The K-functional and Calderón-Zygmund Type Decompositions

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Abstract. The paper is an exposition of some old results on the stability of the K-method and recent results on calculation of the K-functional.

1. Introduction

Since the publication of the classical paper by J.L. Lions and J. Peetre [**LP**], real interpolation theory has developed into a rich theory with applications to many different areas of analysis. In this paper we give a short introduction to the general K-method of interpolation and demonstrate its surprising stability.

A number of applications of interpolation theory, in particular some recent problems in image processing and singular integral operators, require the computation of suitable K-functionals, as well as precise algorithms for constructing nearly optimal minimizers. In this paper we will present an algorithm for constructing nearly optimal minimizers based on a generalization of the classical Calderón-Zygmund decompositions. Our algorithm also leads to new formulas for calculating suitable K-functionals. In particular, we will illustrate our algorithm on the model couple (L_1, Lip_α).

2. Preliminaries

We start by briefly recalling the main notions of interpolation theory (see [**BL**]).

Let X_0 and X_1 be two Banach spaces embedded in some topological vector space X. We will say that the spaces X_0 and X_1 form a Banach couple $\vec{X} = (X_0, X_1)$ if the following "compatibility" condition holds:

- *If the sequence $y_n \in X_0 \cap X_1$, $n = 1, \ldots$ is such that it converges in the norm of X_0 to the element $x_0 \in X_0$ and in the norm of X_1 to the element $x_1 \in X_1$, then $x_0 = x_1$.*

This condition allows us to introduce a Banach structure on the linear spaces $X_0 \cap X_1$ and $X_0 + X_1$, namely

$$\|x\|_{X_0 \cap X_1} = \max(\|x\|_{X_0}, \|x\|_{X_0}), \quad \|x\|_{X_0+X_1} = \inf_{x=x_0+x_1}(\|x_0\|_{X_0} + t\,\|x_1\|_{X_1}).$$

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Let $\vec{X} = (X_0, X_1)$, $\vec{Y} = (Y_0, Y_1)$ be two Banach couples. A linear operator T from $X_0 + X_1$ to $Y_0 + Y_1$ is called a bounded linear operator from the couple $\vec{X}$ to the couple $\vec{Y}$ if the restrictions of T on X_i $(i = 0, 1)$ are bounded linear operators from X_i to Y_i.

A Banach space $X \subset X_0 + X_1$ is called an intermediate space for the couple $\vec{X}$ if the continuous embeddings $X_0 \cap X_1 \subset X \subset X_0 + X_1$ hold.

An intermediate space X is called an interpolation space if for any bounded linear operator T from the couple $\vec{X}$ to itself the restriction of T on X is a bounded linear operator from X to X.

Let X be an intermediate space for the couple $\vec{X}$ and let Y be an intermediate space for the couple $\vec{Y}$. We will say that the spaces X and Y are relative interpolation spaces if a restriction of any bounded linear operator T from the couple $\vec{X}$ to the couple $\vec{Y}$ is a bounded linear operator from X to Y.

3. The K-method of Interpolation: Introduction to a General Theory of K-spaces

The modern theory of real interpolation is based on the notion of the K-functional introduced by J. Peetre. Let us recall its definition.

Let $x \in X_0 + X_1$, then the K-functional of x is a nonnegative concave function on $\mathbb{R}_+ = (0, \infty)$ defined by the formula

$$K(t, x; \vec{X}) = \inf_{x = x_0 + x_1} (\|x_0\|_{X_0} + t\,\|x_1\|_{X_1}), \qquad t > 0.$$

The K-functional can be obtained from a "distance function", the so-called E-functional :

$$E(t, x; \vec{X}) = \inf_{\|x_1\|_{X_1} \leq t} \|x - x_1\|_{X_0}, \quad t > 0.$$

Remark 1. *We deviate somewhat from the standard notation* $E(t, x; \vec{X}) = \inf_{\|x_0\|_{X_0} \leq t} \|x - x_0\|_{X_1}$.

Clearly,

$$K(t, x; \vec{X}) = \inf_{s > 0} (E(t, x; \vec{X}) + ts)$$

and conversely for any Banach couple $\vec{X}$ we also have

$$E(s, x; \vec{X}) = \sup_{t > 0} (K(t, x; \vec{X}) - ts).$$

One of the advantages of using the K-functional instead of the E-functional is that the K-functional possesses several very nice properties that the E-functional does not have.

Let us now list the main properties of the K-functional.

- For a fixed $t > 0$ the expression $K(t, \cdot; \vec{X})$ is a norm on the space $X_0 + X_1$.
- For the couple $\vec{X}^T = (X_1, X_0)$ we have $K(t, x; \vec{X}^T) = tK(t^{-1}, x; \vec{X})$.

The proofs of these properties are simple and direct.

Much less trivial is the following K-divisibility property (see [**BK**], pp. 315-337).

- Let
$$K(\cdot, x; \vec{X}) \le \sum_{i=1}^{\infty} \varphi_i, \qquad \sum_{i=1}^{\infty} \varphi_i(1) < \infty,$$
where φ_i $(i = 1, ...)$ are nonnegative concave functions on $\mathbb{R}_+$. Then there exists a decomposition $x = \sum_{i=1}^{\infty} x_i$ such that
$$K(\cdot, x_i; \vec{X}) \le \gamma \varphi_i, \qquad i = 1, ... \quad , \tag{3.1}$$
where γ is an absolute constant.

Remark 2. *It is known (see* [**BK**] *and* [**CJM**]*) that* $1.5 < \gamma < 6$.

The importance of the K-functional for interpolation arises from the following simple proposition.

Proposition 1. *Let T be a linear bounded operator from the couple $\vec{X} = (X_0, X_1)$ to the couple $\vec{Y} = (Y_0, Y_1)$. Then we have the estimate*
$$K(t, Tx; \vec{Y}) \le \inf_{x = x_0 + x_1} (\|Tx_0\|_{Y_0} + t \|Tx_1\|_{Y_1}) \le \max_{i=0,1} \|T\|_{X_i \to Y_i} \ K(t, x; \vec{X}).$$

On the basis of the K-functional we can construct interpolation spaces (K-spaces). A Banach space Φ of measurable functions on $\mathbb{R}_+$ is called a *parameter* of the K-method if it satisfies the following two properties:

- if $f \in \Phi$ and $|g| \le |f|$ then $g \in \Phi$ and $\|g\|_\Phi \le \|g\|_\Phi$;
- $\min(1, t) \in \Phi$.

The last condition means that Φ contains at least one nonnegative concave function. Then the space $K_\Phi(\vec{X})$ is defined as the set of elements $x \in X_0 + X_1$ such that
$$\|x\|_{K_\Phi(\vec{X})} = \left\| K(\cdot, x; \vec{X}) \right\|_\Phi .$$
It is possible to verify that $K_\Phi(\vec{X})$ is an intermediate space for the couple $\vec{X}$. Moreover, from Proposition 1 we immediately obtain

Theorem 1. *(On interpolation) Let T be a bounded linear operator from the couple $\vec{X} = (X_0, X_1)$ to the couple $\vec{Y} = (Y_0, Y_1)$. Then T is a bounded linear operator from the space $K_\Phi(\vec{X})$ to the space $K_\Phi(\vec{Y})$.*

Remark 3. *As we have seen, the interpolation theorem follows directly from the definitions. This triviality is "compensated" by the difficulty of calculation of spaces $K_\Phi(\vec{X})$ for concrete couples $\vec{X}$.*

For some couples all interpolation spaces are K-spaces and so they can be parameterized by the parameters of the K-method. An important example of such couples is presented in the following theorem.

Theorem 2. *Let $\vec{X} = (L_{p_0}(\omega_0), L_{p_1}(\omega_1))$ be a couple of weighted Lebesque spaces. Then all interpolation spaces of $\vec{X}$ are K-spaces.*

The proof of the theorem follows from the result of G. Sparr which states that the couple $(L_{p_0}(\omega_0), L_{p_1}(\omega_1))$ is a Calderón couple and Lemma 4.1.12 from [**BK**]. Recall that the couple $\vec{X} = (X_0, X_1)$ is called a Calderón couple if from the inequality $K(\cdot, x; \vec{X}) \geq K(\cdot, y; \vec{X})$ it follows that there exists a bounded linear operator $T : \vec{X} \to \vec{X}$ such that $Tx = y$.

3.1. Stability of K-spaces. Now we are ready to formulate the main results of the general theory: reiteration and duality.

To formulate the reiteration theorem first note that different parameters Φ of the K-method can lead to the same space $K_\Phi(\vec{X})$. This happens because the K-functional is a nonnegative concave function and therefore only the restriction of the norm of Φ on the cone of nonnegative concave functions on $\mathbb{R}_+$ is important. For example, if we consider a parameter $\hat{\Phi}$ of the K-method defined by the norm

$$\|f\|_{\hat{\Phi}} = \left\|\hat{f}\right\|_{\Phi},$$

where by $\hat{f}$ we denote the least concave majorant of the function $|f|$, then we have $K_\Phi(\vec{X}) = K_{\hat{\Phi}}(\vec{X})$ for all couples $\vec{X}$ even with the equality of the norms.

The question that is answered in the reiteration theorem is the following.

PROBLEM 1. *Let $\vec{X}$ be a Banach couple. Suppose that the spaces Y_0, Y_1 are obtained by the K-method from a couple $\vec{X}$, i.e. $Y_i = K_{\Phi_i}(\vec{X})$ $(i = 0, 1)$. How can we calculate the space $K_\Phi(\vec{Y})$?*

Surprisingly, the resulting space is again the K-space of the initial couple $\vec{X}$ and a formula for its parameter can be given.

THEOREM 3. *(On reiteration) Let Φ, Φ_0, Φ_1 be parameters of the K-method. Then the following formula is correct:*

$$K_\Phi(K_{\Phi_0}(\vec{X}), K_{\Phi_1}(\vec{X})) = K_\Psi(\vec{X}), \tag{3.2}$$

where $\Psi = K_\Phi(\hat{\Phi}_0, \hat{\Phi}_1)$. The equality of spaces in (3.2) means that they coincide and their norms are equivalent with the constants of equivalence independent of $\vec{X}$.

The proof of the reiteration theorem follows quite easily from the K-divisibility (see [**BK**], Theorem 3.3.11).

Let us now turn to the duality. Let a couple $\vec{X} = (X_0, X_1)$ be regular, i.e. the Banach space $X_0 \cap X_1$ is dense in X_0 and in X_1. For a regular couple the dual spaces X_0', X_1' are embedded in the space $(X_0 \cap X_1)'$ and form a Banach couple (see [**BL**]). Moreover, if X is an intermediate space for the couple $\vec{X}$, then we can define its dual space $X' \subset (X_0 \cap X_1)'$ as a dual of the space X^0, where by X^0 we denote the closure of the set $X_0 \cap X_1$ in X.

The problem of duality can be formulated as follows.

PROBLEM 2. *Suppose that a couple $\vec{X}$ is regular. How can we calculate the dual space to $K_\Phi(\vec{X})$?*

Of course, it is natural to expect that the dual of a K-space is again a K-space for the dual couple $\vec{X}' = (X_0', X_1')$. Unfortunately, this is not correct: the dual to the space $K_\Phi(\vec{X})$ does not have to be an interpolation space for the couple $\vec{X}'$, as can be seen from the proof of Theorem 2.4.17 in [**BK**]. Nevertheless, the expectation is met if we impose some mild conditions on $\vec{X}$ or on the parameter Φ.

Definition 1. *A parameter Φ of the K-method is called nondegenerate if Φ contains at least one nonnegative concave function f such that*

$$\lim_{t\to 0}\frac{f(t)}{t}=\lim_{t\to\infty}f(t)=\infty.$$

Definition 2. *A couple $\vec{X}$ is called relatively complete if the unit ball of the space $X_0\cap X_1$ is a closed subset of the space X_0+X_1.*

To formulate the duality result we will need to consider the Calderón operator

$$(Sf)(t)=\int_0^t f(s)\frac{ds}{s}+t\int_t^\infty f(s)\frac{ds}{s^2},$$

The operator S is defined on the functions f on $\mathbb{R}_+$ that belong to the space $L_1(\omega)$, $\omega=\min(\frac{1}{s},\frac{1}{s^2})$, so the integrals in the definition of S converge absolutely.

Next theorem follows from Theorem 3.5.9, Theorem 3.7.2, and Proposition 3.1.17 from [**BK**].

Theorem 4. *(On duality) Let $\vec{X}$ be a regular couple. Suppose that one of the following conditions is satisfied:*

a) the parameter Φ of the K-method is nondegenerate;

b) $\vec{X}$ is a relatively complete couple.

Then the dual space to $K_\Phi(\vec{X})$ is a K-space for the dual couple and

$$K_\Phi(\vec{X})'=K_\Psi(\vec{X}'),$$

where the norm in the parameter Ψ is given by the expression

$$\|f\|_\Psi=\sup\left\{\int_0^\infty f(t)g(\frac{1}{t})\frac{dt}{t}:\|Sg\|_\Phi\le 1\right\}.$$

4. Calderón-Zygmund type decompositions and K-functional

To apply the theory we need to calculate K-functionals. This is usually a difficult problem and each solved case contains some nontrivial information.

Let us look at some examples.

Example 1. *Let us consider the couple (L_1,L_∞). It is known that*

$$K(t,f;L_1,L_\infty)\approx t(Mf)^*(t), \tag{4.1}$$

where

$$Mf(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f|$$

is a Hardy-Littlewood maximal function. Here and below the constants of equivalence are independent of f and t, and Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Since $L_p=(L_1,L_\infty)_{1-\frac{1}{p},p}$, we have

$$\|f\|_{L_p}\approx\left(\int_0^\infty(t^{-(1-\frac{1}{p})}K(t,f;L_1,L_\infty))^p\frac{dt}{t}\right)^{\frac{1}{p}}=$$
$$\left(\int_0^\infty((Mf)^*(t))^p\,dt\right)^{\frac{1}{p}}=\left(\int_{\mathbb{R}^n}(Mf(x))^p dx\right)^{\frac{1}{p}}=\|Mf\|_{L_p}$$

and we can see that the formula (4.1) leads to the Hardy-Littlewood maximal theorem: $\|f\|_{L_p}\approx\|Mf\|_{L_p}$.

EXAMPLE 2. *Let us consider the couple* $(L_p, \dot{W}_p^k)$, $p \in (1, \infty)$. *It is known that*

$$K(t, f; L_p, \dot{W}_p^k) \approx \omega_k(f, t^{\frac{1}{k}})_p,$$

where $\omega_k(f, t)_p$ *is the* k*-th modulus of continuity in* L_p. *From this formula and the closedness of the unit ball* $\dot{W}_p^k$ *in* L_p *for* $p \in (1, \infty)$ *follows the description of the Sobolev space* $\dot{W}_p^k$ *in terms of the modulus of continuity*

$$\|f\|_{\dot{W}_p^k} \approx \sup_{t>0} \frac{1}{t} K(t, f; L_p, \dot{W}_p^k) \approx \sup_{t>0} \frac{1}{t} \omega_k(f, t^{\frac{1}{k}})_p.$$

For some problems it is important to have an algorithm for constructing a family of elements $u_t \in X_1$ such that

$$K(t, x; X_0, X_1) \approx \|x - u_t\|_{X_0} + t \|u_t\|_{X_1},$$

with the constants of equivalence independent of x and t. We will call such decompositions *near minimizers* for the K-functional. For some couples it is easier to construct near minimizers for the E-functional, i.e. such a family of elements $u_t \in X_1$ that

$$\|u_t\|_{X_1} \le ct \quad \text{and} \quad \|x - u_t\|_{X_0} \le cE(\frac{t}{c}, x; X_0, X_1),$$

with $c \ge 1$ independent of x and $t > 0$. Note that if we take $t = 2c\frac{K(s,x;X_0,X_1)}{s}$ then it is not difficult to show that u_t will be a near minimizer for the K-functional at the point s.

An important example of a problem for which we need to find a near minimizer comes from image processing. In the paper [**ROF**] L. Rudin et al. proposed to reconstruct the geometrical properties of an object from its noisy image by means of calculating the function u_t which minimizes the L-functional

$$L(t, f; L_2, BV) = \inf_{u \in BV} (\|f - u\|_{L_2}^2 + t \|u\|_{BV}),$$

where all functions are defined on a rectangle in $\mathbb{R}^2$ and BV is a space of functions of bounded variations defined by the seminorm

$$\|f\|_{BV} = \sup_{t>0} \frac{1}{t} \omega_1(f, t)_1.$$

Recently this approach to denoising has become quite popular, see, for example, [**TNV**] and the book [**M**].

Note that for $s = tK(t, f; L_2, BV)$ we have

$$L(s, f; L_2, BV) \approx K(t, f; L_2, BV)^2$$

(see [**BK**], p. 520). Therefore instead of the L-functional it is possible to consider the K-functional

$$K(t, f; L_2, BV) = \inf_{u \in BV} (\|f - u\|_{L_2} + t \|u\|_{BV})$$

and it is sufficient to solve the problem of constructing minimizers for the K-functional. A wavelet-based approach to this problem was considered in several papers, see [**CDPH**], [**CDDD**], and [**BDKPW**].

Let us formulate the result for the multivariate Haar system $\mathcal{H}_i$ $(i \in \Delta)$ normalized in the space BV, i.e. $\|\mathcal{H}_i\|_{BV} = 1$ for all i. We let

$$G_N(f) = \sum_{i \in \Delta_N} c_i \mathcal{H}_i, \quad f = \sum_i c_i \mathcal{H},$$

where Δ_N is a subset of N elements of Δ that correspond to the coefficients c_i with the largest absolute values. Then we have

Theorem 5. *(see [**BDKPW**]) Let $p_* = \frac{n}{n-1}$, where $n \geq 2$ is a dimension. Then*

$$K(N^{-\frac{1}{n}}, f; L_{p_*}, BV) \approx \|f - G_N(f)\|_{L_{p_*}} + N^{-\frac{1}{n}} \|G_N(f)\|_{BV}.$$

So we see that a near minimizer for the couple (L_{p_*}, BV) can be constructed using a greedy wavelet algorithm.

Below we will suggest another general approach to the problem of constructing near minimizers and calculating the K-functional. Our approach is based on a generalization of classical Calderón-Zygmund decompositions. These decompositions were used recently to solve some problems in the theory of singular integral operators, see [**KK**], [**KiKr**] and [**KiKr1**].

4.1. Classical Calderón-Zygmund Decompositions and Near Minimizers. In their classical paper [**CZ**], A. Calderón and A. Zygmund suggested a simple construction that proved to be a very powerful and useful tool in harmonic analysis. The decomposition is constructed as follows.

Let $f \in L_1$ and $t > 0$ be fixed. Then using stopping time technique it is possible to construct a family of dyadic cubes $\{Q_i\}_{i \in I}$ with nonoverlapping interiors such that

$$t \leq \frac{1}{|Q_i|} \int_{Q_i} |f| \leq 2^n t, \quad i \in I$$

and

$$\left\| f \chi_{\mathbb{R}^n \setminus \cup Q_i} \right\|_{L_\infty} \leq t.$$

Then the Calderón-Zygmund decomposition is defined as

$$f = f_t + (f - f_t),$$

where the so-called "good" function f_t is given by the formula

$$f_t = \sum_i c_i \chi_{Q_i} + f \chi_{\mathbb{R}^n \setminus \cup Q_i}, \quad c_i = \frac{1}{|Q_i|} \int_{Q_i} f \ , \ i \in I.$$

Clearly, $\|f_t\|_{L_\infty} \leq 2^n t$. More interestingly, the function f_t is a near minimizer for the $E-$functional

$$\|f - f_t\|_{L_1} \leq 4E(\frac{t}{2}, f; L_1, L_\infty).$$

Indeed,

$$\|f - f_t\|_{L_1} \leq \sum_i \int_{Q_i} |f - f_{Q_i}| \leq 2 \sum_i \int_{Q_i} |f| \leq 2t \sum_i |Q_i|$$

and it only remains to note that

$$E(\frac{t}{2},f;L_1,L_\infty)=\inf_{\|g\|_{L_\infty}\le\frac{t}{2}}\|f-g\|_{L_1}\ge\inf_{\|g\|_{L_\infty}\le\frac{t}{2}}\left(\sum_i\int_{Q_i}|f-g|\right)\ge \tag{4.2}$$

$$\inf_{\|g\|_{L_\infty}\le\frac{t}{2}}\left(\sum_i(\int_{Q_i}|f|-\int_{Q_i}|g|)\right)\ge\sum_i(t\,|Q_i|-\frac{t}{2}\,|Q_i|)\ge\frac{t}{2}\sum_i|Q_i|\,.$$

This simple observation suggests that an extension of the Calderón-Zygmund construction for couples different from (L_1,L_∞) could be useful for constructing near minimizers.

4.2. A Generalization of the Calderón-Zygmund Construction. To avoid technicalities we will only consider here the model case (L_1,Lip_α), where the space Lip_α is defined by the seminorm

$$\|f\|_{Lip_\alpha}=\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}.$$

The exposition below follows [**Kr**] . Our algorithm will provide a method to construct near minimizers for the E-functional of the couple (L_1,Lip_α).

Let us fix $f\in L_1$ and $t>0$. Constructing the "good" function $f_t\in Lip_\alpha$ is done in three steps.

4.2.1. *Step 1. Limiting cubes.* In this step we use a stopping time technique to construct a family of cubes that possesses two important properties.

For $x\in\mathbb{R}^n$ let us consider a function

$$\varphi_x(r)=\frac{1}{|Q(x,r)|^{1+\frac{\alpha}{n}}}\inf_c\int_{Q(x,r)}|f-c|\,,$$

where $Q(x,r)$ is a cube in $\mathbb{R}^n$ with its center in x and side lengths equal to r .

Let us then consider a set

$$\Omega=\left\{x\in\mathbb{R}^n:\sup_r\varphi_x(r)>t\right\}.$$

As $\varphi_x(r)\to 0$ when $r\to\infty$, therefore for $x\in\Omega$ it is possible to find $r_x>0$ such that

$$\sup_{r\ge r_x}\varphi_x(r)\le t\qquad\text{and}\qquad\sup_{r\ge\frac{1}{2}r_x}\varphi_x(r)>t.$$

In this case we let

$$Q_x=Q(x,r_x).$$

The resulting family $\{Q_x\}_{x\in\Omega}$ possesses the following important property, similar to (4.2).

Proposition 2. *Let $\pi=\{Q_{x_i}\}$ be a subfamily of $\{Q_x\}_{x\in\Omega}$ which consists of cubes with non-overlapping interiors, i.e. $\mathring{Q}_{x_i}\cap\mathring{Q}_{x_j}=\emptyset$, $i\neq j$. Then*

$$\sum_i|Q_{x_i}|^{1+\frac{\alpha}{n}}\le c\frac{1}{t}E(\frac{t}{c},f;L_1,Lip_\alpha)$$

where the constant $c\ge 1$ is independent of f, t and π.

To construct the cubes Q_x, for $x \in \mathbb{R}^n \setminus \Omega$, let us split $\mathbb{R}^n$ into cubes Q_i, $i = 1, 2, ...$ with volumes equal to 1, and for $x \in \Omega \cap Q_i$ let us take

$$Q_x = Q(x, \varepsilon^i),$$

were $\varepsilon > 0$ is a sufficiently small number. If $\pi = \{Q_{x_i}\}$ is a subfamily of the constructed family $\{Q_x\}_{\mathbb{R}^n \setminus \Omega}$ consisting of cubes with disjoint interiors, then not more than $\frac{1}{\varepsilon^{in}}$cubes from π have their centers in the cube Q_i. Therefore

$$\sum_i |Q_{x_i}|^{1+\frac{\alpha}{n}} \leq c \sum_{i=1}^{\infty} \varepsilon^{i(n+\alpha)} (\frac{1}{\varepsilon^{in}}) \leq c\varepsilon^{\alpha}$$

and we can see that if $\varepsilon > 0$ is small enough then the whole family $\{Q_x\}_{x \in \mathbb{R}^n}$ possesses the following property.

Property 1. Let

$$\left|\{Q_x\}_{x \in \mathbb{R}^n}\right|_{1+\frac{\alpha}{n}} = \sup_{\pi = \{Q_{x_i}\}} (\sum_i |Q_{x_i}|^{1+\frac{\alpha}{n}}), \tag{4.3}$$

where π consists of cubes with disjoint interiors and sup is taken over all subfamilies $\pi = \{Q_{x_i}\}$ of the family $\{Q_x\}_{x \in \mathbb{R}^n}$. Then

$$\left|\{Q_x\}_{x \in \mathbb{R}^n}\right|_{1+\frac{\alpha}{n}} \leq c \frac{1}{t} E(\frac{t}{c}, f; L_1, Lip_\alpha),$$

where the constant $c \geq 1$ independent of $f \in L_1$ and $t > 0$.

Moreover, from the construction of the cubes Q_x we have

Property 2. If a cube Q is not strictly embedded in some cube Q_x then

$$\frac{1}{|Q|^{1+\frac{\alpha}{n}}} \inf_c \int_Q |f - c| \leq t.$$

4.2.2. *Step2. A Covering Theorem.* To formulate the theorem we will need the following definition.

DEFINITION 3. *The family of cubes $\{K_i\}_{i \in I}$ forms a Whitney-Besicovitch covering (WB-covering for short) if the following three properties hold:*

- $\sum_i \chi_{K_i} \leq M(n)$;
- $\cup_i \frac{1}{2} K_i = \cup_i K_i$;
- *if $K_i \cap K_j \neq \emptyset$, then $|K_i \cap K_j| \geq \varepsilon(n) \max(|K_i|, |K_j|)$, where $M(n)$, $\varepsilon(n)$ are some positive constants depending only on the dimension n.*

The main idea of the covering theorem is to construct a WB-covering by enlarging some of the limiting cubes and to keep the properties (1) and (2).

Let $\{Q_x\} = \{Q_x\}_{x \in \mathbb{R}^n}$ be a family of nondegenerate cubes (x is the center of Q_x).

THEOREM 6. *Suppose that (see 4.3) $|\{Q_x\}|_{1+\frac{\alpha}{n}} < \infty$ and $\alpha > 0$. Then it is possible to construct a family of cubes $\{K_i\}_{i \in I}$ that forms a WB-covering and possesses the following properties:*

- *if x_i is the center of K_i then $Q_{x_i} \subset K_i$, $i \in I$;*
- *for any cube Q_x there exists $i = i(x)$ such that $Q_x \subset K_i$;*
- $\sum_{i \in I} |K_i|^{1+\frac{\alpha}{n}} \leq c(n) \left|\{Q_x\}_{x \in \mathbb{R}^n}\right|_{1+\frac{\alpha}{n}}$.

REMARK 4. *The theorem follows from the proof of the covering theorem in* [**Kr1**].

Applying the covering theorem to the family of limiting cubes gives us a family of cubes $\{K_i\}_{i\in I}$ that satisfies three geometrical properties:

- $\cup_i \frac{1}{2}K_i = \mathbb{R}^n$;
- $\sum_i \chi_{K_i} \leq M(n)$;
- *if* $K_i \cap K_j \neq \emptyset$, *then* $|K_i \cap K_j| \geq \varepsilon(n)\max(|K_i|, |K_j|)$;

and two analytical properties:

- $\sum_i |K_i|^{1+\frac{\alpha}{n}} \leq c(n)\frac{1}{t}E(\frac{t}{c(n)}, f; L_1, Lip_\alpha)$;
- *if a cube* Q *is not strictly embedded in some cube* K_i, $i \in I$, *then*

$$\frac{1}{|Q|^{1+\frac{\alpha}{n}}}\inf_c \int_Q |f-c| \leq t.$$

4.2.3. *Construction of a Minimizer for the Couple* (L_1, Lip_α).

Definition 4. *A family of* C^∞ *functions* $\{\psi_i\}$ *will be called a partition of the unity corresponding to the WB-covering* $\{K_i\}$ *if*

i) $0 \leq \psi_i \leq 1$, $\sum_i \psi_i = \chi_{\cup_i K_i}$*;*
ii) $\psi_i = 0$ *outside the cube* $(\frac{2}{3})K_i$ *and* $\psi_i \geq c > 0$ *on* $\frac{1}{2}K_i$ *with the constant* c *depending only on the dimension* n*;*
iii) the following estimate holds for the functions ψ_i:

$$\left|D^{\bar{k}}\psi_i\right| \leq \gamma(n,\bar{k})\frac{1}{|K_i|^{\frac{|\bar{k}|}{n}}}, \qquad D^{\bar{k}} = \frac{\partial^{\bar{k}}}{\partial x_1^{k_1}...\partial x_n^{k_n}}.$$

The construction of such partition of the unity is standard, see, for example, [**S**].

Let us consider a partition of the unity $\{\psi_i\}$ that corresponds to the WB-covering $\{K_i\}$ constructed from the family of limiting cubes. Then the "good" function f_t can be defined by the formula

$$f_t = \sum_i c_i\psi_i, \quad c_i = \frac{1}{\int \psi_i}\int f\psi_i.$$

Now we can formulate the result (see [**Kr**]).

Theorem 7. *The function* f_t *is a minimizer for the* E*-functional for the couple* (L_1, Lip_α).

Remark 5. *The formula for the "good" function* f_t *is similar to the one in the paper of C. Fefferman and E. Stein* [**FS**]. *The main difference is the absence of the term* $f\chi_{\mathbb{R}^n\setminus\cup K_i}$. *The reason for that is that in our case* $\cup K_i = \mathbb{R}^n$.

Remark 6. *The above construction can be generalized in several directions (see* [**Kr1**], [**KrKu**]*).For example, its generalization works for the couple* $(L_q, \dot{W}_p^k)$ *under the condition*

$$\frac{k}{n} + \frac{1}{q} - \frac{1}{p} > 0,$$

and for the couple $(L_1, \mathcal{L}^{1,\lambda})$, *where* $\mathcal{L}^{1,\lambda}$ *is a Morrey space constructed on the base of* L_1. *Recall that the norm in* $\mathcal{L}^{1,\lambda}$ *is given by the expression*

$$\|f\|_{\mathcal{L}^{1,\lambda}} = \sup_Q \frac{1}{|Q|^{1-\frac{\lambda}{n}}}\int_Q |f|, \qquad 1-\frac{\lambda}{n} \in (0,1). \tag{4.4}$$

4.3. Calculation of the K-functional. Construction of minimizers usually gives some formula for the K-functional. Let us consider, for example, the couple $(L_1, \mathcal{L}^{1,\lambda})$ where $\mathcal{L}^{1,\lambda}$ is a Morrey space (see 4.4). Let $M_\lambda f$ be a fractional maximal function

$$M_\lambda f(x) = \sup_{Q\ \ni x} \frac{1}{|Q|^{1-\frac{\lambda}{n}}} \int_Q |f|\,.$$

To formulate the result we need the notion of the Hausdorff capacity. Let Ω be a set in $\mathbb{R}^n$, then the Hausdorff capacity of the set Ω can be defined as

$$\mu_\lambda(\Omega) = \inf_{\Omega\subset\cup Q_i} \sum |Q_i|^{1-\frac{\lambda}{n}}\,,$$

where inf is taken over all the families of cubes $\{Q_i\}$ such that $\Omega \subset \cup Q_i$.

Remark 7. *Standard notation for Hausdorff capacity of the set Ω is $\Lambda^{(\infty)}_{n-\lambda}(\Omega)$.*

Although μ_λ is not a measure, it is still possible to define the decreasing rearrangement of the function f with respect to μ_λ, which we denote by $f^*_{\mu_\lambda}$. By the definition it is a nonincreasing, continuous from the right function on $\mathbb{R}_+$ such that

$$\left|s : f^*_{\mu_\lambda}(s) > t\right| = \mu_\lambda(\{x : |f(x)| > t\}\,.$$

Then the following formula is correct (see [**KrKu1**])

$$K(t, f; L_1, \mathcal{L}^{1,\lambda}) \approx t(M_\lambda f)^*_{\mu_\lambda}(t).$$

The last formula leads immedeately to an analog of Hardy-Littlewod maximal theorem for the fractional maximal operator $M_\lambda f$ (see the discussion in [**KrKu1**] and compare with Example 1):

$$\|f\|_{(L_1,\mathcal{L}^{1,\lambda})_{1-\frac{1}{p},p}} \approx \left(\int_0^\infty (t^{-(1-\frac{1}{p})} K(t, f; L_1, \mathcal{L}^{1,\lambda}))^p \frac{dt}{t}\right)^{\frac{1}{p}} =$$
$$\left(\int_0^\infty ((M_\lambda f)^*_{\mu_\lambda}(t))^p\ dt\right)^{\frac{1}{p}} = \left(p \int_{\mathbb{R}^n} (\mu_\lambda \{x : M_\lambda f > t\}) t^{p-1} dt\right)^{\frac{1}{p}}.$$

References

[BDKPW] P. Bechler, R. DeVore, A. Kamot, G. Petrova, P. Wojtaszczyk, *Greedy Wavelet Projections are Bounded on BV*, Transactions of AMS 359 (2007), 619-635.

[BL] J. Bergh, J. Lofstrom, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin, 1976.

[BK] Ju.A. Brudnyi, N. Kruglyak, *Interpolation Functors and Interpolation Spaces 1,* North Holland, Amsterdam, 1991.

[CZ] A.P. Calderón, A. Zygmund, *On the Existence of Certain Singular Integrals*, Acta Math. 88 (1952), 85-139.

[CDPH] A. Cohen, R. DeVore, P. Petrushev, H Xu, *Nonlinear Approximation and the Space $BV(R^2)$*, American Journal of Mathematics 121 (1999), 587-628.

[CDDD] A. Cohen, W. Dahmen, I. Daubechies, R. DeVore, *Harmonic Analysis of the Space BV*, Rev. Mat. Iberoamericana 19 (2003), 235-263.

[CJM] M. Cwikel, B. Jawerth, M. Milman, *On the fundamental lemma of interpolation theory*, J. Approx. Th. 60 (1990), 70-82.

[FS] C. Fefferman, E.M. Stein, *H^p Spaces of Several Variables*, Acta Math. 129 (1972), 137-193.

[KK] S. Kislyakov, N. Kruglyak, *Stability of Approximation Under the Action of Singular Integral Operators,* Functional Analysis and its Applications 40 (2006), no 4, 285-297.

[KiKr] S. Kislyakov, N. Kruglyak, *Stability of Approximation under Singular Integral Operators and Calderón-Zygmund Type Decompositions II*, preprint 1734 (2005), Erwin Schrödinger International Institute for Mathematical Physics, Vienna.

[KiKr1] S. Kislyakov, N. Kruglyak, *Stability of Approximation under Singular Integral Operators and Calderón-Zygmund Type Decompositions,* preprint 07 (2005), St.-Petersburg Steklov Mathematical Institute.

[Kr] N. Kruglyak, *Investigations on the Real Interpolation Method,* Doctor of Science Thesis, Steklov Institute of Mathematics, St.-Petersburg Division, 1996.

[Kr1] N. Kruglyak, *Smooth Analogs of Calderon-Zygmund Decompositions*, *Quantitative Covering Theorems and the K-functional* (Russian), Algebra i Analiz 8 (1996), no. 4, 110-160, English translation in St.-Petersburg Math. Journal 8 (1997), no. 4, 617-649.

[KrKu] N. Kruglyak, E. Kuznetsov, *Smooth and Nonsmooth Calderón-Zygmund Decompositions for Morrey Spaces,* J. Fourier Analysis and Applications 11 (2005), no. 6, 697-714.

[KrKu1] N. Kruglyak, E. Kuznetsov, *Sharp Integral Estimates for the Fractional Maximal Function and Interpolation,* Arkiv för Matematik 44 (2006), no. 2, 309-326.

[LP] J.L. Lions, J. Peetre, *Sur une classe d'espaces d'interpolation*, Publ. Math. Inst. Hautes Etudes Sci., 19 (1964), 5-68.

[M] Y. Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, University Lecture Series vol. 22, AMS, Providence, 2001.

[ROF] L. Rudin, S. Osher, C. Fatemi, *Nonlinear Total Variation Based Noise Removal Algorithms*, Physica D, vol. 60, 259–268, 1992.

[S] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.

[TNV] E. Tadmor, S. Nezzar, L. Vese, *A Multiscale Image Representation Using Hierarchical* (BV, L_2) *Decompositions*, Multiscale Model. Simul. 2 (2004), no. 4, 554-573.

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