Existence and Decay Rates of Solutions to the Generalized Burgers Equation

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Abstract. In this paper we study the generalized Burgers equation $u_t + (u^2/2)_x = f(t)u_{xx}$, where $f(t) > 0$ for $t > 0$. We show the existence and uniqueness of classical solutions to the initial value problem of the generalized Burgers equation with rough initial data belonging to $L^\infty(\mathbb{R})$, as well it is obtained the decay rates of u in L^p norm are algebra order for $p \in [1, \infty[$.

Key Words: generalized Burgers equation, existence, uniqueness, classical solutions, decay rates

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1. Introduction

In this paper we will consider the following initial value problem for the generalized Burgers equation

$$u_t + (u^2/2)_x = f(t)u_{xx}, \quad x \in \mathbb{R}, \ t > 0 \tag{1}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_0 \in L^\infty(\mathbb{R}), \tag{2}$$

where $f(t)$ is positive for $t > 0$.

The equation (1) is the mathematical model of the propagation of the finite-amplitude sound waves in variable-area ducts (see Crighton [1]), where u is an acoustic variable, with the linear effects of changes in the duct area taken out, and the coefficient $f(t)$ is a positive function that depends on the particular duct chosen. It also can be derived from the system of compressible Navier-Stokes equations with planar, cylindrical, sub-cylindrical, super-cylindrical and spherical symmetry, when the method of multiple scales is used, see Sachdev [12], Leibovich and Seebass [10]. The long time behavior of solutions to the initial value problem has been studied, e.g. by Crighton and Scott [2] as well as Scott [14] under the assumption of the well-posedness of the initial value problem (1), (2) without verification. It is well known that in general solutions to the initial value problem for the inviscid Burgers equation $u_t + (u^2/2)_x = 0$ will develop singularity in finite time even the corresponding initial data is smooth. The equation (1) is a uniformly parabolic equation if $f(t) \geq \nu > 0$ for $t > 0$. The well-posedness of the corresponding initial value problem is well known (see [4][8]). Particularly the Burgers equation $u_t + (u^2/2)_x = \mu u_{xx}$ has been used by Hopf [7] to study the inviscid Burgers equation by letting μ tend to zero. But the equation (1) is a non-uniformly parabolic equation if $f(t)$ has no positive lower bound. To our knowledge, there is no general theory to guarantee the well-posedness of the classical solution of the generalized Burgers equation as a non-uniformly parabolic equation . Wang and Warnecke [16] show the existence and uniqueness of the classical solution to the initial value problem of the generalized Burgers equation with $f(t) = t$. The case $f(t) = t$ is called the cylindrical case in the model equation of nonlinear acoustics (see Crighton [1]). As a next natural step we consider the equation (1) with general form of $f(t)$ in this paper. Then the super-cylindrical case e.g. $f(t) = t^\alpha$ where $1 < \alpha < \infty$ and the sub-cylindrical case e.g. $f(t) = t^\alpha$ where $0 < \alpha < 1$ and cylindrical case serve as its concrete subcases of physical meaning. In fact we will show the initial value problem of the generalized Burgers equation with L^∞ initial data admits a unique classical solution if $f(t)$ is positive for $t > 0$. In other words, the positivity of $f(t)$ prevents the corresponding solution from developing singularity and has a smooth effect on the solution when the initial data is rough no matter how fast $f(t)$ tends to zero as t tends to zero.

In this paper we will show the existence and uniqueness of classical solutions to the initial value problem (1),(2) when the initial data only belong to $L^\infty(\mathbb{R})$. It is straightforward to extend the results obtained in this paper to the type of the equation (1) with a general convex flux function in its convection term instead of the quadratic function considered here. Meanwhile, it is shown that decay rates of u in some norms are algebra order.

In the Section 2, we first show that in the definition of weak solutions to the initial value problem (1), (2) we may use more general test functions that do not have compact support. This allows us to use solutions to the adjoint problem as test functions.The corollary 2.3 describes the relation between the forenamed more general test functions and the test functions with compact support.Although the proof of the corollary is given in [16] we have included it in order make our exposition self-contained. Secondly, the section 2 is devoted to the uniqueness of weak solution. It is shown by a nonlinear version of the Holmgren method, which was used by Oleinik [11] and Hoff [5] for convex conservation laws. We estimate the decay rates of solutions, as well as their derivatives, to the adjoint parabolic equation for the difference of two solutions to the initial value problem (1), (2). Finally we show that the weak solutions of the initial value problem (1), (2) are classical solutions in the sense that they have all of the continuous derivatives occurring in equation (1). The slightly stronger version of a one-sided Lipschitz condition that was given by Tadmor [15] is used in the process of the proof.

The Section 3 is devoted to the decay rates of the solution obtained in the above sections in L^p norm for $p \in]1, \infty[$. It is strongly motivated by the work of M.E. Schonbek [13]. We show that the decay rates are the same as the ones of the solution to the equation without the nonlinear term $(u^2/2)_x$ in the case $0 < f'(t) \leq 1$ for $t \geq 1$. But in the case $f'(t) \geq 0$ and $f''(t) \geq 0$ we have not obtained the decay rates as sharp as ones in aforementioned case. These results are given in Theorem 3.3 and 3.5.

In the last section, we indicate how the existence of the weak solutions to the initial value problem (1), (2) may also be obtained via a finite difference scheme with variable time steps. As matter of fact the scheme can be used as a numerical method for the computation of approximate solutions to this problem. It is interesting to note that for the cylindrical case $f(t) = t$, considered in [16], the first n_0 steps of the scheme proposed here use a constant time step when the Lax-Friedrichs scheme is taken to approximate conservation laws. This number n_0 depends only on the supremum norm of the initial data. The first n_0 steps of the scheme deal with the non-uniform parabolicity as t tends to 0 and ensure that the scheme satisfies the CFL stability condition. Thereafter variable time steps are used in order to be consistent with the generalized Burgers equation (1). But for the super-cylindrical case $f(t) = t^\alpha$ where $1 < \alpha < \infty$, the number n_0 with constant time step is order of $l^{\frac{1}{\alpha}-1}$, here l is space mesh length and for sub-cylindrical case $f(t) = t^\alpha$, where $0 < \alpha < 1$ the variable time steps begin at the second step.

2. Existence and uniqueness of the classical solution

In this section we will investigate the existence and uniqueness of the classical solution to the initial value problem (1) and (2). It is the way that we first obtain the existence and uniqueness of the weak solutions, and then improve the regularity of the weak solution.

2.1 Definition of weak solutions

Definition 2.1 *A bounded measurable function u is called a* **weak solution** *of the initial value problem (1) and (2) if it satisfies the following conditions:*

$$\int_{t_1}^{t_2}\int_{-\infty}^{\infty}[u\phi_t + \frac{u^2}{2}\phi_x + f(t)u\phi_{xx}]\,dxdt - \int_{-\infty}^{\infty}(u\phi)(x,\cdot)|_{t_1}^{t_2}\,dx = 0 \tag{3}$$

for any $0 < t_1 < t_2$ and all $\phi \in C_0^2(\mathbb{R}\times\mathbb{R}_+)$ where $\mathbb{R}_+ = [0,\infty[$. Additionally we assume that the solution u satisfies for any $t > 0$ the one-sided Lipschitz condition

$$L^+[u(\cdot,t)] := ess\sup_{x\neq y}\left(\frac{u(x,t)-u(y,t)}{x-y}\right)^+ \leq \frac{1}{t} \tag{4}$$

and for the initial data we require that $u(\cdot,t)$ tends to u_0 in $L^1_{loc}(\mathbb{R})$ as $t \to 0$. □

2.2 Existence of weak solutions

Theorem 2.2. *Let* $u_0 \in L^\infty(\mathbb{R})$*. Then there exists a weak solution* u *of the initial value problem (1), (2) having the properties that* $u(\cdot, t)$ *converges in* $L^1_{loc}(\mathbb{R})$ *to* u_0 *for* $t \to 0$*, it satisfies a one-sided Lipschitz condition (4) and the bound*

$$|u(x,t)| \leq \|u_0\|_{L^\infty} = M.$$

Proof. We use the vanishing viscosity method. Consider for $\varepsilon > 0$ the nonsingular parabolic equation

$$u_t + (u^2/2)_x = (f(t) + \varepsilon)u_{xx}, \quad 0 < \varepsilon \leq 1.$$

The existence of weak solutions for the uniformly parabolic case $\varepsilon > 0$ and the properties of the singular limit $\varepsilon \to 0$ follow by standard theory along the lines of Oleinik [11] and Kružkov [9] analogously as in the case of conservation laws with convex flux functions. The proof of the one-sided Lipschitz condition can be given along the line of the argument by Tadmor [15] with slight modifications. □

Corollary 2.3. *If equality (3) holds for* $\phi \in C_0^2(\mathbb{R} \times \mathbb{R}_+)$ *this implies that the equality (3) holds for all*

$$\begin{aligned} &\phi \in C^2(\mathbb{R} \times \mathbb{R}_+), \\ &\phi, \phi_x, \phi_{xx} \text{ and } \phi_t \text{ belonging to } L^1(\mathbb{R} \times [t_1, t_2]) \text{ for any } t_2 > t_1 > 0. \end{aligned} \tag{5}$$

Proof. We introduce for $N > 0$ a cut-off function

$$\xi_N(x) = \int_{-\infty}^{\infty} \chi(x-y)\eta_N(y)\, dy$$

by taking a standard non-negative mollifying function $\chi \in C_0^\infty(\mathbb{R})$ with unit mass supported in the interval $[-1, 1]$. The function η_N is the characteristic function of the interval $[-N-1, N+1]$, i.e.

$$\eta_N(x) = \begin{cases} 1, & \text{for } |x| \leq N+1 \\ 0, & \text{for } |x| > N+1. \end{cases}$$

It is easy to see that

$$
\begin{aligned}
&\xi_N \in C_0^\infty(\mathbb{R}),\\
&\xi_N(x) = \begin{cases} 1, & \text{for } |x| \le N \\ 0, & \text{for } |x| \ge N+2. \end{cases} \\
&|\xi_N(x)| \le 1, \quad |\xi_N'(x)| \le C-1, \quad |\xi_N''(x)| \le C_1,
\end{aligned} \tag{6}
$$

where

$$
C := \int_{-\infty}^{\infty} |\chi'(s)|\, ds + 1, \quad C_1 := \int_{-\infty}^{\infty} |\chi''(s)|\, ds. \tag{7}
$$

Now consider the quantity $Q(u,\phi)$ obtained by substituting ϕ with the properties (5), instead of a $\phi \in C_0^2(\mathbb{R}\times\mathbb{R}_+)$, into the left hand side of (3), i.e.

$$
\begin{aligned}
Q(u,\phi) &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (u\phi_t + \frac{u^2}{2}\phi_x + f(t)u\phi_{xx})\, dxdt - \int_{-\infty}^{\infty} (u\phi)(x,\cdot)|_{t_1}^{t_2}\, dx \\
&= Q(u,\phi\xi_N) + Q(u,\phi(1-\xi_N)).
\end{aligned} \tag{8}
$$

It is easy to see from (5), (6) that $\phi\xi_N \in C_0^2(\mathbb{R})$. Therefore it follows by the assumption of the corollary that it is an admissible test function for which

$$
\begin{aligned}
Q(u,\phi\xi_N) &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (u(\phi\xi_N)_t \\
&\quad + \frac{u^2}{2}(\phi\xi_N)_x + f(t)u(\phi\xi_N)_{xx})\, dxdt - \int_{-\infty}^{\infty} (u\phi\xi_N)(x,\cdot)|_{t_1}^{t_2}\, dx \\
&= 0.
\end{aligned} \tag{9}
$$

Now let us estimate $Q(u,\phi(1-\xi_N))$ as follows

$$
\begin{aligned}
|Q(u,\phi(1-\xi_N))| &= \Big| \int_{t_1}^{t_2}\int_{-\infty}^{\infty} [u(\phi(1-\xi_N))_t \\
&\quad + \frac{u^2}{2}(\phi(1-\xi_N))_x + f(t)u(\phi(1-\xi_N))_{xx}]\, dxdt \\
&\quad - \int_{-\infty}^{\infty} (u\phi(1-\xi_N))(x,\cdot)|_{t_1}^{t_2}\, dx \Big| \\
&\le \int_{t_1}^{t_2}\int_{|x|\ge N} |[u(\phi(1-\xi_N))_t| + |\frac{u^2}{2}(\phi(1-\xi_N))_x| \\
&\quad + |f(t)u(\phi(1-\xi_N))_{xx}]|\, dxdt \\
&\quad + \int_{|x|\ge N} |(u\phi(x,\cdot)|_{t_1}^{t_2}(1-\xi_N))|\, dx
\end{aligned}
$$

where (6) was used. We consider the properties of ϕ given by (5), the bound of u, and the estimates on ξ_N in (6). Then we have for any given $\varepsilon > 0$, taking $N = N(\varepsilon)$ large enough,

$$|Q(u, \phi(1 - \xi_N))| \leq \varepsilon. \tag{10}$$

It follows from (8), (9) and the arbitrariness of ε in (10) that $Q(u, \phi) = 0$. □

2.4 Uniqueness of the weak solution

We will show that weak solutions of the initial value problem (1), (2) which satisfy Definition 2.1 are unique. The method of proof we give is a nonlinear version of the method of Holmgren that Hoff [5] and Oleinik [11] used for convex scalar conservation laws.

We take two solutions u and v and modify them using a standard nonnegative mollifying function $\chi \in C_0^\infty(\mathbb{R})$ with unit mass that is supported in $[-1, 1]$ to obtain for any $\delta \in]0, t_1[$

$$\begin{aligned} u_\delta(x,t) &= \frac{1}{\delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi\left(\frac{x-y}{\delta}\right) \chi\left(\frac{t-\tau}{\delta}\right) u(y,\tau)\, dy d\tau, \\ v_\delta(x,t) &= \frac{1}{\delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi\left(\frac{x-y}{\delta}\right) \chi\left(\frac{t-\tau}{\delta}\right) v(y,\tau)\, dy d\tau. \end{aligned}$$

Then we consider the *adjoint problem*, namely the backward initial value problem for any given $0 < t_1 < t_2$

$$\phi_t + \frac{1}{2}(u_\delta + v_\delta)\phi_x + f(t)\phi_{xx} = 0, \qquad t \in]t_1, t_2[, \tag{11}$$

$$\phi(x, t_2) = \psi(x), \tag{12}$$

where $\psi \in C_0^\infty(\mathbb{R})$ is any given function. The equation (11) is a linear uniformly parabolic equation with smooth coefficients. It has smooth solutions and satisfies the maximum principle, see Friedman [4]. We take

$$M_0 := \|\psi(\cdot)\|_{L^\infty}, \quad M_1 := \|\psi'(\cdot)\|_{L^\infty},$$

$$M_2 := \|\psi''(\cdot)\|_{L^\infty}, \quad \text{and} \quad \mathrm{supp}\psi \subset [-K, K].$$

First we give the following properties of the solution to the adjoint problem. Its proof is in the appendix.

Lemma 2.4. *If $f(t)$ is a nondecreasing positive function of t for $t > 0$ then the solution of the adjoint problem (11) and (12) satisfies the following estimates for $t \in [t_1, t_2]$*

$$|\phi(x,t)| \leq M_0 \min\left\{1, \exp\left(C(t_2 - t) + \frac{(M+1)(t_2-t) + K + 2 - |x|}{f(t_2)}\right)\right\}, \tag{13}$$

$$|\phi_x(x,t)| \leq \frac{t_2}{t_1} M_1 \min\left\{1, \exp\left(C(t_2 - t) + \frac{(M+1)(t_2-t) + K + 2 - |x|}{f(t_2)}\right)\right\}, \tag{14}$$

$$|\phi_{xx}(x,t)| \leq \left(\frac{t_2^2}{t_1^2} M_2 + \frac{C_1 t_2^3 M M_1}{t_1^3 \delta^2}\right) \cdot \exp\left(C(t_2 - t) + \frac{(M+1)(t_2-t) + K + 2 - |x|}{f(t_2)}\right) \tag{15}$$

and

$$|\phi_t(x,t)| \leq f(t_2)|\phi_{xx}(x,t)| + M|\phi_x(x,t)|, \tag{16}$$

where the constants C as well as C_1 were specified in (7) and M in Theorem 2.2. Then by Corollary 2.3 the function ϕ is admissible as a test function in (3).

Theorem 2.5. *Let u and v be two solutions of the initial value problem (1), (2) as obtained in Theorem 2.2. Then $u = v$ almost everywhere for $t > 0$ as $f(t)$ is a nondecreasing positive function of t for $t > 0$.*

Proof. Since u and v are weak solutions of the initial value problem (1), (2) we have by (3)

$$\begin{aligned}
\int_{-\infty}^{\infty} (u\phi)(x,\cdot)|_{t_1}^{t_2}\, dx &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (u\phi_t + \frac{u^2}{2}\phi_x + uf(t)\phi_{xx})\, dxdt, \\
\int_{-\infty}^{\infty} (v\phi)(x,\cdot)|_{t_1}^{t_2}\, dx &= \int_{t_1}^{t_2}\int_{-\infty}^{\infty} (v\phi_t + \frac{v^2}{2}\phi_x + vf(t)\phi_{xx})\, dxdt.
\end{aligned}$$

We subtract these two equations and set $d = u - v$. Then we get

$$\int_{-\infty}^{\infty} (d\phi)(x,\cdot)|_{t_1}^{t_2}\, dx = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} d(\phi_t + \frac{u+v}{2}\phi_x + f(t)\phi_{xx})\, dxdt$$

and therefore using the adjoint equation (11)

$$\int_{-\infty}^{\infty} (d\phi)(x,\cdot)|_{t_1}^{t_2}\, dx = \int_{t_1}^{t_2}\int_{-\infty}^{\infty} d\frac{u+v-(u_\delta + v_\delta)}{2}\phi_x\, dxdt. \tag{17}$$

Since $|u|, |v|, |u_\delta|$ and $|v_\delta|$ are bounded by the constant M considered in Theorem 2.2 and ϕ_x tends to 0 exponentially in x uniformly for $t \in [t_1, t_2]$ by Lemma 2.4, we have for any given $\varepsilon > 0$

$$\left| \int_{t_1}^{t_2} \int_{|x| \geq N} d \frac{u + v - (u_\delta + v_\delta)}{2} \phi_x \right| \, dxdt \leq \varepsilon \tag{18}$$

provided that N is large enough. From the estimate (14) for ϕ_x we also obtain on the complementary set $S_N^{t_1,t_2} = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ | \ |x| \leq N, t_1 \leq t \leq t_2\}$ the estimate

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{|x| \leq N} d \frac{u + v - (u_\delta + v_\delta)}{2} \phi_x \, dxdt \\ &\quad \leq M \|\phi_x\|_{L^\infty(S_N^{t_1,t_2})} \left(\|u - u_\delta\|_{L^1(S_N^{t_1,t_2})} + \|v - v_\delta\|_{L^1(S_N^{t_1,t_2})} \right) \\ &\quad \leq \frac{t_2}{t_1} M M_1 \left(\|u - u_\delta\|_{L^1(S_N^{t_1,t_2})} + \|v - v_\delta\|_{L^1(S_N^{t_1,t_2})} \right). \end{aligned} \tag{19}$$

Applying (18) and (19) to (17) and letting $\delta \to 0$ gives

$$\int_{-\infty}^{\infty} (d\phi)(x, \cdot)|_{t_1}^{t_2} \, dx \leq \varepsilon.$$

¿From the arbitrariness of ε we infer that

$$\int_{-\infty}^{\infty} (d\phi)(x, \cdot)|_{t_1}^{t_2} \, dx \leq 0$$

or

$$\int_{-\infty}^{\infty} (d\phi)(x, t_2) \, dx \leq \int_{-\infty}^{\infty} (d\phi)(x, t_1) \, dx. \tag{20}$$

We observe that u and $v \to u_0$ in $L^1_{loc}(\mathbb{R})$ for $t \to 0$ by Theorem 2.2 and deduce from (13) the estimate

$$\phi(x, 0) \leq M_0 \exp\left(\frac{C t_2 f(t_2) + M t_2 + t_2 + (K + 2 - |x|)}{f(t_2)} \right).$$

Then we obtain that the right hand side of (20) tends to zero as $t_1 \to 0$ and therefore

$$\int_{-\infty}^{\infty} d(x, t_2) \psi(x) \, dx \leq 0.$$

By the arbitrariness of t_2 and the fact that the inequality must be satisfied by any test function ψ as well as by $-\psi$ this implies that $u(x,t) = v(x,t)$ almost everywhere on $\mathbb{R} \times \mathbb{R}_+$. □

2.5 The regularity of the weak solution

In this subsection we will show that the weak solutions of the initial value problem (1), (2) have all continuous derivatives occurring in equation (1) even for initial data $u_0 \in L^\infty(\mathbb{R})$.

Theorem 2.6. *Assume $u_0 \in L^\infty(\mathbb{R})$ and $f(t)$ is a nondecreasing positive function of t for $t > 0$. Then the weak solution u to the initial value problem (1), (2) has all continuous derivatives occurring in the equation (1).*

Proof. For any given $t_0 > 0$ consider the function

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & 0 < t \leq t_0, \\ \overline{u}(x,t), & t_0 < t. \end{cases}$$

Here $\overline{u}(x,t)$ is the solution of the following initial value problem

$$\overline{u}_t + (\overline{u}^2/2)_x = f(t)\overline{u}_{xx}, \quad x \in \mathbb{R}, \ t > t_0, \tag{21}$$

$$\overline{u}(x,t_0) = u(x,t_0). \tag{22}$$

The function $\overline{u}(x,t)$, as the solution of a uniformly parabolic equation (21), has all continuous derivatives occurring in (21), i.e. $\overline{u}_x, \overline{u}_{xx}$ and $\overline{u}_t$ are continuous.

Now we shall prove that $\tilde{u}$ is a weak solution of the initial value problem (1), (2). According to Definition 2.1 we only have to prove that $\overline{u}$ satisfies the one-sided Lipchitz condition (4). To the initial value problem (21), (22) we can easily obtain that

$$L^+(\overline{u}(\cdot,t)) \leq \frac{1}{\frac{1}{L^+(u(\cdot,t_0))} + (t - t_0)}, \quad \text{for } t > t_0. \tag{23}$$

along the line of argument given by Tadmor [15, Theorem 3.1] for a parabolic equation. Since u is a weak solution of the initial value problem (1), (2) we have

$$L^+(u(\cdot,t_0)) \leq \frac{1}{t_0}. \tag{24}$$

Substituting (24) into (23) gives

$$L^+(\overline{u}(\cdot,t)) \leq \frac{1}{t} \quad \text{for } t > t_0.$$

By the uniqueness shown in Theorem 2.5 we deduce that

$$u(x,t) = \tilde{u}(x,t), \ t > 0.$$

Therefore, the derivatives u_x, u_{xx}, u_t are continuous for $t > t_0$. The arbitrariness of $t_0 > 0$ implies that u_x, u_{xx}, u_t are continuous for $t > 0$. $\square$

3. Decay rates of the solution

In this section we will obtain decay rates of the solution of (1) and (2) in L^p-norm.

Lemma 3.1 *The solution $u(x,t)$ obtained in Theorem 2.6 satisfies*

$$\int_{-\infty}^{\infty} |u(x,t)|dx \leq \int_{-\infty}^{\infty} |u_0(x)|dx, \ t > 0 \tag{25}$$

if $u_0 \in L^1(\mathbb{R})$.

Proof. Let g be the solution of the adjoint equation $\partial_t g + u/2\partial_x g + f(t)\partial_x^2 g = 0$ with the Cauchy data $g(x,T) = \gamma(x) \in C_0^\infty$ and variable $T - t$. By using the maximum principle in chapter 3 of [6], we have if $|\gamma(x)| \leq 1$ then $|g(x,t)| \leq 1$. Since

$$\int_0^T \int_{-\infty}^{\infty} \partial_t(ug)dxdt = \int_{-\infty}^{\infty} u(x,T)\gamma(x)dx - \int_{-\infty}^{\infty} u(x,0)g(x,0)dx$$

and the equations satisfied by g and u give

$$\int_0^T \int_{-\infty}^{\infty} \partial_t(ug)dxdt = \int_0^T \int_{-\infty}^{\infty} g\partial_t u + u\partial_t g dxdt = 0.$$

We have

$$\int_{-\infty}^{\infty} u(x,T)\gamma(x)dx = \int_{-\infty}^{\infty} u(x,0)g(x,0)dx.$$

Therefore

$$\int_{-\infty}^{\infty} u(x,T)\gamma(x)dx \leq \int_{-\infty}^{\infty} |u_0(x)|dx \tag{26}$$

holds for any given $\gamma(x)$ such that $\gamma(x) \in C_0^\infty$ and $|\gamma(x)| \leq 1$. The arbitrariness of γ and (26) implies (25). $\square$

3.1 The case $f'(t) \geq 0$ and $f''(t) \geq 0$

Lemma 3.2 *If $u_0 \in L^1$, then for all $p = 2^s$ with $s \geq 1$ positive integers, the following estimate holds for the sufficiently large t*

$$\|u(\cdot,t)\|_{L^p} \leq 2^{17/4}\|u_0\|_{L^1}(f'(t))^{\frac{1}{2}(1-\frac{1}{p})}(f(t)^2+1)^{-\frac{1}{2}(1-\frac{1}{p})} \tag{27}$$

as

$$f'(t) \geq 0 \ \text{and} \ f''(t) \geq 0. \tag{28}$$

Proof. We will prove it by using the inductive method. First we show (27) is true for $s = 1$. Multiplying (1) by u and integrating in space yields

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{|u|^2}{2} dx = -f(t) \int_{-\infty}^{\infty} |u_x|^2 dx. \tag{29}$$

Let

$$\hat{u}(\xi, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ix\xi} dx \tag{30}$$

and

$$A(t) = \left\{ \xi : |\xi| > (f(t)^2 + 1)^{-1/2} (f'(t))^{1/2} \right\}.$$

Applying the Fourier transform and the Plancherel's equality, we have

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} \frac{|\hat{u}|^2}{2} d\xi &= -f(t) \int_{-\infty}^{\infty} |\xi|^2 |\hat{u}|^2 d\xi \leq -f(t) \int_{A(t)} |\xi|^2 |\hat{u}|^2 d\xi \\ &\leq -\frac{f(t) f'(t)}{f^2(t) + 1} \int_{A(t)} |\hat{u}|^2 d\xi, \end{aligned}$$

That is,

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{|\hat{u}|^2}{2} d\xi + \frac{f(t) f'(t)}{f^2(t) + 1} \int_{-\infty}^{\infty} |\hat{u}|^2 d\xi \leq \frac{f(t) f'(t)}{f^2(t) + 1} \int_{A^C(t)} |\hat{u}|^2 d\xi. \tag{31}$$

Multiplying (31) by $(f(t)^2 + 1)$ yields

$$\frac{d}{dt}\left[(f(t)^2 + 1) \int_{-\infty}^{\infty} \hat{u}^2 d\xi \right] \leq 2f'(t) f(t) \int_{A^C(t)} |\hat{u}|^2 d\xi. \tag{32}$$

By (30) and Lemma 3.1, we have

$$\|\hat{u}\|_{L^\infty} \leq \int_{-\infty}^{\infty} |u| dx \leq \int_{-\infty}^{\infty} |u_0| dx. \tag{33}$$

By using (33), we can obtain the further estimate of (32)

$$\begin{aligned} \frac{d}{dt}\left[(f(t)^2 + 1) \int_{-\infty}^{\infty} |\hat{u}|^2 d\xi \right] &\leq 2f'(t) f(t) \|\hat{u}\|_{L^\infty}^2 \int_{A^C(t)} d\xi \\ &\leq 4(f'(t))^{3/2} f(t) \|u_0\|_{L^1}^2 (f(t)^2 + 1)^{-1/2}. \end{aligned}$$

Integrating from 0 to t yields

$$
\begin{aligned}
&(f(t)^2+1)\int_{-\infty}^{\infty}|\hat{u}(\xi,t)|^2d\xi \\
&\le \int_{-\infty}^{\infty}|\widehat{u_0}(\xi)|^2d\xi+\int_0^t 4(f'(s))^{3/2}f(s)\|u_0\|_{L^1}^2(f(s)^2+1)^{-1/2}ds. \\
&\le \int_{-\infty}^{\infty}|\widehat{u_0}(\xi)|^2d\xi+4\|u_0\|_{L^1}^2(f'(t))^{1/2}\int_0^t f(s)f'(s)(f(s)^2+1)^{-1/2}ds \\
&\le \int_{-\infty}^{\infty}|\widehat{u_0}(\xi)|^2d\xi+4\|u_0\|_{L^1}^2(f'(t))^{1/2}(f(t)^2+1)^{1/2},
\end{aligned}
$$

By the Plancherel's equality, we have

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^2}^2 &\le [\|u_0\|_{L^2}^2+4\|u_0\|_{L^1}^2(f'(t))^{1/2}(f(t)^2+1)^{1/2}](f(t)^2+1)^{-1} \\
&\le C_{1,1}(f(t)^2+1)^{-1}+4\|u_0\|_{L^1}^2(f'(t))^{1/2}(f(t)^2+1)^{-1/2},
\end{aligned}
$$

where $C_{1,1}$ is positive constant depending on $\|u_0\|_{L^\infty}$ and $\|u_0\|_{L^1}$.

At first we set

$$
C_p=\begin{cases} 2\|u_0\|_{L^1}, & s=1; \\ 2^{(2+3s/2)/2^s}C_{p/2}, & s\ge 2. \end{cases} \tag{34}
$$

Next we suppose that for any $t\in[0,\infty[$,

$$
\begin{aligned}
\int_{-\infty}^{\infty}|u(x,t)|^q dx &\le \sum_{i=1}^{s-1}C_{s-1,i}(f'(t))^{a_{s-1,i}/2}(f(t)^2+1)^{-b_{s-1,i}/2} \\
&\quad +C_q^q(f'(t))^{(q-1)/2}(f(t)^2+1)^{-(q-1)/2},
\end{aligned} \tag{35}
$$

are true for $q=2^{s-1}(s\ge 2)$. Here $C_{s-1,i}$ are positive constants depending on $s-1,u_0$ and $a_{s-1,i},b_{s-1,i}$ also positive constants only depending on s. For example,

$$
C_{1,1}=\|u_0\|_{L^\infty}\|u_0\|_{L^1},\ \ a_{1,1}=0,\ \ b_{1,1}=2;
$$

$$
C_{2,1}=2\|u_0\|_{L^\infty}^3\|u_0\|_{L^1},\ \ a_{2,1}=0,\ \ b_{2,1}=4,
$$

$$
C_{2,2}=32C_{1,1},\ \ a_{2,2}=3,\ \ b_{2,2}=5.
$$

Moreover

$$
a_{s-1,i}<q-1,\quad b_{s,i}>q-1.
$$

We will prove

$$\int_{-\infty}^{\infty} |u(x,t)|^p dx \;\le\; \sum_{i=1}^{s} C_{s,i}(f'(t))^{a_{s,i}/2}(f(t)^2+1)^{-b_{s,i}/2} + C_p^p(f'(t))^{(p-1)/2}(f(t)^2+1)^{-(p-1)/2}, \tag{36}$$

and

$$a_{s,i} < p-1, \quad b_{s,i} > p-1.$$

Multiplying (1) by u^{p-1} and integrating in space implies

$$\frac{1}{p}\frac{d}{dt}\int_{-\infty}^{\infty}|u|^p dx + \int_{-\infty}^{\infty}\left(\frac{u^{p+1}}{p+1}\right)_x dx = f(t)\int_{-\infty}^{\infty} u^{p-1}u_{xx}dx.$$

The second integral on the left-hand side vanishes, hence after an integration by parts in the right-hand, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty}|u|^p dx = -p(p-1)f(t)\int_{-\infty}^{\infty}u^{p-2}u_x^2 dx.$$

Noting that

$$u^{p-2}u_x^2 = (u^{q-1}u_x)^2 = [(\frac{u^q}{q})_x]^2 = \frac{1}{q^2}[(u^q)_x]^2.$$

It follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty}|u|^p dx = -\frac{p(p-1)f(t)}{q^2}\int_{-\infty}^{\infty}[(u^q)_x]^2 dx \le -f(t)\int_{-\infty}^{\infty}[(u^q)_x]^2 dx,$$

where the last inequality follows since $p(p-1)q^{-2} \ge 1$ for $s \ge 2$. Applying Plancherel's theorem to the last inequality yields

$$\frac{d}{dt}\int_{-\infty}^{\infty}|w|^2 d\xi \le -f(t)\int_{-\infty}^{\infty}|\xi|^2|w|^2 d\xi, \tag{37}$$

where we have let $w = \widehat{u^q}$. Let

$$A_q = \left\{ |\xi| : |\xi| > \left(\frac{2qf'(t)}{f(t)^2+1}\right)^{1/2} \right\}. \tag{38}$$

We now split the integral on the right-hand side of (37) into an integral over A_q and one over A_q^C and obtain

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}|w|^2 d\xi &\le -f(t)\int_{-\infty}^{\infty}|\xi|^2|w|^2 d\xi = -f(t)\left(\int_{A_q} + \int_{A_q^C}\right)|\xi|^2|w|^2 d\xi \\
&\le -f(t)\int_{A_q}|\xi|^2|w|^2 d\xi \\
&\le -\frac{2qf'(t)f(t)}{f(t)^2+1}\int_{-\infty}^{\infty}|w|^2 d\xi + \frac{2qf'(t)f(t)}{f(t)^2+1}\int_{A_q^C}|w|^2 d\xi
\end{aligned}$$

The last inequality is now multiplied by the integrating factor $(f(t)^2+1)^q$,

$$\frac{d}{dt}\left[(f(t)^2+1)^q\int_{-\infty}^{\infty}|w|^2d\xi\right]\le(f(t)^2+1)^q\frac{2qf'(t)f(t)}{f(t)^2+1}\int_{A_q^C}|w|^2d\xi. \tag{39}$$

Hence

$$\begin{aligned}&\frac{d}{dt}\left[(f(t)^2+1)^q\int_{-\infty}^{\infty}|w|^2d\xi\right]\\ &\le(f(t)^2+1)^q\|w\|_{L^\infty}^2\frac{2qf(t)f'(t)}{f(t)^2+1}\int_{A_q}d\xi\\ &\le 2(f(t)^2+1)^q\|w\|_{L^\infty}^2\frac{2qf(t)f'(t)}{f(t)^2+1}\left(\frac{2qf'(t)}{f(t)^2+1}\right)^{1/2}\\ &=2(2q)^{3/2}f(t)(f'(t))^{3/2}(f(t)^2+1)^{q-3/2}\|w\|_{L^\infty}^2.\end{aligned} \tag{40}$$

By (35) we have

$$\begin{aligned}\|w\|_{L^\infty}^2 &\le (\int_{-\infty}^{\infty}|u|^qdx)^2\\ &\le 2(s-1)\left(\sum_{i=1}^{s-1}C_{s-1,i}^2(f'(t))^{a_{s-1,i}}(f^2+1)^{-b_{s-1,i}}\right)\\ &\quad +2C_q^{2q}(f'(t)^{q-1}(f(t)^2+1)^{-(q-1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\left[(f(t)^2+1)^q\int_{-\infty}^{\infty}|w|^2d\xi\right]\\ &\le 4(2q)^{3/2}f(t)(f'(t))^{3/2}(f(t)^2+1)^{q-3/2}\cdot\\ &\quad (s-1)\left(\sum_{i=1}^{s-1}C_{s-1,i}^2(f'(t))^{a_{s-1,i}}(f(t)^2+1)^{-b_{s-1,i}}\right)\\ &\quad +4(2q)^{3/2}C_q^{2q}f(t)(f'(t))^{q-1+3/2}(f(t)^2+1)^{-1/2}.\end{aligned} \tag{41}$$

Integrating with respect to t from 0 to t, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|u(x,t)|^pdx &\le \sum_{i=1}^{s}C_{s,i}(f'(t))^{a_{s,i}/2}(f(t)^2+1)^{-b_{s,i}/2}\\ &\quad +2^{2+3s/2}C_q^p(f'(t))^{(p-1)/2}(f(t)^2+1)^{-(p-1)/2}.\end{aligned} \tag{42}$$

And

$$a_{si}=2a_{s-1,i}+1<2(q-1)+1=p-1, \tag{43}$$

$$b_{si}=2b_{s-1,i}+1>2(q-1)+1=p-1. \tag{44}$$

Therefore, the sufficiently large t, from (34) and (42)-(44) we have

$$\|u(\cdot,t)\|_{L^p} \leq 2^{17/4}\|u_0\|_{L^1}(f'(t))^{\frac{1}{2}(1-\frac{1}{p})}(f(t)^2+1)^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{45}$$

□

Theorem 3.3 *Under the conditions in Lemma 3.2, for $p \in]1,\infty[$, the following estimate holds for the sufficiently large t*

$$\|u(\cdot,t)\|_{L^p} \leq 2^{17/4}\|u_0\|_{L^1}(f'(t))^{\frac{1}{2}(1-\frac{1}{p})}(f(t)^2+1)^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{46}$$

Proof. For $m \in]p, 2p[(p = 2^s,\ s = 0, 1, 2, \cdots)$ by the following standard Sobolev interpolation inequality for L^m spaces:

$$\|u\|_{L^m} \leq \|u\|_{L^p}^{(2p-m)/m}\|u\|_{L^{2p}}^{2-2p/m},$$

(46) can be easily obtained from Lemma 3.1 and 3.2. □

Remark 3.1. When t is sufficiently large and $p \in]1,\infty[$ the decay rate for the super-cylindrical case is

$$\|u(\cdot,t)\|_{L^p} \leq \alpha^{1-1/p}2^{17/4}\|u_o\|_{L^1}t^{-\frac{1}{2}(1-\frac{1}{p})(\alpha+1)}$$

for $f(t) = t^\alpha/\alpha$ for $\alpha > 1$. Generally, if $f(t) = Const.t^\alpha(Const., \alpha > 0)$, it can be transformed into this form by setting $x' = \beta x$ and $t' = \beta t$ with the restriction $\beta^{\alpha-1} = Const.\alpha$.

3.2 The case $0 < f'(t) \leq 1$ for $t \geq 1$

Lemma 3.4*If $u_0 \in L^1 \cap L^\infty$, then for all $p = 2^s$ with $s \geq 1$ positive integers, the following estimate holds for the sufficiently large t*

$$\|u(\cdot,t)\|_{L^p} \leq 2^{17/4}\|u_0\|_{L^1}(f(t)^2+1)^{-\frac{1}{2}(1-\frac{1}{p})}, \tag{47}$$

as

$$0 < f'(t) \leq 1 \text{ for } t \geq 1. \tag{48}$$

Proof. The proof is similar to Lemma 3.2. In virtue of (29) we have for $t > 1$ $(f'(t) \leq 1)$

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{|\hat{u}|^2}{2}d\xi &= -f(t)\int_{-\infty}^{\infty}|\xi|^2|\hat{u}|^2d\xi \leq -f(t)\int_{B(t)}|\xi|^2|\hat{u}|^2d\xi \\ &\leq -f(t)f'(t)\int_{B(t)}|\xi|^2|\hat{u}|^2d\xi \leq -\frac{f(t)f'(t)}{f^2(t)+1}\int_{B(t)}|\hat{u}|^2d\xi,\end{aligned}$$

where

$$B(t) = \{\xi : |\xi| > (f(t)^2+1)^{-1/2}\}. \tag{49}$$

That is,

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{|\hat{u}|^2}{2}d\xi + \frac{f(t)f'(t)}{f^2(t)+1}\int_{-\infty}^{\infty}|\hat{u}|^2d\xi \leq \frac{f(t)f'(t)}{f^2(t)+1}\int_{B^C(t)}|\hat{u}|^2d\xi. \tag{50}$$

Multiplying (50) by $(f(t)^2+1)$ yields

$$\frac{d}{dt}\left[(f(t)^2+1)\int_{-\infty}^{\infty}\hat{u}^2d\xi\right] \leq 2f'(t)f(t)\int_{B^C(t)}|\hat{u}|^2d\xi. \tag{51}$$

By (33) and (51), we can obtain the further estimate of (51)

$$\begin{aligned}\frac{d}{dt}\left[(f(t)^2+1)\int_{-\infty}^{\infty}|\hat{u}|^2d\xi\right] &\leq 2f'(t)f(t)\|\hat{u}\|_{L^\infty}^2\int_{B^C(t)}d\xi \\ &\leq 4f'(t)f(t)\|u_0\|_{L^1}^2(f(t)^2+1)^{-1/2}.\end{aligned}$$

Integrating from 1 to t yields

$$\begin{aligned}&(f(t)^2+1)\int_{-\infty}^{\infty}|\hat{u}(\xi,t)|^2d\xi \\ &\leq (f(1)^2+1)\int_{-\infty}^{\infty}|\widehat{u}(\xi,1)|^2d\xi + \int_1^t 4f'(s)f(s)\|u_0\|_{L^1}^2(f(s)^2+1)^{-1/2}ds. \\ &\leq C\|\widehat{u_0}\|_{L^2} + 4\|u_0\|_{L^1}^2\sqrt{f(t)^2+1}.\end{aligned}$$

where we utilized the fact $\|u\|_{L^2} \leq \|u_0\|_{L^2}$ from (29). By the Plancherel's equality, we have

$$\|u(\cdot,t)\|_{L^2}^2 \leq C[\|u_0\|_{L^2}^2 + 4\|u_0\|_{L^1}^2(f(t)^2+1)^{1/2}](f(t)^2+1)^{-1}.$$

where C is a positive constant depending on $\|u_0\|_{L^\infty}$ and $\|u_0\|_{L^1}$ and $f(1)$.

The remaining arguments are similar to the proof of Lemma 3.2 by using

$$B_q = \left\{|\xi| : |\xi| > \left(\frac{2q}{f(t)^2+1}\right)^{1/2}\right\}$$

instead of (38) and integrating with respect to t from 1 to t, instead of form 0 to t. Meanwhile (48) is utilized by the way as the proof of $\|u\|_{L^2}$. □

Theorem 3.5 *Under the conditions in Lemma 3.4, for $p \in]1,\infty[$, the following estimate holds for the sufficiently large t*

$$\|u(\cdot,t)\|_{L^p} \leq 2^{17/4}\|u_0\|_{L^1}(f(t)^2+1)^{-\frac{1}{2}(1-\frac{1}{p})}. \tag{52}$$

The proof of Theorem 3.5 is the same as Theorem 3.3.

Remark 3.2. When t is sufficiently large, and $p \in]1,\infty[$ the decay rate for the sub-cylindrical case and cylindrical case is

$$\|u(\cdot,t)\|_{L^p} \leq \alpha^{1-1/p}2^{17/4}\|u_0\|_{L^1}t^{-(1-\frac{1}{p})\alpha},$$

for $f(t) = t^\alpha/\alpha$ for $0 < \alpha \leq 1$.

Remark 3.3 The decay rate (52) is the same as one of the linear parabolic equation $u_t = f(t)u_{xx}$. For example, $v(x,t) = (1+t^2/2)^{-1/2}e^{-x^2/(1+t^2/2)}$ is the solution of the Cauchy problem

$$v_t = \frac{t}{4}v_{xx}, \quad v(x,0) = e^{-x^2}.$$

By simple calculation, we know

$$\|v(\cdot,t)\|_{L^p} \leq 2^{17/4}\sqrt{\pi}(1+t^2/2)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad p \in]1,\infty[.$$

Remark 3.4. The decay rate obtained in Theorem 3.4 is not optimal. Moreover,due to the nonlinear term, the decay rates in L^p norm of the derivatives of $u(x,t)$ with respect to x can not been obtained in the similar way . We will address these two issues in future.

4. A finite difference method

In this section we shall give an alternative proof of the existence theorem for weak solutions to the initial value problem (1), (2) by a finite difference method. This also gives a means to compute approximate solutions numerically.Consider a fixed mesh size l in space. Using variable time steps h_n, $n \in \mathbb{N}$, let the upper-half plane $t \geq 0$ be discretized using the grid points (x_j, t_n) with $x_j = jl$, $t_n = \sum_{i=1}^n h_i$, for $j \in \mathbb{Z}$, $n \in \mathbb{N}$. Consider the *Lax-Friedrichs scheme*

$$\frac{u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2}}{h_n} + \frac{(u_{j+1}^n)^2/2 - (u_{j-1}^n)^2/2}{2l} = 0, \tag{53}$$

with discrete initial data obtained as the point values given by averaging over the *spatial cells* $[(j-\frac{1}{2})l, (j+\frac{1}{2})l]$ centered around x_j, i.e.

$$u_j^0 = \frac{1}{l}\int_{(j-\frac{1}{2})l}^{(j+\frac{1}{2})l} u_0(x)\, dx. \tag{54}$$

We take the bound M for the data, as considered in Theorem 2.2. It is well known and easily seen that the scheme (53) is monotone and stable if the CFL-condition

$$\frac{Mh_n}{l} \leq 1 \qquad \text{for } n = 1, 2, \cdots \tag{55}$$

holds. For simplicity of notation we will consider the following two typical cases instead of $f(t)$ in general form:

Case 1, $f(t) = t^\alpha/\alpha$ for $1 < \alpha < \infty$;

Case 2, $f(t) = t^\alpha/\alpha$ for $0 < \alpha < 1$.

The case for $\alpha = 1$ has been considered in [16]. We first consider case 1. Now we fix the time step $n_0 := [\left(\frac{\alpha}{2}\frac{M^{\alpha+1}}{l^{\alpha-1}}\right)^{\frac{1}{\alpha}}] + 1 \in \mathbb{N}$. The time steps h_n will be defined as follows

$$h_n = l/M \quad \text{ for } \quad n \leq n_0 \tag{56}$$

$$h_n = \frac{l^2}{2f(\sum_{i=1}^n h_i)} = \frac{l^2}{2f(t_n)} \quad \text{ for } \quad n > n_0. \tag{57}$$

It follows from (57) that $t_n - t_{n-1} = \frac{l^2}{2f(t_n)}$ i.e. $t_n - \frac{l^2}{2f(t_n)} = t_{n-1}$. For the function $F(t) = t - \frac{l^2}{2f(t)} - t_{n-1}$ we have $F'(t) = 1 + \frac{f'(t)l^2}{f^2(t)} > 1$ and $F(t_{n-1}) = -\frac{l^2}{f(t_{n-1})}$. So for any given $t = t_{n-1}$ we can find a unique $t_n > t_{n-1}$ such that $F(t_n) = 0$, i.e.

$$h_n = t_n - t_{n-1} = \frac{l^2}{2f(t_n)} > 0. \tag{58}$$

For any $n > n_0$ we have by (56) and (57) that

$$\begin{aligned} h_n &= \frac{l^2}{2f(\sum_{i=1}^{n} h_i)} < \frac{l^2}{2f(\sum_{i=1}^{n_0} h_i)} \\ &= \frac{\alpha l^2}{2(n_0 \frac{l}{M})^\alpha} \\ &\le \frac{\alpha l^2}{2\left(\left(\frac{\alpha}{2}\frac{M^{\alpha+1}}{l^{\alpha-1}}\right)^{\frac{1}{\alpha}} \frac{l}{M}\right)^\alpha} \\ &\le \frac{l}{M}. \end{aligned} \tag{59}$$

It follows from (56) and (59) that the CFL stability condition (55) is satisfied. The difference scheme (53) can be rewritten in the following equivalent form using (56) and (57) respectively

$$\frac{u_j^{n+1} - u_j^n}{h_n} + \frac{(u_{j+1}^n)^2/2 - (u_{j-1}^n)^2/2}{2l} = Ml\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2l^2} \quad \text{for } n \le n_0 \tag{60}$$

and for $n_0 \le n$

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{h_n} + \frac{(u_{j+1}^n)^2/2 - (u_{j-1}^n)^2/2}{2l} &= \frac{l^2}{h_n}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2l^2} \\ &= f(t_n)\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{l^2}. \end{aligned} \tag{61}$$

The scheme (61) is obviously consistent with the equation (1). Following the same line of arguments given by Oleinik [11, Sections 3 and 5] we can obtain the weak solution u of the initial value problem (1), (2) in the sense of Definition 1.2 as the limit of a subsequence of approximate solutions U_l constructed by the scheme (53) and (54) as step functions by setting

$$U_l(x,t) = u_j^n \quad \text{for} \quad (j - \frac{1}{2})l \le x < (j + \frac{1}{2})l, \quad t_n \le t < t_{n+1},$$

except for the fact that the weak solutions obtained here satisfy the slightly different one-sided Lipschitz condition

$$\operatorname*{ess\,sup}_{x \ne y} \left(\frac{u(x,t) - u(y,t)}{x - y}\right)^+ \le \frac{2}{t} \tag{62}$$

instead of (4). It is easy to see that the weak solutions of the initial value problem (1), (2) satisfying (62) are also unique and that (4) implies (62). So the weak solutions of

(1), (2) obtained by the viscosity method in Section 2.2 and the difference scheme in this section are identical and are the unique classical solutions of (1), (2), due to the results of Sections 2.3 and 2.4.

We would like to remark that for any $l > 0$ the solutions of the difference scheme (53) and (54) with time steps given by (56), (57) are well defined for all $t > 0$. Indeed suppose there exists a finite number $T > 0$ such that $t_{n+n_0} \to T$ as $n \to \infty$. From (57) we have

$$t_{n_0+n} = t_{n_0+n-1} + \frac{l^2}{2f(t_{n_0+n})}. \tag{63}$$

Letting $n \to \infty$ in (63) gives

$$T = T + \frac{l^2}{2f(T)}$$

which contradicts the assumption that T is finite.

Further, all previous arguments remain true with the time steps taken to be (56) and

$$h_n = \frac{l^2}{2f(\sum_{i=1}^{n-1} h_i)} = \frac{l^2}{2f(t_{n-1})} \text{ for } n > n_0. \tag{64}$$

The time step h_n given by (64) is slightly simpler than (58).

Now we consider case 2, i.e. $f(t) = t^\alpha/\alpha$ for $0 < \alpha < 1$. In this case we assume

$$l < \left(\frac{2}{\alpha M^{\alpha+1}}\right)^{1/(1-\alpha)}. \tag{65}$$

The time step h_n will be defined as

$$h_n = \frac{l^2}{2f(\Sigma_{i=1}^n h_i)}, \quad n = 1, 2, 3, \cdots.$$

It is easy to prove the scheme satisfies the CFL condition (55). Since $h_n < h_1 (n > 1)$ by the definition of h_n, we only have to prove that the CFL condition holds for h_1. Now by the definition of h_1 we have $\frac{Mh_1}{l} = M(\frac{\alpha l^{1-\alpha}}{2})^{1/(1+\alpha)} < 1$, here we used the restriction (65). The remaining arguments in this case are the same as the case 1.

It is interesting that the solution of the non-uniformly parabolic equation (1) with the initial data (2) can be approximated by the Lax-Friedrichs scheme in the form (53), (54) if the first finite number of constant finite difference steps are given by (60). These steps deal with the non-uniform parabolicity of the equation (1). Then afterwards the use of specifically chosen variable time steps makes the scheme consistent with equation

(1). By (57) the time steps are decreased with the order $\frac{1}{t}$ for fixed mesh size l in order to compensate for the growing diffusion coefficient. This is needed to keep the scheme stable. As is usual for second order parabolic equations, the time step is also of the order l^2 for $l \to 0$ to remain stable.

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5. Appendix

Proof. The estimate (13): First let us transform the backward initial value problem (11), (12) into the following forward initial value problem

$$L(\phi) = \phi_\tau - \frac{u_\delta + v_\delta}{2}\phi_x - f(t_2 - \tau)\phi_{xx} = 0, \quad \tau \in]0, t_2 - t_1[, \tag{66}$$

$$\phi(x, 0) = \psi(x) \tag{67}$$

with $\tau = t_2 - t$. Let us introduce an auxilliary function

$$\Phi(x, \tau) = \exp\left(C\tau + \frac{(M+1)\tau + G(x) + 1}{f(t_2)}\right) \tag{68}$$

where

$$G(x) = \int_{-\infty}^{\infty} \chi(x - y)(K - |y|)\, dy.$$

Here χ is the mollifying function introduced above. Using (7) the function G is easily seen to satisfy the estimates

$$|G'(x)| \leq 1, \qquad |G''(x)| \leq \int_{-\infty}^{\infty} |\chi'(s)|\, ds = C - 1.$$

A similar type of function Φ as in (68) was considered by Kružkov [9], see also Hörmander [6]. One may also show that

$$K - |x| \leq G(x) + 1 \leq K + 2 - |x|.$$

Since $\tau = t_2 - t$ we have

$$\exp\left(C(t_2-t) + \frac{(M+1)(t_2-t)+K+2-|x|}{f(t_2)}\right) \geq \Phi(x,\tau). \tag{69}$$

Therefore in order to get the estimate (13) it is enough to prove that

$$|\phi(x,\tau)| \leq M_0 \min(1, \Phi(x,\tau)). \tag{70}$$

By straightforward calculation we have

$$\begin{aligned} L(M_0\Phi \pm \phi) \;&=\; M_0\Phi\Big(C + \frac{M+1}{f(t_2)} - \frac{u_\delta + v_\delta}{2}\frac{G'(x)}{f(t_2)} - f(t_2-\tau)\frac{G'(x)^2}{f(t_2)^2} \\ &\quad - f(t_2-\tau)\frac{G''(x)}{f(t_2)}\Big) > M_0\Phi > 0 \end{aligned} \tag{71}$$

for $\tau \in]0, t_2 - t_1]$. Since $\|\psi(\cdot)\|_{L^\infty} \leq M_0$ implies that

$$\inf_{x\in\mathbb{R}} (M_0\Phi \pm \phi)|_{\tau=0} \geq min(\inf_{|x|\leq K} (M_0\Phi \pm \phi)|_{\tau=0}, \inf_{|x|\geq K} (M_0\Phi)|_{\tau=0}) \geq 0$$

we obtain $(M_0\Phi \pm \phi)(x,\tau)$ are the super-solutions of (66) and (67). Therefore (70) is proved.

The estimate (14): We have to prove that

$$|\phi_x(x,\tau)| \leq \frac{t_2}{t_1} M_1 \min(1, \Phi(x,\tau)).$$

Differentiating $L(\phi)$ in (66) with respect to x and setting $\tilde{\phi} = \phi_x$ gives

$$\begin{aligned} &\tilde{\phi}_\tau - \frac{1}{2}(u_\delta + v_\delta)\tilde{\phi}_x - \frac{(u_\delta+v_\delta)_x}{2}\tilde{\phi} - f(t_2-\tau)\tilde{\phi}_{xx} = 0, \\ &\tilde{\phi}(x,0) = \psi'(x). \end{aligned}$$

Then using the transformation $\tilde{\phi} = \frac{t_2}{t_2-\tau}\overline{\phi}$ we have

$$L_1(\overline{\phi}) = \overline{\phi}_\tau - \frac{u_\delta+v_\delta}{2}\overline{\phi}_x - \left[\frac{(u_\delta+v_\delta)_x}{2} - \frac{1}{t_2-\tau}\right]\overline{\phi} - f(t_2-\tau)\overline{\phi}_{xx} = 0, \tag{72}$$

$$\overline{\phi}(x,0) = \psi'(x). \tag{73}$$

The one-sided Lipschitz condition (4) in Definition 2.1 implies that

$$\frac{(u_\delta+v_\delta)_x}{2} \leq \frac{1}{t} = \frac{1}{t_2-\tau}.$$

This allows us to apply the maximum principle to (72), see Friedman [4, Chapter 2]. Therefore we obtain

$$|\overline{\phi}(x,\tau)| \le \|\psi'(\cdot)\|_{L^\infty} = M_1 \tag{74}$$

and analogously to (71)

$$L_1(M_1\Phi \pm \overline{\phi}) > M_1\Phi > 0, \quad \tau \in]0, t_2 - t_1].$$

By the maximum principle we know that a non-positive minimum of $M_1\Phi \pm \overline{\phi}$ cannot be taken for $\tau \in]0, t_2 - t_1]$. But we have

$$\inf_{x\in\mathbb{R}}(M_1\Phi \pm \overline{\phi})|_{\tau=0} \ge \min(\inf_{|x|\le K} M_1 \pm \psi'(x), \inf_{|x|\ge K} M_1\Phi)|_{\tau=0}) = 0.$$

Therefore, we get

$$|\overline{\phi}| \le M_1\Phi. \tag{75}$$

The inequalities (74), (75) and the transformation give

$$|\phi_x(x,\tau)| = |\tilde{\phi}(x,\tau)| \le \frac{t_2}{t_1} M_1 \min(1,\Phi), \tag{76}$$

which implies the estimate (14) by (69).

The estimate (15): Differentiating $L(\phi)$, satisfying (66), twice with respect to x and setting $\hat{\phi} = \phi_{xx}$ gives

$$\begin{aligned}
&\hat{\phi}_\tau - \frac{1}{2}(u_\delta + v_\delta)\hat{\phi}_x - (u_\delta + v_\delta)_x\hat{\phi} - f(t_2 - \tau)\hat{\phi}_{xx} = \frac{1}{2}(u_\delta + v_\delta)_{xx}\phi_x,\\
&\hat{\phi}(x,0) = \psi''(x).
\end{aligned}$$

Then using transformation $\hat{\phi} = \frac{t_2^2}{(t_2-\tau)^2}\underline{\phi}$ we have

$$\begin{aligned}
L_2(\underline{\phi}) &= \underline{\phi}_\tau - \frac{u_\delta + v_\delta}{2}\underline{\phi}_x - \left[(u_\delta + v_\delta)_x - \frac{2}{t_2 - \tau}\right]\underline{\phi} - f(t_2 - \tau)\underline{\phi}_{xx}\\
&= \frac{1}{2}(u_\delta + v_\delta)_{xx}\phi_x \frac{(t_2 - \tau)^2}{t_2^2},\\
\underline{\phi}(x,0) &= \psi''(x).
\end{aligned}$$

First we consider the case of a homogeneous right hand side

$$\begin{aligned}
&L_2(\underline{\phi}_1) = 0,\\
&\underline{\phi}(x,0) = \psi''(x).
\end{aligned}$$

The one-sided Lipschitz condition (4) in Definition 2.1 implies that

$$-\left[(u_\delta+v_\delta)_x-\frac{2}{t_2-\tau}\right]\geq 0. \tag{77}$$

Similar to the arguments to obtain (74) and (75) we get the estimate

$$\|\underline{\phi}_1(\cdot,\cdot)\|_{L^\infty}\leq M_2\min(1,\Phi(x,\tau)). \tag{78}$$

Now we consider the case of homogeneous initial data

$$\begin{aligned}&L_2(\underline{\phi}_2)=\frac{1}{2}(u_\delta+v_\delta)_{xx}\phi_x\frac{(t_2-\tau)^2}{t_2^2},\\&\underline{\phi}_2(x,0)=0.\end{aligned} \tag{79}$$

It follows from the inequality (76) that

$$\left|\frac{1}{2}(u_\delta+v_\delta)_{xx}\phi_x\frac{(t_2-\tau)^2}{t_2^2}\right|\leq\frac{C_1t_2MM_1}{t_1\delta^2}\Phi \tag{80}$$

where C_1 was defined in (7). Straightforward calculation and the inequalities (77), (79) give

$$\begin{aligned}&L_2\left(\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\pm\underline{\phi}_2\right)\\&\qquad>\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\pm\frac{1}{2}(u_\delta+v_\delta)_{xx}\phi_x\frac{(t_2-\tau)^2}{t_2}>0,\end{aligned}$$

for $\tau\in]0,t_2-t_1]$. This implies by the maximum principle that a non-positive minimum of the quantity $\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\pm\underline{\phi}_2$ cannot exist for $\tau\in]0,t_2-t_1]$. But we have

$$\inf_{x\in\mathbb{R}}\left(\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,0)\pm\underline{\phi}_2(x,0)\right)\geq 0.$$

Therefore we obtain

$$|\underline{\phi}_2(x,\tau)|\leq\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau).$$

It follows from (78) and (80) that

$$\begin{aligned}|\phi_{xx}(x,\tau)|&=\left|\frac{t_2^2}{(t_2-\tau)^2}\underline{\phi}(x,\tau)\right|\leq\frac{t_2^2}{t_1^2}\left(|\underline{\phi}_1(x,\tau)|+|\underline{\phi}_2(x,\tau)|\right)\\&\leq\frac{t_2^2}{t_1^2}\left(M_2+\frac{C_1t_2MM_1}{t_1\delta^2}\Phi(x,\tau)\right).\end{aligned}$$

Then the estimate (15) follows from (69).

The estimate (16): This follows from the equation (11) and the fact that $\|u_\delta\|_{L^\infty}\leq M$ and $\|v_\delta\|_{L^\infty}\leq M$. □

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