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A priori error estimates for approximate solutions to convex conservation laws

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Summary. We introduce a new technique for proving a priori error estimates between the entropy weak solution of a scalar conservation law and a finite–difference approximation calculated with the scheme of Engquist-Osher, Lax-Friedrichs, or Godunov. This technique is a discrete counterpart of the duality technique introduced by Tadmor [SIAM J. Numer. Anal. 1991]. The error is related to the consistency error of cell averages of the entropy weak solution. This consistency error can be estimated by exploiting a regularity structure of the entropy weak solution. One ends up with optimal error estimates.

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1 Introduction

In this paper we consider the following convex scalar conservation laws

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, \quad \text{in } \mathbb{R}\times\mathbb{R}^+, \\ u(.,0) &= u_0, \quad \text{in } \mathbb{R}, \end{aligned} \tag{1.1}$$

with $f \in C^2(\mathbb{R})$ being strictly convex; without loss of generality we assume that $f(0) = 0$. The initial data is supposed to satisfy $u_0 \in L^\infty \cap BV(\mathbb{R})$ and especially

$$U_m \le u_0(x) \le U_M \quad \text{for almost all } x \in \mathbb{R}. \tag{1.2}$$

On these conditions it is clear that there exists a uniquely determined entropy solution in $L^\infty \cap BV(\mathbb{R}\times\mathbb{R}^+)$ to (1.1), see, e.g., [11, Theorem 3.1, p. 69].

A commonly used numerical approximation to the entropy weak solution is given by a scheme in conservation form with numerical flux $g : \mathbb{R}^2 \to \mathbb{R}$ satisfying

1. **(Monotony)** g is non-decreasing with respect to its first argument and non-increasing with respect to its second argument.
2. $g \in C^1(\mathbb{R}^2)$. In particular, g is locally Lipschitz continuous, i.e. for every compact interval $I \subset \mathbb{R}$ there exists a constant $L_g > 0$, depending on I and f only, such that for all $u_1, u_2, v_1, v_2 \in I$
$$|g(u_1, v_1) - g(u_2, v_2)| \leq L_g\big[|u_1 - u_2| + |v_1 - v_2|\big].$$
3. **(Consistency)**
$$g(s, s) = f(s) \quad \forall s \in \mathbb{R}.$$

A well-known example for such a numerical flux is the Engquist-Osher flux which is defined by

$$g(v, w) := \int_0^v \max\{f'(s), 0\}\mathrm{d}s + \int_0^w \min\{f'(s), 0\}\mathrm{d}s. \tag{1.3}$$

Let the real line be partitioned by an equidistant grid $(x_{j+1/2})_{j\in\mathbb{Z}}$ with $x_{j+1/2} := (j + 1/2)\Delta x$. The time axis $\mathbb{R}^+$ is partitioned by an equidistant grid of grid size Δt. The numerical scheme then reads as follows.
For $i \in \mathbb{Z}$ set

$$v_i^0 := \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} u_0(y)\mathrm{d}y. \tag{1.4}$$

Let $(v_i^n)_{i\in\mathbb{Z}}$ be given for $n \in \mathbb{N}$. Define

$$v_i^{n+1} := v_i^n - \frac{\Delta t}{\Delta x}\Big[g(v_i^n, v_{i+1}^n) - g(v_{i-1}^n, v_i^n)\Big]. \tag{1.5}$$

Set

$$v_h(x, t) = v_i^n \quad \text{for } (x, t) \in [x_{i-1/2}, x_{i+1/2}[\times[t^n, t^{n+1}[. \tag{1.6}$$

We are interested in a priori error estimates for the error between the numerical approximation v_h and the entropy solution u. In an *a priori* error estimate the error between the entropy weak solution and a numerical approximation is bounded by some expression that solely depends on the entropy weak solution. In contrast, in an *a posteriori* error estimate the error is bounded in terms of the numerical approximation which can be used for designing adaptive methods.

We want to derive an a priori error estimation technique which takes the approximation of the continuous flux f by a numerical flux g into account, that is a technique that relies up on the consistency error which has to be measured appropriately. Up to now we do not know of any such technique in the literature.

It is known that for convex conservation laws the rate of convergence in the $L^\infty(L^1)$ norm is $(\Delta x)^{1/2}$, see [16]. However, this technique, which uses the uniqueness proof of Kruzkov, is in fact an a posteriori technique. It has been turned by Cockburn and Gremaud [1–3] into an a priori technique, but it does not give any relation between the numerical flux and the continuous flux, that is it does not relate the error to the consistency error. However, such a relation would be needed essentially if one would like to prove error estimates for higher order schemes like MUSCL schemes which show convergence rates higher than those for first order schemes in numerical experiments [14, Example 3.5.10].

A different approach to prove error estimates has been proposed by Tadmor and co-workers in the last years [27,19–21,28] which utilizes the uniqueness proof due to Oleinik [22], see also [8, Theorem 3, p. 151]. However, this $Lip^+ - Lip'$ technique again results in an a posteriori approach which has been exploited numerically by Kurganov and Karni [13]. We will cast this method into an a priori technique. It is linked to corresponding techniques for parabolic problems. Let us briefly comment on the techniques to prove error estimates between the solution of a linear parabolic problem and the numerical approximation given by some finite element method, see, e.g., [7, Chapter 16]. The error accumulation is controlled by the solution of a backward in time dual problem. Using the continuous dual problem leads to an a posteriori error estimate while the use of a discrete dual problem gives an a priori error estimate. This is the main idea behind our technique. However, since we are in the hyperbolic case and have to deal with possibly strong nonlinearities, the method is more technical.

Let us assume for a moment that the entropy weak solution u to (1.1) is smooth, i.e. it is a classical C^1 solution to (1.1). Integration of the differential equation over $[x_{i-1/2}, x_{i+1/2}[\times[t^n, t^{n+1}[$ yields

$$
\begin{aligned}
(1.7)\quad & \fint_{x_{i-1/2}}^{x_{i+1/2}} u(y, t^{n+1})\mathrm{d}y - \fint_{x_{i-1/2}}^{x_{i+1/2}} u(y, t^n)\mathrm{d}y \\
&\qquad + \lambda\Big[\fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}, \tau))\mathrm{d}\tau - \fint_{t^n}^{t^{n+1}} f(u(x_{i-1/2}, \tau))\mathrm{d}\tau \Big] = 0,
\end{aligned}
$$

where we used the abbreviation

$$\fint_a^b g(s)\mathrm{d}s = \frac{1}{b-a}\int_a^b g(s)\mathrm{d}s, \quad a, b \in \mathbb{R}, \quad a < b,$$

and $\lambda := \Delta t/\Delta x$. Set

$$u_i^n := \fint_{x_{i-1/2}}^{x_{i+1/2}} u(y, t^n)\mathrm{d}y. \tag{1.8}$$

Hence, we can rewrite (1.7) as follows

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\Big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\Big] + R_i^n(u), \tag{1.9}$$

where

(1.10) $R_i^n(u)$

$$= \frac{\Delta t}{\Delta x}\Big[g\Big(\fint_{x_{i-1/2}}^{x_{i+1/2}} u(., t^n), \fint_{x_{i+1/2}}^{x_{i+3/2}} u(., t^n)\Big) - \fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}, \tau))\mathrm{d}\tau\Big]$$
$$- \frac{\Delta t}{\Delta x}\Big[g\Big(\fint_{x_{i-3/2}}^{x_{i-1/2}} u(., t^n), \fint_{x_{i-1/2}}^{x_{i+1/2}} u(., t^n)\Big) - \fint_{t^n}^{t^{n+1}} f(u(x_{i-1/2}, \tau))\mathrm{d}\tau\Big].$$

Note, that $R_i^n(u)$ is nothing else than the approximation error between the numerical flux and the continuous flux. Usually this consistency error comes up in a pointwise sense, but here it appears in the sense of mean values which is due to our finite volume point of view.

Using that the scheme (1.5) is monotone under a suitable CFL–condition we get with the Crandall-Tartar Lemma [4] that the method is L^1 contractive and hence we get for the error $\mathrm{e}^n := u^n - v^n$ the estimate

$$\begin{aligned}\|e(., t^N)\|_{l^1} &:= \sum_{i\in\mathbb{Z}} \Delta x|u_i^N - v_i^N| \\ &\leq \sum_{i\in\mathbb{Z}} \Delta x|u_i^0 - v_i^0| + \sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}} \Delta x|R_i^n(u)| \\ &\leq \sum_{i\in\mathbb{Z}} \Delta x|u_i^0 - v_i^0| + \mathcal{O}(1),\end{aligned}$$

where in the last inequality we have used a simple Taylor series expansion up to first order to estimate the consistency error in (1.10).

Unfortunately, we do not get a convergence rate, i.e. power of Δx for the second term. The reason for this is that an order of $1/2$ is optimal in case of linear equations as shown by Tang and Teng [32] and with this simple approach we can only expect an integer order of convergence. We have to weaken the sense in which we measure the consistency. As Tadmor suggested, an appropriate space is Lip' to measure the consistency in. Lip' is the dual of Lip which is roughly speaking the space of all L^∞ functions that have bounded difference quotients (for the exact definition see Sect. 2). In Sect. 4 we shall prove that the following weak error estimate holds (see Theorem 10)

$$\|\mathrm{e}^N\|_{lip'} \leq S(t^N) \sup_{n=0,\dots,N-1} \left\| \frac{1}{\Delta t} R^n(u) \right\|_{lip'}, \tag{1.11}$$

where lip' is a discrete version of the space Lip'. The factor $S(t^N)$ measures the stability of the discretization and is related to Oleinik's entropy condition. The second factor measures the consistency of the scheme. The most important point to note is that the consistency error $R^n(u)$ is written in terms of cell averages. Exploiting regularity properties of the entropy weak solution we shall show (Proposition 27) that

$$\left\| \frac{1}{\Delta t} R^n(u) \right\|_{lip'} = \mathcal{O}(\Delta x). \tag{1.12}$$

By interpolating between lip' and bv which are discrete analogies to Lip' and BV one can control the l^1 error and we end up with the well-known result, see, e.g., [19, Thm. 3.1] or [11, Thm. A.1, p.162]

Theorem 1 *Assume the the initial data* $u_0 \in Lip^+ \cap BV(\mathbb{R})$ *and that the entropy solution* u *to* (1.1) *with strictly convex* f *is piecewise smooth and satisfies the regularity Assumption 24 below (see Sect. 7). The numerical approximation is defined by the scheme* (1.4), (1.5) *with the numerical flux of Engquist-Osher* (1.3) *where the following CFL–condition is supposed to hold*

$$\frac{\Delta t}{\Delta x} \max_{[U_m, U_M]} |f'| \leq \frac{1}{2}. \tag{1.13}$$

Then, for all $n \in \mathbb{N}$ *there exists a constant* $C > 0$ *depending only on* t^n, f *and* u_0 *such that*

$$\|u^n - v^n\|_{l^1} \leq C(\Delta x)^{1/2}, \tag{1.14}$$

where u^n *are the cell averages of the entropy weak solution defined in* (1.8) *and the norms and spaces are defined in the next section.*

Remark 2 1. For the ease of presentation we have formulated this result in case of the Engquist-Osher scheme. However, the explicit form of the numerical flux enters only at one specific point, see Lemma 14. Hence one has to check a similar results for a given numerical flux. We indicate the argumentation for the fluxes of Lax-Friedrichs and Godunov, see Lemma 15 below.

2. Note, that the error estimate (1.14) can be casted into an error estimate for the corresponding functions as follows. Define

$$u_h(x,t) = u_i^n \quad \text{for } (x,t) \in [x_{i-1/2}, x_{i+1/2}[\times[t^n, t^{n+1}[,$$

and similarly define v_h from $(v^n)_{n\in\mathbb{N}}$. It is straightforward to check that with a constant $C > 0$

$$\|u(.,t) - u_h(.,t)\|_{L^1(\mathbb{R})} \leq C\Delta x|u(.,t)|_{BV(\mathbb{R})} \leq C\Delta x|u_0|_{BV(\mathbb{R})}.$$

Using the triangle inequality, one gets from (1.14)

$$\|u(.,t^n) - v_h(.,t^n)\|_{L^1(\mathbb{R})} \leq C(\Delta x + (\Delta x)^{1/2}).$$

3. If the continuous flux f is linear Tang and Teng [32] have proved that the order $1/2$ is optimal. If f is strictly convex, the case we consider here, Teng and Zhang [33] proved that for special solutions which are piecewise constant and consist only of shock discontinuities one can achieve an order 1 of convergence. Finally, Sabac [23] proved that in general the order $1/2$ is optimal.

4. The condition that $u_0 \in Lip^+$ means that we exclude the case of initial data corresponding to an initial rarefaction, that is initial data

$$u_0(x) = \begin{cases} u_l & \text{if} \quad x < 0, \\ u_r & \text{if} \quad x \geq 0, \end{cases}$$

with $u_l < u_r$, $u_l, u_r \in \mathbb{R}$. Nessyahu and Tassa [21] have modified Tadmor's a posteriori approach and have been able to prove in the case of Lip^+ unbounded initial data a convergence rate of the error between the entropy weak solution of (1.1) and a viscosity approximation of order $\mathcal{O}(\varepsilon^{1/2}|\ln\varepsilon|)$. The main step in their proof is a sharp stability estimate for $|f'(u(.,t)|_{Lip^+}$. Up to now, similar results using this duality technique have not been proven for monotone schemes. We are able to treat this case within the framework of our a priori technique based upon duality techniques. If $f(w) = cw^2$ with a positive constant c then we get the same convergence rate $\mathcal{O}((\Delta x)^{1/2}|\ln\Delta x|)$ for approximations defined by the Engquist-Osher scheme. However, in the general case we have only been able to prove the non-optimal convergence rate of $\mathcal{O}((\Delta x)^{1/3})$, see [15].

The rest of this paper is organized as follows. First, we introduce some notations in Sect. 2. Then, we present a discrete interpolation result (Theorem 7) which guides the further program. In order to apply this interpolation result, we need a weak error estimate, i.e. an estimate in a negative norm, see Theorem 10. For this result, we need a stability result of a discrete dual problem which reduces to Oleinik's entropy condition for the entropy solution and the numerical solution, see Sect. 5 and 6. Finally, the weak error estimate mentioned before gives a relation to the approximation error between the numerical flux and the continuous flux. This consistency error is estimated in Sect. 7. In Sect. 8 we put all results together and prove our main result Theorem 1. We end with a conclusion and an outlook.

2 Notations

We denote spaces of functions by capital letters and spaces of sequences by lower case letters.

Let $I \subseteq \mathbb{R}$ be an open interval. By $L^p(I)$, $1 \leq p \leq \infty$, we denote the well-known Lebesgue space of functions $f : I \to \mathbb{R}$ with finite $\|.\|_{L^p}$ norm, see e. g. [9]. Furthermore, we need the set of functions with bounded variation, see e. g. [9, 11], which is defined as follows

$$BV(I) := \{f \in L^1(I) | \quad |f|_{BV(I)} < \infty\}, \tag{2.1}$$

where the BV semi-norm is given by

$$|f|_{BV(I)} := \sup\Big\{ \int_I f(x)g'(x)\mathrm{d}x | \quad g \in C_0^1(I),\ \|g\|_{L^\infty(I)} \leq 1\Big\}. \tag{2.2}$$

Here, $C_0^1(I)$ denotes the set of continuously differentiable functions $f : I \to \mathbb{R}$ with compact support (in I). Finally, we need the spaces Lip and Lip^+ which are defined as follows

$$Lip := \{f \in L^\infty(\mathbb{R}) \mid \quad |f|_{Lip} < \infty\},$$
$$Lip^+ := \{f \in L^\infty(\mathbb{R}) \mid \quad |f|_{Lip^+} < \infty\},$$

where the corresponding semi-norms are given by

$$|f|_{Lip} := \operatorname*{ess\,sup}_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|,$$
$$|f|_{Lip^+} := \operatorname*{ess\,sup}_{x \neq y} \left(\frac{f(x) - f(y)}{x - y} \right)^+, \quad (a)^+ := \max\{a, 0\} \quad a \in \mathbb{R}.$$

By $\mathbb{N}$ we denote the set of positive integers and by $\mathbb{Z}$ the set of integers. Let $(\phi_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers that is associated with a partition of $\mathbb{R}$ into intervals I_i, $i \in \mathbb{Z}$, of length $\Delta x > 0$. Define for $1 \leq p < \infty$

$$\|\phi\|_{l^p(\Delta x)} := \Big(\sum_{i\in\mathbb{Z}} \Delta x |\phi_i|^p \Big)^{1/p}. \tag{2.3}$$

The discrete maximum norm is denoted by

$$\|\phi\|_{l^\infty(\Delta x)} := \sup_{i\in\mathbb{Z}} |\phi_i|. \tag{2.4}$$

We shall need a discrete space of sequences with zero mass. We define

$$l_0^1(\Delta x) := \left\{ (\phi_i)_{i\in\mathbb{Z}} \in l^1(\Delta x) \mid \quad \sum_{i\in\mathbb{Z}} \Delta x \phi_i = 0 \right\}. \tag{2.5}$$

Moreover, we shall need discrete counterparts of BV, Lip, Lip^+, Lip' and $W^{-1,1}$ (for the definitions of these spaces see [28]). Define

$$\begin{aligned}
bv(\Delta x) &:= \{\phi \in l^1(\Delta x) \mid \quad |\phi|_{bv(\Delta x)} < \infty\},\\
lip(\Delta x) &:= \{\phi \in l^\infty(\Delta x) \mid \quad |\phi|_{lip(\Delta x)} < \infty\},\\
lip^+(\Delta x) &:= \{\phi \in l^\infty(\Delta x) \mid \quad |\phi|_{lip^+(\Delta x)} < \infty\},\\
lip'(\Delta x) &:= \{\phi \in l_0^1(\Delta x) \mid \quad \|\phi\|_{lip'(\Delta x)} < \infty\},\\
w^{-1,1}(\Delta x) &:= \{\phi \in l_0^1(\Delta x) \mid \quad \|\phi\|_{w^{-1,1}(\Delta x)} < \infty\},
\end{aligned}$$

where the norms and semi-norms are defined as follows

$$\begin{aligned}
|\phi|_{bv(\Delta x)} &:= \sum_{i\in\mathbb{Z}} |\phi_i - \phi_{i-1}|,\\
|\phi|_{lip(\Delta x)} &:= \sup_{i\in\mathbb{Z}} \frac{|\phi_i - \phi_{i-1}|}{\Delta x},\\
|\phi|_{lip^+(\Delta x)} &:= \sup_{i\in\mathbb{Z}} \frac{(\phi_i - \phi_{i-1})^+}{\Delta x}, \quad (a)^+ := \max\{a, 0\},\\
\|\phi\|_{lip'(\Delta x)} &:= \sup_{\psi\in lip(\Delta x),\ |\psi|_{lip}=1} |(\phi,\psi)_{\Delta x}|,\\
\|\phi\|_{w^{-1,1}(\Delta x)} &:= \sum_{i\in\mathbb{Z}} \Delta x \Big| \sum_{j\leq i-1} \Delta x \phi_j \Big|,
\end{aligned}$$

where we used the abbreviation

$$(\phi,\psi)_{\Delta x} := \sum_{i\in\mathbb{Z}} \Delta x\ \phi_i \psi_i. \tag{2.6}$$

For ease of notation we shall omit the argument Δx in the discrete spaces whenever it is obvious from the context that there is an underlying grid with grid-size Δx associated with the sequences.

Subsequently, we shall need a relation between the spaces $lip'(\Delta x)$ and $w^{-1,1}(\Delta x)$. We start with the following result on summation by parts.

Lemma 3 *Let $\phi \in l_0^1(\Delta x)$ and $\psi \in l^\infty(\Delta x)$. Then*

$$(\phi, \psi)_{\Delta x} = -\sum_{i \in \mathbb{Z}} (\psi_i - \psi_{i-1}) \sum_{j \le i-1} \Delta x \phi_j. \tag{2.7}$$

Proof. Let $\phi \in l_0^1(\Delta x)$ and $\psi \in l^\infty(\Delta x)$ be given. For $n \in \mathbb{N}$ we get by using summation by parts

$$\begin{aligned}\sum_{i=-n}^{n} \Delta x \phi_i \psi_i = &- \sum_{i=-n}^{n} \Delta x (\psi_i - \psi_{i-1}) \sum_{j \le i-1} \phi_j \\ &+ \psi_n \sum_{j \le n} \Delta x \phi_j - \psi_{-n-1} \sum_{j \le -n-1} \Delta x \phi_j.\end{aligned}$$

Letting n tend to infinity gives the assertion. $\square$

Using this result we can prove

Lemma 4 *If $\phi \in w^{-1,1}(\Delta x)$ then $\phi \in lip'(\Delta x)$ and*

$$\|\phi\|_{lip'(\Delta x)} \le \|\phi\|_{w^{-1,1}(\Delta x)}. \tag{2.8}$$

Proof. Let $\phi \in w^{-1,1}(\Delta x)$. Then, using summation by parts (Lemma 3) we get for any $\psi \in lip$

$$\begin{aligned}(\phi, \psi)_{\Delta x} &= -\sum_{i \in \mathbb{Z}} \Delta x \frac{1}{\Delta x} [\psi_i - \psi_{i-1}] \Delta x \sum_{j \le i-1} \phi_j \\ &\le \|\phi\|_{w^{-1,1}} |\psi|_{lip}.\end{aligned}$$

Consequently, we have

$$\|\phi\|_{lip'} \le \|\phi\|_{w^{-1,1}}.$$

$\square$

3 Interpolation between $lip'(\Delta x)$ and $bv(\Delta x)$

In this section we prove a discrete interpolation results between the spaces lip' and bv which is a discrete counterpart of Corollary 3.5 in [27].

One main idea to do so is to interpret sequences of real numbers as piecewise constant functions (this is not surprising since this is the way numerical approximations are defined, see (1.6)).

First, we need some relation between BV and bv.

It is easy to prove the following equivalence between $|.|_{BV}$ and $|.|_{bv}$.

Proposition 5 *Let $(u_i)_{i\in\mathbb{Z}} \in bv(\Delta x)$ be given. Set*

$$u_h(x) := u_i \quad \textit{for } x \in [x_{i-1/2}, x_{i+1/2}[,\ i \in \mathbb{Z}.$$

Then,

$$u_h \in BV(\mathbb{R}) \quad \textit{and } |u_h|_{BV(\mathbb{R})} = |u|_{bv(\Delta x)}. \tag{3.1}$$

The next result is well-known in the literature, see, e.g., [10, Lemma 6.9].

Lemma 6 *Let $f \in BV(\mathbb{R})$ and $\eta \in \mathbb{R}$. Then*

$$\|f(.+\eta) - f\|_{L^1(\mathbb{R})} \leq |\eta|\ |f|_{BV(\mathbb{R})}. \tag{3.2}$$

Now, we can state our interpolation result between $lip'(\Delta x)$ and $bv(\Delta x)$.

Theorem 7 *Let $e \in lip'(\Delta x) \cap bv(\Delta x)$. Then, there exists a constant $K > 0$ independent from Δx and e such that the following estimate holds*

$$\|e\|_{l^1(\Delta x)} \leq K \|e\|^{1/2}_{lip'(\Delta x)} |e|^{1/2}_{bv(\Delta x)}. \tag{3.3}$$

Proof. We follow mostly the continuous analogue in [27, Corollary 3.5]. For ease of notation we omit the argument Δx in discrete norms and spaces.

Let $\zeta \in C_0^1(-1,1)$ be an even and positive function with $\int_{\mathbb{R}} \zeta = 1$. For $\delta > 0$ set

$$\zeta^\delta(x) := \frac{1}{\delta}\zeta\left(\frac{x}{\delta}\right).$$

Let $\phi \in l^\infty$ be given. Set $\phi_h(x) := \phi_i$ for $x \in [x_{i-1/2}, x_{i+1/2}[$, $i \in \mathbb{Z}$. Obviously, we have

$$\|\phi_h\|_{L^\infty} = \|\phi\|_{l^\infty} \quad \text{and} \quad \|\phi_h\|_{L^1} = \|\phi\|_{l^1}. \tag{3.4}$$

Set

$$\phi_h^\delta := \phi_h * \zeta^\delta,$$

where $*$ denotes the convolution. Define $\phi^\delta \in l^\infty$ by $\phi_i^\delta := \phi_h^\delta(x_i)$.

We consider

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i &= \sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i^\delta + \sum_{i\in\mathbb{Z}} \Delta x e_i (\phi_i - \phi_i^\delta) \\ &=: T_1 + T_2.\end{aligned} \tag{3.5}$$

By definition of the lip' norm we get for T_1.

$$T_1 \leq |\phi^\delta|_{lip} \|e\|_{lip'}.$$

At this stage it is not clear that $|\phi^\delta|_{lip} < \infty$, but this point will be clarified in the next step.

Using the definition of ϕ_i^δ we calculate

$$\begin{aligned}\phi_i^\delta - \phi_{i-1}^\delta &= \int\limits_{\mathbb{R}} \phi_h(y)\big[\zeta^\delta(x_i - y) - \zeta^\delta(x_{i-1} - y)\big]\mathrm{d}y \\ &\leq \|\phi_h\|_{L^\infty(\mathbb{R})}\|\zeta^\delta(x_i - .) - \zeta^\delta(x_{i-1} - .)\|_{L^1(\mathbb{R})} \\ &\leq \|\phi\|_{l^\infty}\frac{\Delta x}{\delta}\|\zeta'\|_{L^1(\mathbb{R})}.\end{aligned}$$

Hence, we get

$$|\phi^\delta|_{lip} \leq \frac{1}{\delta}\|\phi\|_{l^\infty}\|\zeta'\|_{L^1(\mathbb{R})}. \tag{3.6}$$

Next, we consider the term T_2 in (3.5). Decompose T_2 as follows

$$T_2 = \sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i - \sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i^\delta =: T_3 - T_4.$$

First, we rewrite the term T_4. Let us use the following abbreviation: for $j \in \mathbb{Z}$ set $I_j := [x_{j-1/2}, x_{j+1/2}[$. Using again the definition of ϕ_i^δ and some substitution we get

$$\begin{aligned}T_4 &= \sum_{i\in\mathbb{Z}} \Delta x e_i \sum_{j\in\mathbb{Z}} \phi_j \int\limits_{I_j} \zeta^\delta(x_i - y)\mathrm{d}y \\ &= \sum_{i,j} \Delta x e_i \phi_j \int\limits_{I_{i-j}} \zeta^\delta(z)\mathrm{d}z \\ &= \sum_{i,j} \Delta x e_{i+j} \phi_i \int\limits_{I_j} \zeta^\delta(z)\mathrm{d}z.\end{aligned}$$

Since $\int\limits_{\mathbb{R}} \zeta^\delta = 1$ we can continue and estimate

$$\begin{aligned}T_3 - T_4 &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \Delta x(e_i - e_{i+j})\phi_i \int\limits_{I_j} \zeta^\delta(z)\mathrm{d}z \\ &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \Delta x(e_h(x_i) - e_h(x_{i+j}))\phi_i \int\limits_{I_j} \zeta^\delta(z)\mathrm{d}z \\ &\leq \|\phi\|_{l^\infty}\Big(\sup_{|z|\leq\delta} \sum_{i\in\mathbb{Z}} \int\limits_{I_i} |e_h(x) - e_h(x+z)|\mathrm{d}x\Big)\|\zeta^\delta\|_{L^1(\mathbb{R})}.\end{aligned}$$

Using Lemma 6 and Proposition 5 we get

$$T_3 - T_4 \leq \|\phi\|_{l^\infty}\delta|e|_{bv}\|\zeta\|_{L^1(\mathbb{R})}. \tag{3.7}$$

Using (3.6) and (3.7) we can estimate

$$\sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i \le \frac{1}{\delta}\|\phi\|_{l^\infty}\|e\|_{lip'}\|\zeta'\|_{L^1(\mathbb{R})} + \|\phi\|_{l^\infty}\delta|e|_{bv}\|\zeta\|_{L^1(\mathbb{R})}$$

$$\le \max\{\|\zeta'\|_{L^1(\mathbb{R})}, 1\}\Big(\frac{1}{\delta}\|e\|_{lip'} + \delta|e|_{bv}\Big)\|\phi\|_{l^\infty}. \tag{3.8}$$

With the choice $\delta = (\|e\|_{lip'}/|e|_{bv})^{1/2}$ we get

$$\sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i \le K\big(\|e\|_{lip'}|e|_{bv}\big)^{1/2}\|\phi\|_{l^\infty}, \tag{3.9}$$

where $K := 2\max\{\|\zeta'\|_{L^1(\mathbb{R})}, 1\}$.

Assertion (3.3) now follows easily from (3.9)

$$\|e\|_{l^1} = \sup_{\|\phi\|_{l^\infty}\le 1} \sum_{i\in\mathbb{Z}} \Delta x e_i \phi_i \le K\big(\|e\|_{lip'}|e|_{bv}\big)^{1/2}.$$

□

Remark 8 Note, that this theorem is exactly tailored to our situation. The sequence e will be replaced by the error between the cell averages of the entropy weak solution to (1.1) and some numerical approximation at some time t^n, $n \in \mathbb{N}$. We will get some consistency error in some weak norm, namely $\|.\|_{lip'}$, which will give us some power of Δx. In a higher order norm, the discrete BV semi-norm $|.|_{bv(\Delta x)}$, we will get stability.

4 A weak error estimate

We recall that the numerical approximation is defined by

$$v_i^{n+1} := v_i^n - \frac{\Delta t}{\Delta x}\Big[g(v_i^n, v_{i+1}^n) - g(v_{i-1}^n, v_i^n)\Big], \tag{4.1}$$

where the numerical flux is assumed to be in C^1. The discrete initial data is given by

$$v_i^0 := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u_0(x)\mathrm{d}x, \quad i \in \mathbb{Z}. \tag{4.2}$$

The cell averages (see (1.8)) of the entropy weak solution are supposed to satisfy

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\Big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\Big] + R_i^n(u) \tag{4.3}$$

with some error term $R_i^n(u)$ which will be specified in Sect. 7.

Set $e_i^n := u_i^n - v_i^n$. Since the local averaging operator and the continuous evolution operator are conservative and since the numerical scheme (4.1) is in conservation form we easily get the following result taking the definition of the discrete initial data (4.2) into account.

Lemma 9 *Let* u *be the entropy solution to* (1.1), $(u_i^n)_{i\in\mathbb{Z}}$ *its local mean values at time* t^n, $n \in \mathbb{N}$, *defined in* (1.8) *and* $(v_i^n)_{i\in\mathbb{Z}}$ *the numerical approximation given by* (4.1), (4.2). *The error is defined by* $e_i^n := u_i^n - v_i^n$ *for* $i \in \mathbb{Z}$. *Then, for all* $n \in \mathbb{N}$ *we have*

$$\sum_{i\in\mathbb{Z}} \Delta x e_i^n = 0. \tag{4.4}$$

Furthermore, we have for all $i \in \mathbb{Z}$ *that* $e_i^0 = 0$.

We shall now derive an error equation. Subtracting (4.1) from (4.3) gives

$$\begin{aligned} e_i^{n+1} = e_i^n - \frac{\Delta t}{\Delta x}\Big[& g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) \\ & - \big(g(v_i^n, v_{i+1}^n) - g(v_{i-1}^n, v_i^n)\big)\Big] + R_i^n(u). \end{aligned}$$

Since we assume that the numerical flux g is C^1 we can use the mean value theorem and get

$$\begin{aligned} & g(u_1, u_2) - g(v_1, v_2) \\ & = \int_0^1 \nabla g(v_1 + \theta(u_1 - v_1), v_2 + \theta(u_2 - v_2))\mathrm{d}\theta \cdot \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix}. \end{aligned}$$

We set

$$G_{i+1/2}^{n,j} := \int_0^1 \partial_j g(v_i^n + \theta(u_i^n - v_i^n), v_{i+1}^n + \theta(u_{i+1}^n - v_{i+1}^n))\mathrm{d}\theta, \tag{4.5}$$

where ∂_j, $j \in \{1,2\}$, denotes the partial derivative of g with respect to the j-th argument.

Hence, we can continue and get the following error representation formula

$$\begin{aligned} & \frac{1}{\Delta t}\left[e_i^{n+1} - e_i^n\right] + \frac{1}{\Delta x}\Big[G_{i+1/2}^{n,1} e_i^n + G_{i+1/2}^{n,2} e_{i+1}^n \\ & \quad - G_{i-1/2}^{n,1} e_{i-1}^n - G_{i-1/2}^{n,2} e_i^n\Big] = \frac{1}{\Delta t} R_i^n(u). \end{aligned} \tag{4.6}$$

Multiply (4.6) by $\Delta t \Delta x \phi_i^{n+1}$, where $\phi^{n+1} \in l^\infty$ is a test sequence. Then summation over $n = 0, \ldots, N-1$ and $i \in \mathbb{Z}$ gives after some reordering (which we justify below)

$$\begin{aligned} &\sum_{i\in\mathbb{Z}} \Delta x e_i^N \phi_i^N - \sum_{i\in\mathbb{Z}} \Delta x e_i^0 \phi_i^0 + \sum_{n=0}^{N-1} \Delta t \sum_{i\in\mathbb{Z}} \Delta x e_i^n \Big\{ \frac{1}{\Delta t} \big[\phi_i^n - \phi_i^{n+1}\big] \\ &+ \frac{1}{\Delta x} \big[G^{n,1}_{i+1/2} \phi_i^{n+1} + G^{n,2}_{i-1/2} \phi_{i-1}^{n+1} - G^{n,1}_{i+1/2} \phi_{i+1}^{n+1} - G^{n,2}_{i-1/2} \phi_i^{n+1} \big] \Big\} \\ &= \sum_{n=0}^{N-1} \Delta t \sum_{i\in\mathbb{Z}} \Delta x \phi_i^{n+1} \frac{1}{\Delta t} R_i^n(u). \end{aligned} \tag{4.7}$$

It remains to justify that we can reorder the summation of the terms coming from the left hand side of (4.6). It is well-known, see, e.g., [10, Prop. 5.3 and Lemma 5.3], that the entropy weak solution satisfies for all $t > 0$

$$u(.,t) \in L^1(\mathbb{R}) \quad \text{and } \|u(.,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})},$$

therefore, the cell averages satisfy for all $n \in \mathbb{N}$

$$u^n \in l^1(\Delta x) \cap l^\infty(\Delta x).$$

If the CFL–condition

$$\lambda \sup_{i\in\mathbb{Z}} |f'(v_i^n)| \leq 1 \tag{4.8}$$

holds for all $n \in \mathbb{N}$ then the numerical approximation defined in (4.1) with the numerical flux of Engquist and Osher satisfies for all $n \in \mathbb{N}$

$$v^n \in l^1(\Delta x) \quad \text{and } U_m \leq v_i^n \leq U_M, \quad \text{for all } i \in \mathbb{Z}.$$

From this we conclude that for all $n \in \mathbb{N}$ and $j = 1, 2$

$$\mathrm{e}^n \in l^1(\Delta x) \quad \text{and } (G^{n,j}_{i+1/2})_{i\in\mathbb{Z}} \in l^\infty(\Delta x).$$

Therefore, the reordering of the summation which leads to (4.7) is justified.

From equation (4.7) we see how we should choose the discrete test function ϕ^n, namely such that the term in curly brackets vanishes, that is ϕ_i^n should be chosen as solution of a discrete, linear and backward in time problem. Let $(\phi_i^N)_{i\in\mathbb{Z}} \in lip$ be given. For $n = N-1, \ldots, 0$ set

$$\phi_i^n := \phi_i^{n+1} - \frac{\Delta t}{\Delta x} \Big[G^{n,1}_{i+1/2} (\phi_i^{n+1} - \phi_{i+1}^{n+1}) + G^{n,2}_{i-1/2} (\phi_{i-1}^{n+1} - \phi_i^{n+1}) \Big]. \tag{4.9}$$

Then, using the notation (2.6) and $\mathrm{e}^0 = 0$ due to (4.2), the error representation formula reduces to

$$(\mathrm{e}^N, \phi^N)_{\Delta x} = \sum_{n=0}^{N-1} \Delta t \Big(\phi_i^{n+1}, \frac{1}{\Delta t} R_i^n(u) \Big)_{\Delta x}. \tag{4.10}$$

Using Lemma 4 we deduce the following weak error estimate which is the discrete analogue to [19, Theorem 2.1].

Theorem 10 *Assume that the consistency error satisfies $R^n(u) \in lip'$ for all $n \in \mathbb{N}$. Then, the error between the numerical solution defined in* (4.1) *and the cell averages of the entropy weak solution to* (1.1) *at time t^N satisfies the following error estimate:*

$$\|\mathrm{e}^N\|_{lip'} \leq \sup_{0 \neq \phi^N \in lip} \big(S(\phi^N) \big) \sup_{n=0,\dots,N-1} \Big\| \frac{1}{\Delta t} R^n(u) \Big\|_{lip'}, \tag{4.11}$$

where the stability factor $S(\phi^N)$ *is defined by*

$$S(\phi^N) := \frac{1}{|\phi^N|_{lip}} \sum_{n=0}^{N-1} \Delta t |\phi^{n+1}|_{lip} \tag{4.12}$$

and ϕ^{n+1}, $n = 0, \dots, N-2$, is defined in (4.9) *and depends on ϕ^N.*

Hence, we have to study the stability properties of the discrete dual problem (4.9) and we have to examine the consistency error $R^n(u)$.

5 Discrete dual problem

In this Section we study the discrete linear backward problem (4.9). Recall, that for given $(\phi_i^N)_{i \in \mathbb{Z}} \in lip$ we define for $n = N-1, \dots, 0$

$$\phi_i^n := \phi_i^{n+1} - \frac{\Delta t}{\Delta x} \Big[G_{i+1/2}^{n,1} (\phi_i^{n+1} - \phi_{i+1}^{n+1}) + G_{i-1/2}^{n,2} (\phi_{i-1}^{n+1} - \phi_i^{n+1}) \Big]. \tag{5.1}$$

Since we want to estimate $|\phi^n|_{lip}$ let us consider differences of the solution of the backward problem (5.1). Set $z_i^n := \phi_i^n - \phi_{i-1}^n$. Then z_i^n satisfies

$$\begin{aligned} z_i^n = z_i^{n+1} - \frac{\Delta t}{\Delta x} \Big[& - G_{i+1/2}^{n,1} z_{i+1}^{n+1} - G_{i-1/2}^{n,2} z_i^{n+1} \\ & + G_{i-1/2}^{n,1} z_i^{n+1} + G_{i-3/2}^{n,2} z_{i-1}^{n+1} \Big]. \end{aligned} \tag{5.2}$$

This linear difference equation has the following properties.

Lemma 11 *If the CFL–condition*

$$1 - \frac{\Delta t}{\Delta x}\big(G^{n,1}_{i-1/2} - G^{n,2}_{i-1/2}\big) \geq 0 \tag{5.3}$$

holds, then the scheme (5.2) *is monotone, that is* z_i^n *depends from* z_{i-1}^{n+1}*,* z_i^{n+1} *and* z_{i+1}^{n+1} *in a monotone increasing manner.*

Remark 12 If the numerical flux g is the flux of Engquist-Osher, then condition (5.3) is satisfied if

$$\frac{\Delta t}{\Delta x} \max_{[U_m, U_M]} |f'| \leq \frac{1}{2}.$$

That is, we have to strengthen the CFL–condition (4.8) in order to guarantee that (5.2) is monotone.

For estimating $|\phi^n|_{lip}$ it is sufficient to estimate $\|z\|_{l^\infty}$. Rewrite (5.2) with $\lambda = \Delta t/\Delta x$ as

$$\begin{aligned} z_i^n = &\Big(1 - \lambda(G^{n,1}_{i-1/2} - G^{n,2}_{i-1/2})\Big) z_i^{n+1} \\ &+ \lambda G^{n,1}_{i+1/2} z_{i+1}^{n+1} + \lambda(-G^{n,2}_{i-3/2}) z_{i-1}^{n+1}. \end{aligned}$$

Due to the monotony of g and the CFL–condition (5.3) we have

$$\begin{aligned} \|z^n\|_{l^\infty} &\leq \sup_{i\in\mathbb{Z}} [1 + \lambda(A_i^n - A_{i-1}^n)] \|z^{n+1}\|_{l^\infty} \\ &\leq (1 + \Delta t |A^n|_{lip^+}) \|z^{n+1}\|_{l^\infty} \\ &\leq \exp\big(\Delta t |A^n|_{lip^+}\big) \|z^{n+1}\|_{l^\infty}, \end{aligned}$$

where

$$A_i^n := G^{n,1}_{i+1/2} + G^{n,2}_{i-1/2}. \tag{5.4}$$

Hence, we have proven the following result which is the discrete counterpart of [27, Theorem 2.2].

Proposition 13 *Let* $\phi^N \in lip$ *be given and* ϕ^n *be given by* (5.1) *for* $n = N-1, \ldots, 0$. *Let* A_i^n *be defined in* (5.4). *Then the following stability estimate holds for* $n \in \{0, 1, \ldots, N\}$

$$|\phi^n|_{lip} \leq \exp\left(\sum_{k=n}^{N-1} \Delta t |A^k|_{lip^+}\right) |\phi^N|_{lip}. \tag{5.5}$$

If the numerical flux g in (4.1) is the Engquist-Osher flux we have that

$$A_i^n = \int_0^1 f'(u_i^n + \theta(v_i^n - u_i^n))\mathrm{d}\theta.$$

Using the convexity of f and the uniform L^∞ bound for u^n and v^n we get

$$\begin{aligned}
&A_i^n - A_{i-1}^n \\
&= \int_0^1 f'(u_i^n + \theta(v_i^n - u_i^n))\mathrm{d}\theta - \int_0^1 f'(u_{i-1}^n + \theta(v_{i-1}^n - u_{i-1}^n))\mathrm{d}\theta \\
&\le \max_{[U_m,U_M]} f'' \int_0^1 \Big((u_i^n + \theta(v_i^n - u_i^n)) - (u_{i-1}^n + \theta(v_{i-1}^n - u_{i-1}^n)\Big)^+ \mathrm{d}\theta \\
&\le \max_{[U_m,U_M]} f'' \int_0^1 \Big((1-\theta)(u_i^n - u_{i-1}^n)^+ + \theta(v_i^n - v_{i-1}^n)^+\Big)\mathrm{d}\theta \\
&= \frac{1}{2} \max_{[U_m,U_M]} f''\Big((u_i^n - u_{i-1}^n)^+ + (v_i^n - v_{i-1}^n)^+\Big).
\end{aligned}$$

Therefore, we have proven

Lemma 14 *Let the numerical flux in* (4.1) *be the Engquist-Osher flux. Then, we have*

$$|A^n|_{lip^+} \le \frac{1}{2} \max_{[U_m,U_M]} f''\big(|u^n|_{lip^+} + |v^n|_{lip^+}\big), \tag{5.6}$$

where A^n is defined in (5.4).

Therefore, we have reduced the stability problem of the discrete backward problem to a one-sided Lipschitz-estimate for the cell averages of the entropy weak solution and for the numerical solution. This one-sided Lipschitz estimate is nothing else than Oleinik's entropy-condition [22].

Finally, we comment on other numerical fluxes like Godunov's flux and the Lax-Friedrichs flux. Let us start with the numerical flux of Lax and Friedrichs which is defined by, see, e.g., [11, p.111]

$$g(u,v) = \frac{1}{2}[f(u) + f(v)] - \frac{\Delta x}{2\Delta t}(v - u).$$

Obviously, we have that $g \in C^2(\mathbb{R}^2)$ and

$$A_i^n = \int_0^1 f'(u_i^n + \theta(v_i^n - u_i^n))\mathrm{d}\theta$$

which is the same as for the Engquist-Osher flux and we can proceed as before.

Godunov's numerical flux is defined for convex f by, see, e.g., [14, p.70]

$$g(u,v) = \begin{cases} f(u) & u \geq v, \quad f(u) \geq f(v), \\ f(v) & u \geq v, \quad f(u) < f(v), \\ f(u) & u < v, \quad f'(u) \geq 0, \\ f(v) & u < v, \quad f'(v) \leq 0, \\ f((f')^{-1}(0)) & u < v, \quad f'(u) < 0 < f'(v). \end{cases}$$

Hence, Godunov's numerical flux is Lipschitz continuous, but not globally differentiable, which means that our assumption $g \in C^1(\mathbb{R}^2)$, stated in the introduction, is not satisfied. However, since it is Lipschitz continuous and piecewise C^1 we can again apply the mean value theorem to deduce the error formula (4.6) and the derivatives of the numerical flux in (4.5) are defined almost everywhere with respect to the two dimensional Lebesgue measure. In order to write its derivatives in a compact form we introduce two sets

$$\begin{aligned} A^+ &:= \{(u,v) \in \mathbb{R}^2 | u \geq v, \ f(u) \geq f(v)\} \\ &\quad \cup \{(u,v) \in \mathbb{R}^2 | u < v, \ f'(u) \geq 0\}, \\ A^- &:= \{(u,v) \in \mathbb{R}^2 | u \geq v, \ f(u) < f(v)\} \\ &\quad \cup \{(u,v) \in \mathbb{R}^2 | u < v, \ f'(v) \leq 0\}. \end{aligned}$$

The characteristic functions associated with these sets are given by

$$\mathbb{I}_{A^\pm}(u,v) := \begin{cases} 1 & (u,v) \in A^\pm, \\ 0 & \text{else.} \end{cases}$$

Hence, we can calculate the derivatives of Godunov's flux explicitly and get

$$\begin{aligned} \partial_1 g(u,v) &= f'(u)\mathbb{I}_{A^+}(u,v) \geq 0, \\ \partial_2 g(u,v) &= f'(v)\mathbb{I}_{A^-}(u,v) \leq 0, \end{aligned}$$

where the sign of the derivatives can be deduced from the definition of the sets $A^\pm$ and taking into account that f is convex. With $w_i^n(\theta) := v_i^n + \theta(u_i^n - v_i^n)$ we get

$$\begin{aligned} A_i^n = &\int_0^1 f'(w_i^n(\theta))\mathbb{I}_{A^+}(w_i^n(\theta), w_{i+1}^n(\theta))\mathrm{d}\theta \\ &+ \int_0^1 f'(w_i^n(\theta))\mathbb{I}_{A^-}(w_{i-1}^n(\theta), w_i^n(\theta))\mathrm{d}\theta. \end{aligned}$$

The following observation on the sign of the integrands in the two integrals is important: if the integrand in the first integral is positive then the integrand in the second integral is zero. And similarly, if the integrand in the second integral is negative then the integrand in the first integral must vanish. Using this observation in a distinction of cases one can show that (5.6) holds for Godunov's flux as well.

Summarizing, we have proven

Lemma 15 *Let the numerical flux in* (4.1) *be the Lax-Friedrichs or Godunov flux. Then the estimate* (5.6) *holds.*

Remark 16 Note that it is not obvious if one can deduce the estimate (5.6) for *any* monotone numerical flux from Lemma 15. It is known [26] that any monotone numerical flux can be written as a convex combination of the numerical fluxes of Lax-Friedrichs and Godunov. The problem is that the weights in this convex combination depend on the arguments of the numerical flux.

6 Stability properties

In this section we recall and derive the stability properties for the numerical solution, defined in (4.1), and for the cell averages of the entropy weak solution to (1.1). We need lip^+ bounds in order to treat the stability of the discrete dual problem (see also Theorem 10, Proposition 13 and Lemma 14) and we need bv bounds for our interpolation result in Theorem 7

6.1 Stability properties of the cell averages of the entropy weak solution

The lip^+ bound for the mean values of the entropy weak solution follows from a Lip^+ bound for the solution. We get from [27, p. 899] that the following estimate holds for the entropy weak solution to (1.1)

$$|u(.,t)|_{Lip^+} \leq \frac{1}{\min_{x\in\mathbb{R}} f''(x)} \frac{1}{|f'(u_0)|^{-1}_{Lip^+} + t}, \quad t \geq 0. \tag{6.1}$$

We need a relation between the discrete lip^+ semi-norm and the continuous Lip^+ semi-norm.

Lemma 17 *Let $w \in Lip^+$. Recall that the cell averages are defined as follows: for $i \in \mathbb{Z}$ set*

$$\overline{w}_i = \frac{1}{|I_i|} \int\limits_{I_i} w(y)\mathrm{d}y, \quad I_i := [x_{i-1/2}, x_{i+1/2}[. \tag{6.2}$$

Then,

$$|\overline{w}|_{lip^+} \leq |w|_{Lip^+}. \tag{6.3}$$

Proof. Using the definition of $\overline{w}_i$ we get

$$\overline{w}_i - \overline{w}_{i-1} = \frac{1}{(\Delta x)^2} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{x_{i-3/2}}^{x_{i-1/2}} (w(x) - w(y)) \mathrm{d}y \mathrm{d}x.$$

In the range of integration we have that $x - y \geq 0$. Consequently, we get

$$\begin{aligned}(\frac{\overline{w}_i - \overline{w}_{i-1}}{\Delta x})^+ &= \frac{1}{(\Delta x)^3} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{x_{i-3/2}}^{x_{i-1/2}} (x-y) \left(\frac{(w(x) - w(y))}{x-y} \right)^+ \mathrm{d}y \mathrm{d}x \\ &\leq |w|_{Lip^+} \frac{1}{(\Delta x)^3} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{x_{i-3/2}}^{x_{i-1/2}} (x-y) \mathrm{d}y \mathrm{d}x \\ &= |w|_{Lip^+}.\end{aligned}$$

□

Putting (6.1) and Lemma 17 together proves

Proposition 18 *Let u be the unique entropy weak solution to* (1.1) *and* $(u_i^n)_{i\in\mathbb{Z}}$ *its cell averages defined in* (1.8). *Then, for* $n \in \mathbb{N}$

$$|u^n|_{lip^+} \leq \frac{1}{\min_{x\in\mathbb{R}} f''(x)} \frac{1}{|f'(u_0)|_{Lip^+}^{-1} + t^n}. \tag{6.4}$$

For the discrete bv semi-norm of the cell averages of the entropy weak solution we get the following estimate.

Proposition 19 *Let u denote the entropy solution to* (1.1)*, where the initial data $u_0 \in L^\infty \cap BV(\mathbb{R})$. For $n \in \mathbb{N}$ let $(u_i^n)_{i\in\mathbb{Z}}$ denote the cell averages of $u(.,t^n)$ defined in* (1.8). *Then, for all $n \in \mathbb{N}$*

$$|u^n|_{bv} \leq |u_0|_{BV(\mathbb{R})}. \tag{6.5}$$

Proof. It is straightforward to prove, see, e.g., [10, Remark 5.4, Lemma 6.4], that for any $n \in \mathbb{N}$

$$|u^n|_{bv} = \sum_{i\in\mathbb{Z}} |u_i^n - u_{i-1}^n| \leq |u(.,t^n)|_{BV(\mathbb{R})}. \tag{6.6}$$

The entropy weak solution satisfies for all $t > 0$ the following stability in BV, see, e.g., [11, Theorem 3.1, p. 69]

$$|u(.,t)|_{BV(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})}. \tag{6.7}$$

Putting (6.6) and (6.7) together concludes the proof. □

6.2 Stability properties of the numerical solution

The lip^+ bound for numerical solutions can be found in a paper by Tadmor [19, p.1514–1515] which we quote in

Proposition 20 *Let* v^n, $n \in \mathbb{N}$, *be defined in* (4.1) *with the numerical flux of Engquist and Osher and assume that the following CFL–condition holds*

$$\frac{\Delta t}{\Delta x} \max_{[U_m, U_M]} |f'| \leq \frac{1}{2}. \tag{6.8}$$

Then, the following estimate for the lip^+ *semi-norm of the numerical approximation holds*

$$|v^n|_{lip^+} \leq \frac{1}{|v^0|_{lip^+}^{-1} + \beta t^n}, \quad n \in \mathbb{N}, \tag{6.9}$$

where $\beta := \frac{1}{4} \min_x f''(x)$.

Remark 21 Similar estimates hold for the Lax-Friedrichs and the Godunov scheme [25,12].

The second stability result concerns the bv stability. This is a well-known result following from the TVD property of monotone schemes, see, e.g., [11,14].

Proposition 22 *Let* v^n *be generated by* (4.1) *with the numerical flux of Engquist and Osher and let the CFL–condition*

$$\frac{\Delta t}{\Delta x} \max_{x \in [U_m, U_M]} |f'(x)| \leq 1 \tag{6.10}$$

be satisfied. Then,

$$|v^n|_{bv} \leq |v^0|_{bv}. \tag{6.11}$$

6.3 Estimation of the stability factor $S(\phi^N)$

Recall that the stability factor defined in theorem 10 is given by

$$S(\phi^N) := \frac{1}{|\phi^N|_{lip}} \sum_{n=0}^{N-1} \Delta t |\phi^{n+1}|_{lip},$$

where $\phi^N \in lip$, $N \in \mathbb{N}$, is given and ϕ^n, $n \in \{1, \ldots, N-1\}$, is defined in (5.1).

Proposition 23 *Let $u_0 \in Lip^+ \cap BV(\mathbb{R})$. Assume that the following CFL–condition is satisfied in the numerical scheme* (4.1)

$$\frac{\Delta t}{\Delta x} \max_{[U_m, U_M]} |f'| \leq \frac{1}{2}.$$

Then, for all $N \in \mathbb{N}$ and $\phi^N \in lip$ the estimate holds

$$S(\phi^N) \leq C, \tag{6.12}$$

where $C > 0$ depends from t^N, f and u_0 only.

Proof. From Proposition 13 we know that for $n \in \{1, \ldots, N-1\}$

$$|\phi^n|_{lip} \leq \exp\left(\sum_{k=n}^{N-1} \Delta t |A^k|_{lip^+} \right) |\phi^N|_{lip}.$$

Using Lemma 14, Proposition 18 and Proposition 20 we can estimate for $k \in \mathbb{N}$

$$\begin{aligned} |A^k|_{lip^+} &\leq \frac{1}{2} \max_{[U_m, U_M]} f'' \Big[|u^k|_{lip^+} + |v^k|_{lip^+} \Big] \\ &\leq \frac{\gamma}{\alpha + \beta t^k}, \end{aligned}$$

where

$$\begin{aligned} \alpha &:= \min \left\{ \min_{\mathbb{R}} f'' |f'(u_0)|^{-1}_{Lip^+},\ |u_0|^{-1}_{Lip^+} \right\}, \\ \beta &:= \frac{1}{4} \min_{x \in \mathbb{R}} f''(x), \\ \gamma &:= \max_{[U_m, U_M]} f''. \end{aligned}$$

Note, that $\alpha > 0$ since we have assumed that $u_0 \in Lip^+$ and $f \in C^2(\mathbb{R})$ is strictly convex. For $n \in \{1, \ldots, N\}$ we get from this estimate

$$\begin{aligned} \sum_{k=n}^{N-1} \Delta t |A^k|_{lip^+} &\leq \sum_{k=n}^{N-1} \int_{t^{k-1}}^{t^k} \frac{\gamma}{\alpha + \beta t} \mathrm{d}t \\ &= \frac{\gamma}{\beta} \ln \left(\frac{\alpha + \beta t^{N-1}}{\alpha + \beta t^{n-1}} \right) \\ &\leq \ln \left(1 + \frac{\beta}{\alpha} t^N \right)^{\gamma/\beta}. \end{aligned}$$

Inserting this we get

$$S(\phi^N) \le t^N \left(1 + \frac{\beta}{\alpha} t^N\right)^{\gamma/\beta},$$

where the right hand side in this estimate defines the constant C in (6.12). □

7 Regularity properties of the entropy weak solution

In this section we first derive the explicit form of the consistency error $R^n(u)$ in (4.3), and then we exploit regularity properties of the entropy solution to estimate this term.

It is well-known that solutions to (1.1) in general exhibit discontinuities. Therefore, one cannot expect a regularizing effect for hyperbolic problems such as, for example, for parabolic initial value problems. Nevertheless, solutions of hyperbolic problems have a special regularity structure.

First results on the structure of the entropy weak solution to (1.1) go back to Lax [17], Oleinik [22], Dafermos [5] and Schaeffer [24]. For smooth initial data, we know from these results that the entropy weak solution is continuous except on the union of an at most countable set of Lipschitz continuous shock curves. The complement of the shock set is open and from each point (x, t) in this open set one can trace a straight line backward in time to $t = 0$, and along that line the solution is constant, i.e. it is given by the initial data.

More recently, regularity results have been obtained by Tadmor and Tassa [30], DeVore and Lucier [6] and Tang and Teng [31].

We make the following structural assumption on the entropy weak solution to (1.1).

Assumption 24 *We assume the entropy solution has the following properties*

1. *The entropy weak solution u is piecewise C^1. To be more precise, there exist a finite number of Lipschitz continuous curves $S_k = (s_k(t), t) \subset \mathbb{R} \times \mathbb{R}^+$, $k = 1, \ldots, K$, that partition $\mathbb{R} \times \mathbb{R}^+$ in a finite number of sets $D_l \subset \mathbb{R} \times \mathbb{R}^+$, $l = 1, \ldots, L$, and $u|_{D_l} \in C^1(D_l)$ and $u|_{D_l}$ can be continuously extended up to the boundary of D_l. Furthermore, across such a Lipschitz curve u is either only continuous, but not differentiable, (which corresponds to a rarefaction) or discontinuous (which corresponds to a shock).*
2. *For any Lipschitz curve $S = (s(t), t)$ the* Rankine-Hugoniot condition *is satisfied, that is for almost all $t > 0$*

$$\begin{aligned} &s'(t)\big[u(s(t)^+, t) - u(s(t)^-, t)\big] \\ &= f(u(s(t)^+, t)) - f(u(s(t)^-, t)), \end{aligned} \tag{7.1}$$

where

$$u(s(t)^{\pm}, t) := \lim_{\varepsilon \downarrow 0} u(s(t) \pm \varepsilon, t). \tag{7.2}$$

Remark 25 1. Assumption 24 is fulfilled, for example, if the following conditions are met [30, Thm. 2.1 and Thm. 4.1]:

- $u_0 \in L^\infty \cap BV(\mathbb{R})$ is piecewise C^1, that is u_0 is C^1 except in a finite number of points.
- The initial data have the following behavior at infinity

$$\lim_{|x| \to \infty} \frac{d}{dx} f'(u_0(x)) = 0$$

and the number of negative minima of $\frac{d}{dx} f'(u_0(x))$ is finite. The derivative has to be understood in a generalized sense, for the details we refer to [30].

2. Assumption 24 is also satisfied for Riemann initial data.

Now, we are able to state the explicit form of the consistency error term $R^n(u)$ in (4.3). It is straightforward, using the Rankine-Hugoniot condition (7.1) and the piecewise smoothness of the entropy weak solution, to prove

Proposition 26 *Let u be the entropy weak solution to* (1.1) *subject to an initial condition u_0 such that Assumption 24 holds. Then, the following* *integral formula holds for any* $n \in \mathbb{N}$, $i \in \mathbb{Z}$*:*

$$0 = \fint_{x_{i-1/2}}^{x_{i+1/2}} u(y, t^{n+1})\mathrm{d}y - \fint_{x_{i-1/2}}^{x_{i+1/2}} u(y, t^n)\mathrm{d}y + \frac{\Delta t}{\Delta x}\left[\fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2} - 0, \tau))\mathrm{d}\tau - \fint_{t^n}^{t^{n+1}} f(u(x_{i-1/2} + 0, \tau))\mathrm{d}\tau \right]. \tag{7.3}$$

Next, we will estimate the consistency error which appears in Theorem 10.

Proposition 27 *Assume that the entropy weak solution* u *to* (1.1) *satisfies Assumption 24. Then,* R^n*, which can be interpreted as a consistency error in the mean, satisfies for* $n \in \mathbb{N}$

$$\|R^n(u)\|_{lip'} \leq C \Delta x \Delta t |u_0|_{BV(\mathbb{R})}, \tag{7.4}$$

where the constant $C > 0$ *depends on* f *and* u_0 *only.*

Proof. We proceed in several steps. It shall turn out that in fact $R^n(u) \in w^{-1,1}$ and with Lemma 4 we can estimate

$$\|R^n(u)\|_{lip'} \leq \|R^n(u)\|_{w^{-1,1}}.$$

Step 1: By Proposition 26 we know that the integral formula (7.3) holds. Hence,

$$\begin{aligned} &R_i^n(u) \\ &= \frac{\Delta t}{\Delta x}\left[g\Big(\fint_{x_{i-1/2}}^{x_{i+1/2}} u(., t^n), \fint_{x_{i+1/2}}^{x_{i+3/2}} u(., t^n) \Big) - \fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}^-, \tau))\mathrm{d}\tau \right] \\ &\quad - \frac{\Delta t}{\Delta x}\left[g\Big(\fint_{x_{i-3/2}}^{x_{i-1/2}} u(., t^n), \fint_{x_{i-1/2}}^{x_{i+1/2}} u(., t^n) \Big) - \fint_{t^n}^{t^{n+1}} f(u(x_{i-1/2}^+, \tau))\mathrm{d}\tau \right]. \end{aligned}$$

We set

$$
\begin{aligned}
\mathcal{R}_i^n(u) &:= \sum_{j\le i} \Delta x R_j^n(u) \\
&= \Delta t \sum_{j\le i} \Bigg[g\Big(\fint_{x_{j-1/2}}^{x_{j+1/2}} u(.,t^n), \fint_{x_{j+1/2}}^{x_{j+3/2}} u(.,t^n) \Big) \\
&\qquad - g\Big(\fint_{x_{j-3/2}}^{x_{j-1/2}} u(.,t^n), \fint_{x_{j-1/2}}^{x_{j+1/2}} u(.,t^n) \Big) \Bigg] \\
&\quad - \Delta t \sum_{j\le i} \Bigg[\fint_{t^n}^{t^{n+1}} f(u(x_{j+1/2}^-,\tau))\mathrm{d}\tau - \fint_{t^n}^{t^{n+1}} f(u(x_{j-1/2}^+,\tau))\mathrm{d}\tau \Bigg] \\
&=: T_1 + T_2.
\end{aligned}
$$

Since the sum T_1 is bounded by $C\|u_0\|_{L^1}$, $C > 0$ a constant depending from Δt and f, we can reorder terms and get

$$
T_1 = \Delta t \, g\Big(\fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,t^n), \fint_{x_{i+1/2}}^{x_{i+3/2}} u(.,t^n) \Big).
$$

Similarly, we can reorder the terms in T_2 since the BV stability estimate (6.7) holds. Using the Rankine-Hugoniot condition (7.1) across the curves $S = (x_{j+1/2}, t)$, $j \le i$, we conclude that

$$
T_2 = -\Delta t \fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}^-,\tau))\mathrm{d}\tau.
$$

We summarize that

$$
\begin{aligned}
\mathcal{R}_i^n(u) = \Delta t \Bigg[& g\Big(\fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,t^n), \fint_{x_{i+1/2}}^{x_{i+3/2}} u(.,t^n) \Big) \\
& - \fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}^-,\tau))\mathrm{d}\tau \Bigg].
\end{aligned}
\tag{7.5}
$$

Step 2: The idea to estimate the right hand side in (7.5) is to use the piecewise smoothness of the entropy solution and to employ techniques resembling Taylor expansion.

Since $f \in C^1$ we have (recall that u_i^n is the cell average of $u(.,t^n)$ over $[x_{i-1/2}, x_{i+1/2}[$, see (1.8))

$$f(u(x_{i+1/2}^-,\tau)) = f(u_i^n) + \int_{u_i^n}^{u(x_{i+1/2}^-,\tau)} f'(s)\mathrm{d}s.$$

Since u satisfies the maximum principle we can estimate

$$\begin{aligned}
&\fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}^-,\tau))\mathrm{d}\tau \\
&\leq f(u_i^n) + \max_{[U_m,U_M]} |f'| \fint_{t^n}^{t^{n+1}} \left| u(x_{i+1/2}^-,\tau) - u_i^n \right| \mathrm{d}\tau.
\end{aligned}$$

We continue with the last term and insert some term. With the triangle inequality we get

$$\begin{aligned}
&\fint_{t^n}^{t^{n+1}} \left| u(x_{i+1/2}^-,\tau) - u_i^n \right| \mathrm{d}\tau \\
&\leq \fint_{t^n}^{t^{n+1}} \left| u(x_{i+1/2}^-,\tau) - \fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,\tau) \right| \mathrm{d}\tau \\
&\quad + \fint_{t^n}^{t^{n+1}} \left| \fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,\tau) - \fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,t^n) \right| \mathrm{d}\tau \\
&=: T_3 + T_4.
\end{aligned}$$

In order to estimate T_3 note, that $u(.,t)$ is piecewise smooth for all $t > 0$ and due to the regularity Assumption 24 there exists for all $t > 0$ and all $i \in \mathbb{Z}$ a positive integer $K_i(t)$ and a partition $x_{i-1/2} = x_{i,1} < x_{i,2} < \ldots < x_{i,K_i(t)} = x_{i+1/2}$ such that

$$u(.,t)|_{I_{i,k}} \in C^1 \quad I_{i,k} := [x_{i,k}, x_{i,k+1}[, \quad k = 1, \ldots, K_i(t) - 1.$$

Using this partition we get for any $t > 0$

$$u(x_{i+1/2}^-, t) - \fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,t) = \sum_{k=1}^{K_i(t)-1} \frac{|I_{i,k}|}{\Delta x} \fint_{I_{i,k}} [u(x_{i+1/2}^-, t) - u(y,t)]\mathrm{d}y$$

$$\leq \sum_{k=1}^{K_i(t)-1} \frac{|I_{i,k}|}{\Delta x} \int_{x_{i,k}}^{x_{i+1/2}^-} |\partial_x u(y,t)|\mathrm{d}y$$

$$\leq |u(.,t)|_{BV(x_{i-1/2},x_{i+1/2})}.$$

Hence, we have

$$T_3 \leq \fint_{t^n}^{t^{n+1}} |u(.,t)|_{BV(x_{i-1/2},x_{i+1/2})}.$$

Step 3: It remains to estimate

$$T_5 := g(u_i^n, u_{i+1}^n) - f(u_i^n).$$

Since the numerical flux is consistent and Lipschitz we get with the mean value theorem (with a density argument analogous to [10, Lemma 6.4])

$$T_5 \leq \max_{[U_m,U_M]} |f'| \Big| \fint_{x_{i-1/2}}^{x_{i+1/2}} u(.,t^n) - \fint_{x_{i+1/2}}^{x_{i+3/2}} u(.,t^n) \Big|$$

$$\leq \max_{[U_m,U_M]} |f'| |u(.,t^n)|_{BV(x_{i-1/2},x_{i+3/2})}.$$

Step 4: Let us put the estimates from Step 2 and 3 together.

$$\mathcal{R}_i^n(u) = \Delta t \left[g(u_i^n, u_{i+1}^n) - \fint_{t^n}^{t^{n+1}} f(u(x_{i+1/2}^-, \tau))\mathrm{d}\tau \right]$$

$$\leq \Delta t \Big[T_5 + \max_{[U_m,U_M]} |f'|(T_3 + T_4) \Big]$$

$$\leq \Delta t \max_{[U_m,U_M]} |f'| \Bigg\{ |u(.,t^n)|_{BV(x_{i-1/2},x_{i+3/2})}$$

$$+ \fint_{t^n}^{t^{n+1}} |u(.,\tau)|_{BV(x_{i-1/2},x_{i+1/2})} \mathrm{d}\tau$$

$$+ \fint_{t^n}^{t^{n+1}} \fint_{x_{i-1/2}}^{x_{i+1/2}} |u(y,\tau) - u(y,t^n)|\mathrm{d}y\mathrm{d}\tau \Bigg\}.$$

Consequently,

$$\begin{aligned}\|R^n(u)\|_{w^{-1,1}} &= \sum_{i\in\mathbb{Z}} \Delta x|\mathcal{R}_i^n(u)| \\ &\le \Delta x\Delta t \max_{[U_m,U_M]} |f'| \Bigg\{ 2|u(.,t^n)|_{BV(\mathbb{R})} + \int_{t^n}^{t^{n+1}} |u(.,\tau)|_{BV(\mathbb{R})} \mathrm{d}\tau \\ &\quad + \int_{t^n}^{t^{n+1}} \frac{1}{\Delta x}\|u(.,\tau)-u(.,t^n)\|_{L^1(\mathbb{R})}\mathrm{d}\tau \Bigg\}.\end{aligned}$$

From [11, Thm. 3.1, p. 69] or [18, Thm. 4.22] we get that there exists a constant $C>0$, depending on f and u_0 only, such that for all $t_1,t_2\ge 0$ the following estimate holds

$$\|u(.,t_1)-u(.,t_2)\|_{L^1(\mathbb{R})} \le C|u_0|_{BV(\mathbb{R})}|t_1-t_2|.$$

Using this estimate together with the BV stability of the entropy solution, see (6.7), we get the assertion. □

8 Proof of our main result

In this section we prove our main result Theorem 1.

Proof. Let u be the entropy weak solution and let $(u_i^n)_{i\in\mathbb{Z}}$ denote its mean values at time t^n, $n\in\mathbb{N}$, which are defined in (1.8). Let $(v_i^n)_{i\in\mathbb{Z}}$ denote the numerical approximation defined in (1.5) with the numerical flux of Engquist-Osher (1.3). For $n\in\mathbb{N}$ and $i\in\mathbb{Z}$ set $e_i^n := u_i^n - v_i^n$. Let $N\in\mathbb{N}$ be fixed. Using Lemma 9 we know by our interpolation result Theorem 7 that

$$\|\mathrm{e}^N\|_{l^1} \le K\left[\|\mathrm{e}^N\|_{lip'}|\mathrm{e}^N|_{bv}\right]^{1/2}.$$

From Theorem 10 and Proposition 23 we know that there exists a constant $C>0$, depending from t^N, f and u_0 only such that

$$\|\mathrm{e}^N\|_{lip'} \le C \sup_{n=0,\dots,N-1} \left\|\frac{1}{\Delta t}R^n(u)\right\|_{lip'}.$$

The consistency error $R^n(u)$ is estimated in Proposition 27. Hence, we get

$$\|\mathrm{e}^N\|_{lip'} \le C|u_0|_{BV(\mathbb{R})}\Delta x.$$

The bv bound for e^N follows from Proposition 19 and Proposition 22, and we get with the triangle inequality

$$|\mathrm{e}^N|_{bv} \le |u^N|_{bv} + |v^N|_{bv} \le 2|u_0|_{BV(\mathbb{R})}.$$

This concludes the proof. □

9 Conclusion and outlook

In this paper we have introduced a technique to prove *a priori* error estimates using the concept of stability and consistency. The stability is related to the entropy condition of Oleinik. The consistency error comes up in the sense of cell averages and is measured in a negative norm. This technique is the discrete analogue to the duality technique of Tadmor [27]. We have proven error estimates between the entropy weak solution and numerical approximations given by the (first order) schemes of Engquist-Osher, Lax-Friedrichs and Godunov.

Now, it is natural to ask if the use of cell averages limits this technique to first order schemes. A formal argumentation shows that the consistency error for second order schemes of MUSCL type, see [11, Chapter 4], is of second order which would lead to a convergence rate of one in the l^1 norm. However, we have not been able to prove the required stability estimates for those second order schemes.

Finally, one may deduce discrete counterparts of several results that have been proven using the duality technique of Tadmor, for example one may try to prove a discrete analogue to the one-sided interpolation result in [29, Lemma 2.1], see also [19], which has been the key result in proving pointwise error estimates. This issue will be studied in future.

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