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CONCENTRATION FACTORS FOR FUNCTIONS WITH HARMONIC BOUNDED MEAN VARIATION

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Abstract. We discuss determination of jumps for functions with generalized bounded variation. The questions are motivated by A. Gelb and E. Tadmor [1], F. Móricz [5] and [6] and Q. L. Shi and X. L. Shi [7]. Corollary 1 improves the results proved in B. I. Golubov [2] and G. Kvernadze [3].

§1. Introduction

Set $T := [-\pi, \pi)$. Let $L(T)$ denote the set of all periodic and integrable functions with period 2π. For any $f \in L(T)$ denote by

$$S[f](x) := \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{1.1}$$

and

$$\widetilde{S}[f](x) := \sum_{k=1}^{\infty} \widetilde{A_k}(x) = \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx) \tag{1.2}$$

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its Fourier series and conjugate Fourier series, respectively, where

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt \quad \text{and} \quad b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt,$$

$k = 1, 2, 3, \dots$. The n-th partial sum of the series (1.1) and (1.2) are denoted by

$$S_n(f, x) := \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x) \quad \text{and} \quad \widetilde{S_n}(f, x) := \sum_{k=1}^{n} \widetilde{A_k}(x),$$

respectively. It is well known that the jump of a function $f \in L(T)$ at its simple discontinuity $x = \xi$ can be determined in terms of the spectral data a_k and b_k, $k = 1, 2, 3 \dots$. Indeed, in 1920 F. Lukács [4] proved that *if the finite limit*

$$d_\xi(f) := \lim_{t \to 0^+} \left[f(\xi + t) - f(\xi - t) \right] \tag{1.3}$$

exists at some point $\xi \in (-\pi, \pi]$, *then*

$$\lim_{n \to \infty} -\frac{\pi \widetilde{S_n}(f, \xi)}{\ln n} = d_\xi(f). \tag{1.4}$$

(see A. Zygmund [10].)

The convergence in this way, however, is at the unacceptably slow rate of order $O(1/\ln n)$. To improve the convergence rate, in 1999 A. Gelb and E. Tadmor [1] introduced the method of concentration factors.

Let σ be a continuous function on $[0, 1]$. Denote

$$\widetilde{S}_n^\sigma(f, x) := \sum_{k=1}^{n} \sigma\left(\frac{k}{n}\right) \widetilde{A_k}(x).$$

If the limit (1.3) exists at ξ and $\lim_{n\to\infty} \widetilde{S}_n^\sigma(f, \xi) = d_\xi(f)$, then we call

$$\left\{ \sigma\left(\frac{k}{n}\right) \right\}_{k=1,\dots,n;\ n=1,2,\dots}$$

the concentration factors of f at the point ξ.

A. Gelb and E. Tadmor [1] established a criterion of concentration factors. Later Q. L. Shi and X. L. Shi [7] proved the following improvement.

THEOREM A. *Assume $\xi \in T$ and $\sigma \in \mathrm{Lip}_1[0,1]$. Then for any $f \in D_\xi$, the factors $\left\{\sigma\left(\frac{k}{n}\right)\right\}_{k=1,\dots,n;\ n=1,2,\dots}$ are concentration factors of f at $x = \xi$ if and only if*

$$\int_0^1 \frac{\sigma(x)}{x}\,dx = -\pi, \tag{1.5}$$

where D_ξ denotes the set of functions of $f \in L(T)$ that satisfy

(i) *$d_\xi(f)$ exists,*

and

(ii) $\frac{f(\xi+t)-f(\xi-t)-d_\xi(f)}{t} \in L[0,\pi]$.

It is not hard to see that a BV function is not necessary to satisfy the condition (ii). The aim of the present paper is to establish a criterion of concentration factors for functions which have some kind of BV property. To state the results we introduce some definitions first.

Let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers that satisfy $\Sigma_{n=1}^{+\infty} 1/\lambda_n = \infty$. Suppose that f is a real function defined on an interval $[a,b]$. $\{I_n\}$ will denote a sequence of non-overlapping intervals $I_n = [a_n, b_n]$, $[a_n, b_n] \subset [a,b]$ and write $f(I_n) = f(b_n) - f(a_n)$.

A function f is said to be of Λ-bounded variation $\big(\Lambda\mathrm{BV}\,[a,b]\big)$ if

$$\sup_{\{I_n\}} \sum_{n=1}^{+\infty} \big|f(I_n)\big|/\lambda_n < \infty.$$

For $\Lambda = \{n\}$, i.e.

$$\sup_{\{I_n\}} \sum_{n=1}^{+\infty} \big|f(I_n)\big|/n < \infty,$$

we say that f is of harmonic bounded variation $\big(\mathrm{HBV}\,[a,b]\big)$.

The concept ΛBV was introduced by D. Waterman [9] in 1972. Later, in 1985 the second author generalized this class to ΛBMV.

A function f is said to be of Λ-bounded mean variation $\big(\Lambda\mathrm{BMV}\,[a,b]\big)$ if

$$\sup_{\{I_n\}} \sum_{n=1}^{+\infty} \mu_{I_n}(f)/\lambda_n < \infty,$$

where

$$\mu_{I_n}(f) = \frac{1}{|I_n|}\int_{I_n} \big|f(x) - f_{I_n}\big|\,dx, \quad f_{I_n} = \frac{1}{|I_n|}\int_{I_n} f(x)\,dx.$$

For $\Lambda = \{n\}$, i.e.

$$\sup_{\{I_n\}} \sum_{n=1}^{+\infty} \mu_{I_n}(f)/n < \infty,$$

we say that f is of harmonic bounded mean variation $\bigl(\mathrm{HBMV}\,[a,b]\bigr)$. It was proved in [8] and [9] that if $\lambda_n \uparrow \infty$ then $\mathrm{BV} \subsetneqq \Lambda\mathrm{BV} \subsetneqq \Lambda\mathrm{BMV}$.

In Section 3 we will prove the following.

THEOREM 1. *Assume that* $\sigma \in \mathrm{Lip}_1[0,1]$ *satisfies* $\int_0^1 \frac{\sigma(x)}{x}\,dx = -\pi$ *and* $\xi \in T$. *If* $f \in L(T) \cap \mathrm{HBMV}\,[\xi-\delta, \xi+\delta]$ *for some* $\delta > 0$ *and the limit* $d_\xi(f)$ *exists, then* $\bigl\{\sigma\bigl(\frac{k}{n}\bigr)\bigr\}_{k=1,\dots,n;\ n=1,2,\dots}$ *are concentration factors of* f *at the point* ξ.

In Theorem 1 the class HBMV is best possible in the following sense.

THEOREM 2. *If* $\mathrm{HBMV} \subsetneqq \Lambda\mathrm{BMV}$, *then the conclusion of Theorem* 1 *is not true when we replace* $\mathrm{HBMV}\,[\xi-\delta,\xi+\delta]$ *by* $\Lambda\mathrm{BMV}\,[\xi-\delta,\xi+\delta]$.

REMARK 1. The function

$$f(x) = \begin{cases} 1, & \text{if } 1/2 < x \leqq \pi; \\ 1/|\ln x|, & \text{if } 0 < x \leqq 1/2; \\ 0, & \text{if } -\pi < x \leqq 0; \\ f(x+2\pi), & \text{if } x \in R. \end{cases}$$

has bounded variation on T, but it is not in D_0.

REMARK 2. Let V_p, $p \geqq 1$, denote the set of all functions f that satisfy

$$\sup_{I_n \subset (-\pi,\pi]} \Bigl(\sum_n \bigl|f(I_n)\bigr|^p\Bigr)^{\frac{1}{p}} < \infty,$$

where the "sup" is taken over all non-overlapping intervals I_n. B. I. Golubov [2] proved that if $f \in V_p$, $p \geqq 1$, then the identities

$$\lim_{n\to\infty} \frac{(-1)^r(2r+1)\pi}{n^{2r+1}} S_n(f,\xi)^{(2r+1)} = d_\xi(f) \tag{1.6}$$

and

$$\lim_{n\to\infty} \frac{(-1)^{r+1}2r\pi}{n^{2r}} \widetilde{S_n}(f,\xi)^{(2r)} = d_\xi(f) \tag{1.7}$$

hold. Later, G. Kvernadze [3] proved (1.6) and (1.7) for $f \in \mathrm{HBV}$. Since $V_p \subsetneqq \mathrm{HBV}$, Kvernadze improved Golubov's results. The identities (1.6) and (1.7) can be rewritten as one formula, i.e.

$$\lim_{n\to\infty} \frac{-p\pi}{n^p} \sum_{k=1}^{n} k^p \widetilde{A_k}(\xi) = d_\xi(f), \tag{1.8}$$

where p is any natural number. By our Theorem 1 we can prove that the identity (1.8) holds for any positive p, i.e. we have the following

COROLLARY 1. *Assume that* $\xi \in T$, $f \in L(T)$ *and the finite limit* (1.3) *exists. If* $f \in D_\xi$ *or* $f \in \mathrm{HBMV}\,[\xi-\delta, \xi+\delta]$ *for some* $\delta > 0$, *then* (1.8) *holds for any* $p > 0$.

The proof of Corollary 1 will be given in Section 3.

REMARK 3. Indeed we proved that for different p, all (1.8) are equivalent to each other (see Lemma 5). Therefore (1.6) and (1.7) are equivalent.

REMARK 4. If $f \in \Lambda\mathrm{BV}$ then the finite limit (1.3) exists everywhere. But this proposition is not true for the class $\Lambda\mathrm{BMV}$ (see D. Waterman [9] and X. L. Shi [8]). Hence in Theorem 1 and Corollary 1 we assume that the limit (1.3) exists.

Based on F. Móricz's results in [5] and [6], Q. L. Shi and X. L. Shi [7] introduced the concept of "concentration factors of Abel–Poisson type". Let μ be a continuous function on $[0,\infty)$ that satisfies $\mu(0) = 0$ and

$$\bigl|\mu(x)\bigr| = O\bigl((1+x)^M\bigr), \qquad \text{as} \quad x \to \infty,$$

where $M \geqq 0$. For $f \in L(T)$, the series

$$\widetilde{P}_r^\mu(f,x) := \sum_{k=1}^{\infty} \mu\bigl((1-r)k\bigr)\, \widetilde{A_k}(x) r^k, \qquad 0 \leqq r < 1,$$

is convergent everywhere. If the limit (1.3) exists at ξ and

$$\lim_{r\to 1-0} \widetilde{P}_r^\mu(f,\xi) = d_\xi(f),$$

then we call

$$\bigl\{\mu\bigl((1-r)k\bigr)\bigr\}_{k=1,2,\dots;\ 0\leqq r<1} \tag{1.9}$$

the concentration factors of Abel–Poisson type for f at the point ξ. For $x > 0$ denote by $L_\mu(x)$ the Lipschitz norm of μ on $[0,x]$, i.e.

$$L_\mu(x) := \sup_{y,z\in[0,x],\ y\neq z} \left|\frac{\mu(y)-\mu(z)}{y-z}\right| + \sup_{0\leqq y\leqq x} \bigl|\mu(y)\bigr|.$$

Let Ω denote the set of all functions μ on $[0,\infty)$ that satisfy the following conditions:
(iii) $\mu(0)=0$,
(iv) there exists $M \geqq 0$ such that

$$L_\mu(x) = O\big((1+x)^M\big), \qquad \text{as} \quad x \to \infty. \tag{1.10}$$

Q. L. Shi and X. L. Shi proved the following:

THEOREM B. *Let $\xi \in T$, $f \in D_\xi$ and $\mu \in \Omega$. Then the factors* (1.9) *are concentration factors of Abel–Poisson type for f at the point ξ if and only if*

$$\lim_{r\to 1-0} \sum_{k=1}^{\infty} \frac{\mu\big((1-r)k\big)}{k} r^k = -\pi. \tag{1.11}$$

In Section 3 we will prove the following:

THEOREM 3. *Assume that $\mu \in \Omega$ satisfies* (1.11) *and $\xi \in T$. If $f \in L(T) \cap \mathrm{HBMV}\,[\xi-\delta,\xi+\delta]$ for some $\delta > 0$ and the limit $d_\xi(f)$ exists, then the factors* (1.9) *are concentration factors of Abel–Poisson type for f at the point ξ.*

THEOREM 4. *If $\mathrm{HBMV} \subsetneqq \Lambda\mathrm{BMV}$, then the conclusion of Theorem 2 is not true if we replace $\mathrm{HBMV}\,[\xi-\delta,\xi+\delta]$ by $\Lambda\mathrm{BMV}\,[\xi-\delta,\xi+\delta]$.*

§2. Some lemmas

We need several preliminary lemmas.

LEMMA 1. *If $f \in L[a,b]$ then*

$$\lim_{\lambda\to\infty} \int_a^b f(t)e^{i\lambda t}\,dt = 0.$$

This is the well-known Riemann–Lebesgue lemma.

LEMMA 2. *Let $k, n \geqq 1$ and $0 < t < \pi$. Then*

$$\frac{1-\cos kt}{\tan\frac{t}{2}} = O(k^2 t), \quad \frac{t - 2\tan\frac{t}{2}}{2t\tan\frac{t}{2}} = O(1), \quad \frac{1}{n}\sum_{k=1}^{n}(\ln n - \ln k) = O(1). \tag{2.1}$$

The proof of this lemma is easy, so we omit it.

LEMMA 3. *Under the assumption of Theorem 1, σ has the following property:*

$$\sigma(t) = O(t), \qquad (t \to 0).$$

LEMMA 4. *For* $f \in \mathrm{HBMV}\,[a,b]$ *and* $0<\delta<b-a$, *denote*

$$\triangle\big(f,[a,a+\delta]\big) := \sup_{\{I_n\}} \sum_{n=1}^{+\infty} \mu_{I_n}(f)/\lambda_n,$$

where the "sup" is taken over all non-overlapping $I_n \subset [a,a+\delta]$. *If* f *is right continuous at* $x=a$, *then*

$$\lim_{\delta\to 0^+} \Delta\big(f,[a,a+\delta]\big) = 0$$

(see X. L. Shi [8]).

Let

$$\tau := \sum_{n=1}^{\infty} t_n,$$

and $p>0$. Denote

$$\tau_n^p := \frac{p}{n^p}\sum_{k=1}^{n} k^p t_k.$$

We have the following

LEMMA 5. *If* $p_1, p_2 > 0$ *and* $p_1 \neq p_2$ *then*

$$\lim_{n\to\infty} \tau_n^{p_1} = l \tag{2.2}$$

if and only if

$$\lim_{n\to\infty} \tau_n^{p_2} = l. \tag{2.3}$$

PROOF. It is enough to prove one direction, i.e. (2.2) implies (2.3). If (2.2) holds, then by Abel transformation

$$\tau_n^{p_2} = \frac{p_2}{n^{p_2}}\sum_{k=1}^{n} k^{p_2-p_1}k^{p_1}t_k$$

$$= \frac{p_2}{n^{p_2}}\sum_{k=1}^{n-1}\big[k^{p_2-p_1}-(k+1)^{p_2-p_1}\big]\sum_{j=1}^{k} j^{p_1}t_k + \frac{p_2}{n^{p_2}}n^{p_2-p_1}\sum_{j=1}^{n} j^{p_1}t_k$$

$$= \frac{p_2}{p_1 n^{p_2}}\sum_{k=1}^{n-1}\big[k^{p_2-p_1}-(k+1)^{p_2-p_1}\big]k^{p_1}\tau_k^{p_1} + \frac{p_2}{p_1 n^{p_2}}n^{p_2-p_1}n^{p_1}\tau_n^{p_1}.$$

By (2.2) we have $\tau_k^{p_1} = l + o(1)$. Hence

$$\tau_n^{p_2} = \left(\frac{p_2}{p_1 n^{p_2}} \sum_{k=1}^{n-1} \left[k^{p_2-p_1} - (k+1)^{p_2-p_1} \right] k^{p_1} l + \frac{p_2}{p_1} l \right) \tag{2.4}$$

$$+ \left(\frac{p_2}{p_1 n^{p_2}} \sum_{k=1}^{n-1} \left[k^{p_2-p_1} - (k+1)^{p_2-p_1} \right] k^{p_1} o(1) + \frac{p_2}{p_1} o(1) \right) = I_1 + I_2,$$

say. Since

$$k^{p_2-p_1} - (k+1)^{p_2-p_1} = O\big(k^{p_2-p_1-1}\big),$$

for positive p_2 we have

$$I_2 = \frac{1}{n^{p_2}} \sum_{k=1}^{n-1} O\big(k^{p_2-1}\big) o(1) + o(1) = o(1). \tag{2.5}$$

Next let us consider I_1:

$$I_1 = \frac{p_2}{p_1 n^{p_2}} \sum_{k=1}^{n-1} \left[k^{p_2} - (k+1)^{p_2} \right] l + \frac{p_2}{p_1} l \tag{2.6}$$

$$+ \frac{p_2}{p_1 n^{p_2}} \sum_{k=1}^{n-1} (k+1)^{p_2-p_1} \left[(k+1)^{p_1} - k^{p_1} \right] l$$

$$= \frac{p_2}{p_1} \frac{l}{n^{p_2}} + \frac{p_2}{p_1 n^{p_2}} \sum_{k=1}^{n-1} (k+1)^{p_2} \left(1 - \left(\frac{k}{k+1} \right)^{p_1} \right) l$$

$$= o(1) + \frac{p_2}{p_1 n^{p_2}} \sum_{k=1}^{n-1} (k+1)^{p_2} \left(\frac{p_1}{k+1} + O\left(\frac{1}{k^2} \right) \right) l = l + o(1).$$

By combining (2.4)–(2.6) we obtain (2.3). □

Lemma 6. *Let $\tau := \sum_{n=1}^{\infty} t_n$. Assume that $\mu \in \Omega$ and satisfies* (1.11). *If*

$$\lim_{n\to\infty} \tau_n^1 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} k t_k = l, \tag{2.7}$$

then

$$Q(r) := \sum_{k=1}^{\infty} \mu\big((1-r)k\big) t_k r^k \to -\pi l \quad as \quad r \to 1-0. \tag{2.8}$$

PROOF. By Abel transformation

$$Q(r)=\sum_{k=1}^{\infty}\left[\frac{\mu\big((1-r)k\big)}{k}-\frac{\mu\big((1-r)(k+1)\big)}{k+1}r\right]k\tau_k^1 r^k.$$

By (2.7), $\tau_n^1=l+\varepsilon_n$ as $n\to\infty$ with $\varepsilon_n=o(1)$, hence

$$Q(r)=l\sum_{k=1}^{\infty}\left[\frac{\mu\big((1-r)k\big)}{k}-\frac{\mu\big((1-r)(k+1)\big)}{k+1}r\right]kr^k \tag{2.9}$$

$$+\sum_{k=1}^{\infty}\left[\frac{\mu\big((1-r)k\big)}{k}-\frac{\mu\big((1-r)(k+1)\big)}{k+1}r\right]k\varepsilon_k r^k=Q_1+Q_2.$$

Under the assumption on μ we see that

$$\mu\big((1-r)k\big)=\begin{cases}O\big((1-r)k\big) & \text{if } (1-r)k\leqq 1,\\ O\big((1-r)^M k^M\big) & \text{if } (1-r)k\geqq 1,\end{cases} \tag{2.10}$$

and

$$\mu\big((1-r)k\big)-\mu\big((1-r)(k+1)\big) \tag{2.11}$$

$$=\begin{cases}O\big((1-r)\big), & \text{if } (1-r)k\leqq 1,\\ O\big((1-r)(1-r)^M k^M\big), & \text{if } (1-r)k\geqq 1.\end{cases}$$

If we write

$$\mu_{r,k}:=\left[\frac{\mu\big((1-r)k\big)}{k}-\frac{\mu\big((1-r)(k+1)\big)}{k+1}r\right]k$$

$$=\Big[\mu\big((1-r)k\big)-\mu\big((1-r)(k+1)\big)\Big]+\left[\mu\big((1-r)(k+1)\big)\left(\frac{1}{k}-\frac{1}{k+1}\right)\right]k$$

$$+\left[\frac{\mu\big((1-r)(k+1)\big)}{k+1}(1-r)\right]k,$$

then by (2.10) and (2.11) we obtain

$$\mu_{r,k}=\begin{cases}O\big((1-r)\big), & \text{if } (1-r)k\leqq 1,\\ O\big((1-r)^M k^M\big), & \text{if } (1-r)k\geqq 1.\end{cases}$$

Thus we have

(2.12)
$$|Q_2| = O\Bigg((1-r)\sum_{k\leqq\left[\frac{1}{1-r}\right]}|\varepsilon_k|\Bigg) + O\Bigg((1-r)^{M+1}\sum_{k>\left[\frac{1}{1-r}\right]}k^M|\varepsilon_k|\Bigg) = o(1),$$

as $r\to 1-0$. Next we calculate Q_1. It is not hard to see that

(2.13)
$$Q_1 = l\sum_{k=1}^{\infty}\Big[\mu\big((1-r)k\big) - \mu\big((1-r)(k+1)\big)\,r\Big]r^k$$

$$+\,l\sum_{k=1}^{\infty}\frac{\mu\big((1-r)(k+1)\big)}{k+1}r^{k+1} = l\sum_{k=1}^{\infty}\frac{\mu\big((1-r)k\big)}{k}r^k = -\pi l + o(1),$$

as $r\to 1-0$. By combining (2.9), (2.12) and (2.13) we obtain (2.8). □

§3. Proofs of the results

3.1. Proof of Theorem 1. Set

$$\phi(x) = \begin{cases} \dfrac{\pi - x}{2\pi} & \text{if } \ 0 < x < 2\pi; \\ 0, & \text{if } \ x = 0; \\ \phi(x+2\pi), & \text{if } \ x\in R. \end{cases}$$

and $g(x) = d_\xi(f)\phi(x-\xi)$. Write $f(x) = g(x) + h(x)$. Then we have

$$\widetilde{S_n^\sigma}(f,\xi) = \widetilde{S_n^\sigma}(g,\xi) + \widetilde{S_n^\sigma}(h,\xi).$$

It is clear that

(3.1)
$$\widetilde{S_n^\sigma}(g,\xi) = -\frac{d_\xi(f)}{\pi}\sum_{k=1}^{n}\frac{\sigma\left(\frac{k}{n}\right)}{k} + o(1) = -\frac{d_\xi(f)}{\pi}\int_0^1\frac{\sigma(x)}{x}\,dx + o(1),$$

and hence $\lim_{n\to\infty}\widetilde{S_n^\sigma}(g,\xi) = d_\xi(f)$. Therefore what we need to show is

(3.2)
$$\lim_{n\to\infty}\widetilde{S_n^\sigma}(h,\xi) = 0.$$

By Abel transformation

$$(3.3)\qquad \widetilde{S_n^\sigma}(h,\xi)=\sum_{k=1}^{n-1}\left(\sigma\left(\frac{k}{n}\right)-\sigma\left(\frac{k+1}{n}\right)\right)\widetilde{S_k}(h,\xi)+\sigma(1)\widetilde{S_n}(h,\xi).$$

Let us estimate $\widetilde{S_k}(h,\xi)$ first. By the Dirichlet representation of the conjugate partial sum we have

$$(3.4)\qquad \widetilde{S_k}(h,\xi)=-\frac{1}{\pi}\int_0^\pi \psi_\xi(t)\left(\frac{1-\cos kt}{2\tan\frac{t}{2}}+\frac{1}{2}\sin kt\right)dt,$$

where $\psi_\xi(t)=h(\xi+t)-h(\xi-t)$. Write

$$(3.5)\quad \widetilde{S_k}(h,\xi)=-\frac{1}{\pi}\int_0^{\pi/k}\psi_\xi(t)\left(\frac{1}{2\tan\frac{t}{2}}-\frac{\cos kt}{t}\right)dt-\frac{1}{\pi}\int_{\pi/n}^{\pi}\frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt$$

$$+\frac{1}{\pi}\int_{\pi/n}^{\pi/k}\frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt+\frac{1}{\pi}\int_0^{\pi}\psi_\xi(t)\left(\frac{1}{2\tan\frac{t}{2}}-\frac{1}{t}\right)\cos kt\,dt$$

$$+\frac{1}{\pi}\int_{\pi/k}^{\pi}\frac{\psi_\xi(t)}{t}\cos kt\,dt-\frac{1}{\pi}\int_0^{\pi}\psi_\xi(t)\frac{1}{2}\sin kt\,dt$$

$$=I_1+I_2+I_3+I_4+I_5+I_6.$$

By Lemma 1, we obtain

$$(3.6)\qquad \lim_{k\to\infty}\big(|I_4|+|I_6|\big)=0.$$

By Lemma 2,

$$(3.7)\qquad |I_1|=\frac{1}{\pi}\left|\int_0^{\pi/k}\psi_\xi(t)\left(\frac{t-2\tan\frac{t}{2}\cos kt}{2t\tan\frac{t}{2}}\right)dt\right|$$

$$\leqq\frac{1}{\pi}\int_0^{\pi/k}\big|\psi_\xi(t)\big|\left(\left|\frac{1-\cos kt}{2\tan\frac{t}{2}}\right|+\left|\frac{t-2\tan\frac{t}{2}}{2t\tan\frac{t}{2}}\right|\right)dt.$$

Since $d_\xi(h)=0$, we have

$$(3.8)\qquad \lim_{t\to 0}\big|\psi_\xi(t)\big|=0,$$

hence it follows from Lemma 2 that

$$|I_1| = o(1)\int_0^{\pi/k} O(k^2t)\,dt + o(1)\int_0^{\pi/k} O(1)\,dt = o(1), \tag{3.9}$$

as $k\to\infty$. Thus we obtain

$$\widetilde{S_k}(h,\xi) = -\frac{1}{\pi}\int_{\pi/n}^{\pi} \frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt + \frac{1}{\pi}\int_{\pi/n}^{\pi/k} \frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt + \frac{1}{\pi}\int_{\pi/k}^{\pi} \frac{\psi_\xi(t)}{t}\cos kt\,dt + o(1), \tag{3.10}$$

as $k\to\infty$. By combining (3.3) and (3.10) we get

$$\begin{aligned}\widetilde{S_n^\sigma}(h,\xi) &= \sum_{k=1}^{n-1}\left(\sigma\left(\frac{k}{n}\right)-\sigma\left(\frac{k+1}{n}\right)\right)\frac{-1}{\pi}\int_{\pi/n}^{\pi}\frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt \\ &+ \sum_{k=1}^{n-1}\left(\sigma\left(\frac{k}{n}\right)-\sigma\left(\frac{k+1}{n}\right)\right)\frac{1}{\pi}\int_{\pi/n}^{\pi/k}\frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt \\ &+ \sum_{k=1}^{n-1}\left(\sigma\left(\frac{k}{n}\right)-\sigma\left(\frac{k+1}{n}\right)\right)\frac{1}{\pi}\int_{\pi/k}^{\pi}\frac{\psi_\xi(t)}{t}\cos kt\,dt \\ &+ \sum_{k=1}^{n-1}\left(\sigma\left(\frac{k}{n}\right)-\sigma\left(\frac{k+1}{n}\right)\right)o(1) - \frac{\sigma(1)}{\pi}\int_{\pi/n}^{\pi}\frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt \\ &+ \frac{\sigma(1)}{\pi}\int_{\pi/n}^{\pi}\frac{\psi_\xi(t)}{t}\cos nt\,dt + o(1) = J_1+J_2+J_3+J_4+J_5+J_6+o(1).\end{aligned} \tag{3.11}$$

By (3.8) and Lemma 3 we get

$$J_1+J_5 = \frac{\sigma\left(\frac{1}{n}\right)}{\pi}\int_{\pi/n}^{\pi}\frac{\psi_\xi(t)}{2\tan\frac{t}{2}}\,dt = O\left(\frac{1}{n}\right)o(\ln n) = o(1) \quad \text{as} \quad n\to\infty. \tag{3.12}$$

By (3.8) for $\varepsilon>0$ there exists $\eta_1>0$ such that $\left|\psi_\xi(t)\right|<\varepsilon$, if $0<t<\eta_1$. Thus

$$|J_2| \leqq \sum_{k<\pi/\eta_1} O\left(\frac{1}{n}\right)\int_{\pi/n}^{\pi}\frac{\left|\psi_\xi(t)\right|}{t}\,dt + \sum_{\pi/\eta_1\leqq k\leqq n} O\left(\frac{1}{n}\right)\int_{\pi/n}^{\pi/k}\frac{O(\varepsilon)}{t}\,dt \tag{3.13}$$

$$= o(1) + O(\varepsilon) \sum_{\pi/\eta_1 \leqq k \leqq n} O\left(\frac{1}{n}\right) \ln \frac{n}{k}.$$

It follows from (2.1) and (3.13) that $J_2 = o(1) + O(\varepsilon)$, and hence

$$\lim_{n\to\infty} J_2 = 0. \tag{3.14}$$

In order to estimate J_3 and J_6 we consider the integral

$$P_k := \int_{\pi/k}^{\pi} \frac{\psi_\xi(t)}{t} \cos kt\, dt. \tag{3.15}$$

By Lemma 4 there exists $\eta \in (0, \eta_1)$ such that

$$\Delta\big(\psi_\xi(t), [0,\eta]\big) < \varepsilon. \tag{3.16}$$

Set $k_0 = \left[\frac{\pi}{\eta}\right]$ and for $k \geqq k_0 + 1$

$$m = m_\eta(k) = \left[\frac{1}{2}\left(\frac{k\eta}{\pi} - 1\right)\right].$$

Now

$$\begin{aligned} P_k &:= \int_{\pi/k}^{\eta} \frac{\psi_\xi(t)}{t} \cos kt\, dt + \int_{\eta}^{\pi} \frac{\psi_\xi(t)}{t} \cos kt\, dt \\ &= \sum_{j=1}^{m} \frac{k}{(2j-1)\pi} \int_{(2j-1)\pi/k}^{(2j+1)\pi/k} \psi_\xi(t) \cos kt\, dt \\ &+ \sum_{j=1}^{m} \int_{(2j-1)\pi/k}^{(2j+1)\pi/k} \left(\frac{1}{t} - \frac{k}{(2j-1)\pi}\right) \psi_\xi(t) \cos kt\, dt \\ &+ \int_{(2m+1)\pi/k}^{\eta} \frac{\psi_\xi(t)}{t} \cos kt\, dt + \int_{\eta}^{\pi} \frac{\psi_\xi(t)}{t} \cos kt\, dt. \\ &= P_{k1} + P_{k2} + P_{k3} + P_{k4}. \end{aligned} \tag{3.17}$$

Denote $I_{j,k} = \left[\frac{(2j-1)\pi}{k}, \frac{(2j+1)\pi}{k}\right]$, then

$$\left|\frac{k}{\pi} \int_{(2j-1)\pi/k}^{(2j+1)\pi/k} \psi_\xi(t) \cos kt\, dt\right| = \frac{2}{|I_{j,k}|} \left|\int_{I_{j,k}} (\psi_\xi(t) - \psi_{j,k}) \cos kt\, dt\right| \tag{3.18}$$

$$\leqq 2\mu_{I_{j,k}}(\psi_\xi)$$

where $\psi_{j,k} = \frac{1}{|I_{j,k}|} \int_{I_{j,k}} \psi_\xi(t)\, dt$. Hence by (3.18) we get

$$|P_{k1}| \leqq 2 \sum_{j=1}^{m} \frac{\mu_{I_{j,k}(\psi_\xi)}}{j}.$$

Since for $t \in \left[\frac{(2j-1)\pi}{k}, \frac{(2j+1)\pi}{k}\right]$ we have

$$\frac{1}{t} - \frac{k}{(2j-1)\pi} = O\left(\frac{1}{kt^2}\right).$$

Therefore

$$P_{k2} = O\left(\frac{1}{k}\right) \int_{\pi/k}^{\eta} \frac{\bigl|\psi_\xi(t)\bigr|}{t^2}\, dt = o(1). \tag{3.19}$$

It is evident that

$$P_{k3} + P_{k4} = o(1), \tag{3.20}$$

Hence by (3.15)–(3.20),

$$|P_k| \leqq O(1) \sum_{j=1}^{m} \frac{\mu_{I_{j,k}(\psi_\xi)}}{j} + o(1) = O(\varepsilon) + o(1). \tag{3.21}$$

It follows that

$$|J_3| \leqq \left| \sum_{k \leqq k_0} \left(\sigma\left(\frac{k}{n}\right) - \sigma\left(\frac{k+1}{n}\right)\right) P_k \right|$$

$$+ \left| \sum_{k \geqq k_0} \left(\sigma\left(\frac{k}{n}\right) - \sigma\left(\frac{k+1}{n}\right)\right) P_k \right|$$

$$= \sum_{k \leqq k_0} O\left(\frac{1}{n}\right) O(1) + \sum_{k \geqq k_0}^{n} O\left(\frac{1}{n}\right) \bigl(O(\varepsilon) + o(1)\bigr) = O(\varepsilon) + o(1),$$

as $n \to \infty$. Therefore we have

$$\lim_{n\to\infty} J_3 = 0. \tag{3.22}$$

It follows from (3.21) that we also have

$$\lim_{n\to\infty} J_6 = 0. \tag{3.23}$$

By combining (3.11), (3.12), (3.14), (3.22) and (3.23) we obtain (3.2). □

3.2. Proof of Corollary 1. Set $p = 1$ and $\sigma(x) = -\pi x$. Then σ satisfies the assumptions of Theorem 1. By Theorem 1 we get (1.8) with $p = 1$. By Lemma 5 we see that this is equivalent to (1.8) with arbitrary $p > 0$. □

3.3. Proof of Theorem 2. In [3], Kvernadze constructed an example $f \in C \cap \Lambda\mathrm{BV}$ such that

$$\lim_{n\to\infty} \sup \frac{\left| S_n(f,0)^{(2r+1)} \right|}{n^{2r+1}} > 0,$$

hence (1.6) does not hold. Since $\Lambda\mathrm{BV} \subset \Lambda\mathrm{BMV}$, this example works for proving Theorem 2. We omit the details. □

3.4. Proof of Theorem 3. By Corollary 1, (1.8) holds for $p = 1$. Then by Lemma 6 we get the conclusion of Theorem 3. □

3.5. Proof of Theorem 4. The same counterexample as Theorem 2 can be used to prove Theorem 4. We omit the details. □

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