_0 = 1$, and $\rho_0(x) = 2$ if $x \le 0.15$ and $\rho_0(x) = 1$ if $x > 0.15$. We use piecewise linear continuous finite elements in space on a uniform grid over the interval $[0,1]$ and Runge–Kutta 3 to step in time. We set $\boldsymbol{f} = \kappa|\boldsymbol{u}|h\partial_x\rho$ and $\boldsymbol{G} = \frac{1}{2}(|\boldsymbol{u}| + \sqrt{\gamma T})\partial_x\boldsymbol{u}$ in (5.12)–(5.14), where h is the mesh size. The problem is solved until $t = 0.3$ on a mesh composed of 400 grid cells with $\kappa = 5$ (left panel of Figure 1) and $\kappa = \frac{1}{2}$ (right panel of Figure 1).



(a) $\kappa = 5$. (b) $\kappa = \frac{1}{2}$. (c) $\kappa = 0$.

(d) $\kappa = \frac{1}{2}$, zoom. (e) $\kappa = \frac{1}{2}$, zoom.

FIG. 2. *Navier–Stokes regularization. Density profile,* $t = 0.3$.

The solution is a viscous wave, as claimed above. The contact is smeared as expected, but there is no undershoot or overshoot. It can indeed be proved that in the particular case of a contact wave, the local maximum principle is satisfied for all values $\kappa \geq \frac{1}{2}$, under appropriate CFL condition. It can also be proved that, in general, the density is positive for all values $\kappa \geq \frac{1}{2}$ under appropriate CFL condition.

We now test the Navier–Stokes regularization (2.16)–(2.18) with $\boldsymbol{h} = \kappa|\boldsymbol{u}|h\partial_x T$ and $\boldsymbol{G} = \frac{1}{2}(|\boldsymbol{u}| + \sqrt{\gamma T})\partial_x \boldsymbol{u}$. The results are shown in Figure 2 with $\kappa = 5$, $\kappa = \frac{1}{2}$, and $\kappa = 0$. We observe first that spurious numerical oscillations are created when $\kappa = 0$, thus confirming the argument made at the end of subsection 2.3. Second, we observe that the numerical solution is incompatible with the contact wave; two extra viscous waves propagate outward from the contact line. This phenomenon is amplified as κ grows.

This example shows the superiority of the proposed regularization (5.12)–(5.14) over the more traditional Navier–Stokes regularization (2.16)–(2.18). More details on how the proposed technique can be implemented numerically with continuous finite elements and the proofs of the statements above regarding positivity of the approximate density will be reported in a forthcoming paper.

Appendix A. Primer in thermodynamics. We collect in this appendix standard results from thermodynamics that are used in the paper.

A.1. Chain rule. Let $\Phi : \mathbb{R}^2 \ni (\alpha, \beta) \longmapsto \Phi(\alpha, \beta) = (\phi(\alpha, \beta), \psi(\alpha, \beta)) \in \mathbb{R}^2$ be a $\mathcal{C}^1$-diffeomorphism. The following holds:

$$\frac{1}{\partial_\alpha \phi \partial_\beta \psi - \partial_\beta \phi \partial_\alpha \psi} \begin{pmatrix} \partial_\beta \psi & -\partial_\beta \phi \\ -\partial_\alpha \psi & \partial_\alpha \phi \end{pmatrix} = \begin{pmatrix} \partial_\phi \alpha & \partial_\psi \alpha \\ \partial_\phi \beta & \partial_\psi \beta \end{pmatrix}. \tag{A.1}$$

In particular, if $\phi(\alpha,\beta)=\alpha$, we have

$$\partial_\alpha\beta(\alpha,\psi) = -\frac{\partial_\alpha\psi(\alpha,\beta)}{\partial_\beta\psi(\alpha,\beta)}, \qquad \partial_\psi\beta(\alpha,\psi) = \frac{1}{\partial_\beta\psi(\alpha,\beta)}. \tag{A.2}$$

A.2. Speed of sound. The square of the speed of sound is defined to be

$$c^2 := \partial_\rho p(\rho,s); \tag{A.3}$$

i.e., c^2 is the partial derivative of the pressure as a function of the density and the specific entropy. Using the chain rule, this definition is equivalent to

$$c^2 = \partial_\rho p(\rho,s) = \partial_\rho p(\rho,e) + \partial_e p(\rho,e)\partial_\rho e(\rho,s), \tag{A.4}$$

and, using (A.2) with $\alpha=\rho$, $\beta=e$, $\psi=s$, one obtains

$$c^2 = p_\rho - \frac{s_\rho}{s_e}p_e(\rho,e). \tag{A.5}$$

Using the following representations of p_e and p_ρ,

$$p_e = \rho^2 s_e^{-2}(s_\rho s_{ee} - s_e s_{\rho e}), \qquad p_\rho = s_e^{-2}(\rho^2 s_\rho s_{\rho e} - s_e\partial(\rho^2 s_\rho)), \tag{A.6}$$

the expression (A.5) also gives

$$c^2 = \rho^2 s_e^{-3}(2s_e s_\rho s_{\rho e} - s_e^2\rho^{-2}\partial(\rho^2 s_\rho) - s_\rho^2 s_{ee}). \tag{A.7}$$

A.3. Convexity of the entropy, det($\mathbb{\Sigma}$). Let us define the matrix

$$\mathbb{\Sigma} := \rho\begin{pmatrix}\rho^{-2}\partial_\rho(\rho^2 s_\rho) & s_{\rho e}\\ s_{\rho e} & s_{ee}\end{pmatrix}, \tag{A.8}$$

which, up to the ρ factor, is the Hessian of the entropy with respect to the variables (ρ^{-1},e). The convexity assumption on the entropy implies that s_{ee} and $\rho^{-1}\partial_\rho(\rho^2 s_\rho)$ are negative. We have the following characterization of the determinant of $\mathbb{\Sigma}$:

$$\det(\mathbb{\Sigma}) = s_e^3(p_\rho T_e - p_e T_\rho). \tag{A.9}$$

To prove this statement, we observe that the following holds owing to (A.6):

$$\begin{aligned} s_e^2 T_e &= -s_{ee}, & s_e^2 T_\rho &= -s_{\rho e},\\ s_e^2 p_e &= \rho^2(s_\rho s_{ee} - s_e s_{\rho e}) & s_e^2 p_\rho &= \rho^2\left(s_\rho s_{\rho e} - s_e\rho^{-2}\partial_\rho(\rho^2 s_\rho)\right). \end{aligned}$$

The result is now evident.

A.4. Specific heat at constant pressure. The specific heat at constant pressure is defined to be $c_p(\rho,e) = T\partial_T s(T,p)$.

LEMMA A.1. *The quantities* $\det(\mathbb{\Sigma})$, c^2, *and* c_p *are related by*

$$c_p\det(\mathbb{\Sigma}) = s_e^3 c^2. \tag{A.10}$$

Proof. Using the chain rule, we can rewrite the above definition as follows:

$$c_p(\rho,e) = s_e^{-1}(s_\rho\rho_T(p,T) + s_e e_T(p,T)).$$

The change-of-variables formula (A.1) with the convention ($\alpha = \rho, \beta = e$) and ($\phi = p, \psi = T$) gives

$$\rho_T(p,T) = \frac{-p_e}{p_\rho T_e - p_e T_\rho}, \qquad e_T(p,T) = \frac{p_\rho}{p_\rho T_e - p_e T_\rho}.$$

We then have the following expression for c_p:

$$c_p = s_e^{-1} \frac{(p_\rho s_e - p_e s_\rho)}{p_\rho T_e - p_e T_\rho}. \tag{A.11}$$

Then, using the expression of c^2 in (A.5) and the relation (A.9), we arrive at the desired expression. ∎

LEMMA A.2. *The following holds:*

$$c_p T_e > 1. \tag{A.12}$$

Proof. The definition of c_p implies that we need to estimate $T s_T(p,T) T_e(\rho,e)$. The chain rule implies

$$1 = T s_e(\rho,e) = T s_p(p,T) p_e(\rho,e) + T s_T(p,T) T_e(\rho,e).$$

The result will be established if we can prove that $s_p(p,T)p_e(\rho,e) < 0$. We now calculate $s_p(p,T)$. The chain rule implies again that

$$(s_p(p,T))^{-1} = p_s(s,T) = p_\rho(\rho,e)\rho_s(s,T) + p_e(\rho,e)e_s(s,T).$$

Then using (A.1) with the convention ($\alpha = \rho, \beta = e$) and ($\phi = s, \psi = T$) gives

$$\rho_s(s,T) = \frac{T_e}{s_\rho T_e - s_e T_\rho}, \qquad e_s(s,T) = \frac{-T_\rho}{s_\rho T_e - s_e T_\rho}.$$

This in turn implies that

$$(s_p(p,T))^{-1} = p_s(s,T) = \frac{p_\rho T_e - p_e T_\rho}{s_\rho T_e - s_e T_\rho} = -\frac{s_e^{-3}\det(\Sigma)}{\rho^{-2} p_e},$$

since $s_\rho T_e - s_e T_\rho = s_e^{-2}(-s_\rho s_{ee} + s_e s_{\rho e}) = -\rho^{-2} p_e$, where we used (A.6). In conclusion, $s_p(p,T)p_e(\rho,e) = -s_e^3 p_e^2 \rho^{-2} \det(\Sigma)^{-1} < 0$, owing to (2.8) and (2.10), which concludes the proof. ∎

Remark A.1. Note in passing that the convexity assumption (2.8) implies that $T_e > 0$, which, owing to (A.12), implies that $c_p > 0$. This in turn implies that $c^2 > 0$, owing to (A.10); i.e., the Euler system (2.1)–(2.4) is hyperbolic under the convexity assumption (2.8) and the positivity assumption on the temperature (2.10). Positivity of the pressure is not needed to establish this fact.

A.5. Matrix $\mathbb{H}_2$. Investigations on entropy inequalities involve the quadratic form induced by the matrix $\mathbb{H}_2$:

$$\mathbb{H}_2 := \begin{pmatrix} s_\rho^2 c_P^{-1} + \rho^{-2}\partial_\rho(\rho^2 s_\rho) & s_\rho s_e c_P^{-1} + s_{\rho e} \\ s_\rho s_e c_P^{-1} + s_{\rho e} & s_e^2 c_P^{-1} + s_{ee} \end{pmatrix}.$$

Some key properties of this matrix are collected in the following lemma.

LEMMA A.3. *The following properties hold:*
(i) $\det(\mathbb{H}_2) = 0$.
(ii) $\mathbb{H}_2$ *is negative semidefinite.*
(iii) $h_{22}p_\rho - h_{12}p_e = 0$.

Proof. (i) Using the expressions (A.7) and (A.9) for the speed of sound, c^2, and $\det(\Sigma)$, and the relation (A.10), the determinant of $\mathbb{H}_2$ is rewritten as follows:

$$\begin{aligned}\det(\mathbb{H}_2) &= (s_\rho^2 c_P^{-1} + \rho^{-2}\partial_\rho(\rho^2 s_\rho))(s_e^2 c_P^{-1} + s_{ee}) - (s_\rho s_e c_P^{-1} + s_{\rho e})^2 \\ &= \rho^{-2}\det(\Sigma) + c_P^{-1}(s_e^2\rho^{-2}\partial_\rho(\rho^2 s_\rho) + s_\rho^2 s_{ee} - 2s_\rho s_e s_{\rho e}) \\ &= \rho^{-2}\det(\Sigma) - c_P^{-1}c^2\rho^{-2}s_e^3 = 0.\end{aligned}$$

This is essentially the result established in [9, p. 2126].

(ii) Owing to the inequality $1 < c_p T_e$ established in (A.12), we infer that $h_{22} = s_e^2 c_P^{-1} + s_{ee} < 0$, which together with (i) proves statement (ii).

(iii) Let us compute $s_e^{-2}(h_{22}p_\rho - h_{12}p_e)$,

$$\begin{aligned}s_e^{-2}(h_{22}p_\rho - h_{12}p_e) &= (c_p^{-1} - T_e)p_\rho - (s_\rho s_e^{-1} c_p^{-1} - T_\rho)p_e \\ &= p_e T_\rho - p_\rho T_e + c_p^{-1}s_e^{-1}(s_e p_\rho - s_\rho p_e).\end{aligned}$$

This proves that $s_e^{-2}(h_{22}p_\rho - h_{12}p_e) = 0$, owing to (A.11). ∎

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REFERENCES

[1] H. BRENNER, *Fluid mechanics revisited*, Phys. A, 370 (2006), pp. 190–224.
[2] B. EINFELDT, C.-D. MUNZ, P. L. ROE, AND B. SJÖGREEN, *On Godunov-type methods near low densities*, J. Comput. Phys., 92 (1991), pp. 273–295.
[3] E. FEIREISL AND A. VASSEUR, *New perspectives in fluid dynamics: Mathematical analysis of a model proposed by Howard Brenner*, in New Directions in Mathematical Fluid Mechanics, Adv. Math. Fluid Mech., Birkhäuser-Verlag, Basel, 2010, pp. 153–179.
[4] K. O. FRIEDRICHS AND P. D. LAX, *Systems of conservation equations with a convex extension*, Proc. Natl. Acad. Sci. USA, 68 (1971), pp. 1686–1688.
[5] E. GODLEWSKI AND P.-A. RAVIART, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Appl. Math. Sci. 118, Springer-Verlag, New York, 1996.
[6] S. K. GODUNOV, *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Mat. Sb. (N.S.), 47 (1959), pp. 271–306.
[7] S. K. GODUNOV, *Thermodynamics of gases and differential equations*, Uspekhi Mat. Nauk, 14 (1959), pp. 97–116.
[8] A. HARTEN, *On the symmetric form of systems of conservation laws with entropy*, J. Comput. Phys., 49 (1983), pp. 151–164.
[9] A. HARTEN, P. D. LAX, C. D. LEVERMORE, AND W. J. MOROKOFF, *Convex entropies and hyperbolicity for general Euler equations*, SIAM J. Numer. Anal., 35 (1998), pp. 2117–2127.
[10] T. J. R. HUGHES, L. P. FRANCA, AND M. MALLET, *A new finite element formulation for computational fluid dynamics.* I. *Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics*, Comput. Methods Appl. Mech. Engrg., 54 (1986), pp. 223–234.
[11] P. LAX, *Shock waves and entropy*, in Contributions to Nonlinear Functional Analysis (Proc. Sympos., Math. Res. Center, University of Wisconsin, Madison, WI, 1971), Academic Press, New York, 1971, pp. 603–634.
[12] P. D. LAX, *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math., 7 (1954), pp. 159–193.
[13] H. C. ÖTTINGER, *Beyond Equilibrium Thermodynamics*, Wiley, Hoboken, NJ, 2005.

[14] B. Perthame and C.-W. Shu, *On positivity preserving finite volume schemes for Euler equations*, Numer. Math., 73 (1996), pp. 119–130.

[15] D. Serre, *Systèmes de lois de conservation* I: *Hyperbolicité, entropies, ondes de choc*, Diderot Editeur, Paris, 1996.

[16] D. Serre, *Viscous system of conservation laws: Singular limits*, in Nonlinear Conservation Laws and Applications, IMA Vol. Math. Appl. 153, Springer, New York, 2011, pp. 433–445.

[17] E. Tadmor, *A minimum entropy principle in the gas dynamics equations*, Appl. Numer. Math., 2 (1986), pp. 211–219.

[18] E. Tadmor, *Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems*, Acta Numer., 12 (2003), pp. 451–512.

[19] H.-Z. Tang and K. Xu, *Positivity-preserving analysis of explicit and implicit Lax-Friedrichs schemes for compressible Euler equations*, J. Sci. Comput., 15 (2000), pp. 19–28.