



# Thresholds in three-dimensional restricted Euler–Poisson equations



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## HIGHLIGHTS

- We identify sub-thresholds for global existence for multi-dimensional restricted Euler–Poisson (REP) equations.
- We also identify upper-thresholds for finite time blow up of solutions to these REP equations.
- For repulsive REP equations, the identified thresholds depend on the spectral gap of the initial velocity gradient.

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## ABSTRACT

This work provides a description of the critical threshold phenomenon in multi-dimensional restricted Euler–Poisson (REP) equations, introduced in [H. Liu, E. Tadmor, Spectral dynamics of the velocity gradient field in restricted fluid flows, *Comm. Math. Phys.* 228 (2002) 435–466]. For three-dimensional REP equations, we identified both upper thresholds for the finite-time blow up of solutions and subthresholds for the global existence of solutions, with the thresholds depending on the relative size of the eigenvalues of the initial velocity gradient matrix and the initial density. For the attractive forcing case, these one-sided threshold conditions of the initial configurations are optimal, and the corresponding results also hold for arbitrary  $n$  dimensions ( $n \geq 3$ ).

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## 1. Introduction

We are concerned with the critical threshold phenomenon in multi-dimensional Euler–Poisson equations. In this paper, we consider a localized version of the following  $n$ -dimensional ( $n$ D) Euler–Poisson (EP) equations,

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= k \nabla \Delta^{-1}(\rho - c_b), \end{aligned} \quad (1.1)$$

which govern the unknown local density  $\rho = \rho(t, x)$  and velocity field  $\mathbf{u} = \mathbf{u}(t, x)$ , subject to initial conditions  $\rho(0, x) = \rho_0(x)$  and  $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ . They involve two constants: constant  $k$ , which signifies the property of the underlying repulsive  $k > 0$  or attractive  $k < 0$  forcing, governed by the Poisson potential  $\Delta^{-1}(\rho - c_b)$ , and constant  $c_b > 0$ , which denotes the background state.

This hyperbolic system (1.1) with non-local forcing describes the dynamic behavior of many important physical flows, including charge transport [1], plasma with collision [2], cosmological waves [3], and expansion of cold ions [4], as well as the collapse of stars due to self-gravitation ( $k < 0$ ) [5–7].

There is a considerable amount of literature available on the solution behavior of Euler–Poisson equations. Let us mention the

study of steady-state solutions [5,8–12] and the global existence of weak solutions [13–16]. Global existence due to damping relaxation and with non-zero background can be found in [17–19].

For the question of global behavior of strong solutions, however, the choice of the initial data and/or damping forces is decisive. With a repulsive force  $k > 0$ , we refer to [20,21] for the global existence of classical solutions with initial data close to the stable steady states, and [22] for the non-existence of global solutions; with attractive force  $k < 0$ , we refer to [23,24] for non-existence results. These results rely on some energy methods using small or large enough initial energy.

The non-local forcing in (1.1) dictated by the Poisson potential is only weakly dissipative. As a result, the steady state may be only conditionally stable. Indeed, for a class of one-dimensional Euler–Poisson equations and multi-dimensional equations with spherical symmetry, it was shown in [25] that the persistence of the global features of the solutions hinges on a delicate balance between the nonlinear convection and the non-local forcing. In other words, the persistence of the global features of solutions does not fall into any particular category (global smooth solution, finite-time breakdown, etc.), but, instead, these features depend on crossing a critical threshold associated with the initial configuration of underlying problems – the so-called critical threshold (CT) phenomenon. The study of such a remarkable CT phenomenon opens a new avenue to address the fundamental question of persistence of the  $C^1$  solution regularity for the EP system and related models.

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The concept of critical threshold and the associated methodology originated and was developed in a series of papers by Engelberg, Liu, and Tadmor [25], Liu and Tadmor [26–29], and others. It first appears in [25] with respect to pointwise criteria for  $C^1$  solution regularity of a 1D EP system. The critical threshold obtained therein describes the conditional stability of 1D EP systems, where the answer to the question of global versus local existence depends on whether the initial data crosses a critical threshold. Moving to the multi-dimensional setup, one has to identify the proper quantities to describe the critical threshold phenomenon. Liu and Tadmor, in [26], introduce the method of spectral dynamics, which relies on the dynamical system governing eigenvalues of the velocity gradient matrix,  $M := \nabla \mathbf{u}$ , along particle paths. To illustrate this, we differentiate the second equation of (1.1), obtaining formally

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = kR[\rho - c_b],$$

where  $R[\cdot]$  is the Riesz matrix operator, defined as

$$R[f] := \nabla \otimes \nabla \Delta^{-1}[f].$$

Now, the Euler–Poisson equations are recast into the coupled system

$$M' + M^2 = kR[\rho - c_b], \quad (1.2a)$$

$$\rho' + \rho \operatorname{tr} M = 0, \quad (1.2b)$$

with  $'$  standing for the usual convective derivative,  $\partial_t + \mathbf{u} \cdot \nabla$ . The global nature of the Riesz matrix,  $R[\rho - c_b]$ , makes the issue of regularity for Euler–Poisson equations such an intricate question to solve.

To gain better understanding of the dynamics of the velocity gradient  $M$  governed by (1.2a)–(1.2b), in [26], Liu and Tadmor introduce the restricted Euler–Poisson (REP) system (1.3), which is obtained from (1.2a) by restricting attention to the local isotropic trace  $\frac{k}{n}(\rho - c_b)I_{n \times n}$  of the global coupling term  $kR[\rho - c_b]$ , namely,

$$M' + M^2 = \frac{k}{n}(\rho - c_b)I_{n \times n}, \quad (1.3a)$$

$$\rho' + \rho \operatorname{tr} M = 0, \quad (1.3b)$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

This localization was motivated by the so-called restricted Euler equations proposed in [30] as a localized alternative to the incompressible Euler equation.

For global existence of solutions to 2D REP system, i.e., (1.3) with  $n = 2$ , a complete description of the critical threshold criterion was obtained in [27]. Beyond the pointwise threshold results obtained in [25–28] for one-dimensional or restricted models, effort has been made to extend the critical threshold argument to more general models. For the 1D EP system with pressure, Tadmor and Wei [31] obtain thresholds through tracking  $(u_x, \rho)$  along two characteristic fields. Chae and Tadmor [32] obtain the blow up result for multi-dimensional full Euler–Poisson systems (1.3) with attractive forcing  $k < 0$ . Cheng and Tadmor [33] obtained (2.9), which improved the result of [32]. For proofs of the results in [32,33], the vanishing initial vorticity condition which amounts to the symmetry of  $M$  is essential to ensure the key inequality (2.8).

In this work, we further investigate the 3D REP system (1.3), as well as the  $n$ D REP system. Our results reveal threshold conditions on the initial data that lead to the finite-time blow up or global boundedness of  $M$ . They quantify the balance between density  $\rho$  and eigenvalues  $\lambda(M) = \{\lambda_i\}_{i=1}^n$ . Without loss of generality, we shall label the initial eigenvalues in terms of the real part of each eigenvalue such that

$$\operatorname{Re}(\lambda_{10}) \leq \operatorname{Re}(\lambda_{20}) \leq \dots \leq \operatorname{Re}(\lambda_{n0}).$$

The main results are summarized as follows. For the  $n$ D REP system (1.3) with non-zero background  $c_b > 0$  and initial density  $\rho_0 > 0$ , we have the following.

- (Attractive case  $k < 0$ ) If  $\lambda_{10}$  is real, and there exists  $\Lambda_n(k, \rho_0)$  such that

$$\lambda_{10} > \Lambda_n(k, \rho_0), \quad n \geq 3,$$

then the solution remains bounded for all time. If all  $\{\lambda_{i0}\}_{i=1}^n$  are real, and

$$\lambda_{n0} < \Lambda_n(k, \rho_0),$$

then the solution will blow up in finite time.

- (Repulsive case  $k > 0$ ) Suppose that all eigenvalues are initially real. The solution remains bounded for all time if all eigenvalues are initially identical. If the spectral gap

$$\lambda_{20} - \lambda_{10} > \Gamma_n(k, \rho_0),$$

where  $\Gamma_n$  denotes the gap thresholds, then the solution of the  $n$ D REP system will blow up in finite time for  $n = 3, 4$ .

These results are more precisely stated in Section 2, together with relevant remarks: Theorems 2.1–2.2 ( $n = 3$ ) and Theorems 2.7–2.8 ( $n > 3$ ) for  $k < 0$ ; Theorems 2.3–2.4 ( $n = 3$ ) and Theorems 2.9–2.10 ( $n > 3$ ) for  $k > 0$ .

In Section 3, we prove both global existence and finite-time blow up of solutions to the REP system with attractive forcing. In Section 4, we study the thresholds for the REP system with repulsive forcing. Extension to the  $n$ -dimensional case is carried out in Section 5.

## 2. Statement of main results

We first present results which quantify the balance between density  $\rho$  and eigenvalues  $\lambda(M) = \{\lambda_i\}_{i=1}^3$ . These results, as a generalization of those in [27], also hold in arbitrary dimensions ( $n > 3$ ) when  $k < 0$ , for which further discussion is given after the statement of the 3D theorems.

**Theorem 2.1** (Global Existence for 3D REP with  $k < 0$ ). Consider the 3D attractive REP system (1.3) with  $k < 0$  and with non-zero background  $c_b > 0$ . If  $\lambda_{10} \in \mathbb{R}$ , then the solution of the 3D REP system remains bounded for all time provided that  $\rho_0 > 0$  and

$$\lambda_{10} > \operatorname{sgn}(\rho_0 - c_b) \sqrt{k \left( c_b^{\frac{1}{3}} \rho_0^{\frac{2}{3}} - \frac{2}{3} \rho_0 - \frac{1}{3} c_b \right)}. \quad (2.1)$$

**Theorem 2.2** (Finite-Time Blow Up for 3D REP with  $k < 0$ ). Consider the 3D attractive REP system (1.3) with  $k < 0$  and with non-zero background  $c_b > 0$ . Assume that  $\lambda(M_0) \in \mathbb{R}$ . The solution of the 3D REP system will blow up in finite time if  $\rho_0 > 0$  and

$$\lambda_{30} < \operatorname{sgn}(\rho_0 - c_b) \sqrt{k \left( c_b^{\frac{1}{3}} \rho_0^{\frac{2}{3}} - \frac{2}{3} \rho_0 - \frac{1}{3} c_b \right)}. \quad (2.2)$$

**Theorem 2.3** (Global Existence for 3D REP with  $k > 0$ ). Consider the 3D repulsive REP system (1.3) with  $k > 0$  and with non-zero background  $c_b > 0$ . The solution of the 3D REP system remains bounded for all time if  $\lambda_{10} = \lambda_{20} = \lambda_{30}$ .

**Theorem 2.4** (Finite-Time Blow Up for 3D REP with  $k > 0$ ). Consider the 3D repulsive REP system (1.3) with  $k > 0$  and with non-zero background  $c_b > 0$ . Assume that  $\lambda(M_0) \in \mathbb{R}$ . The solution of the 3D REP system will blow up in finite time provided that  $\rho_0 > 0$  and one of the following three conditions is fulfilled.

$$(i) \quad \lambda_{20} - \lambda_{10} > \left( \frac{k^3 \rho_0^4}{4c_b} \right)^{\frac{1}{6}}.$$

$$(ii) \quad \lambda_{20} - \lambda_{10} = \left( \frac{k^3 \rho_0^4}{4c_b} \right)^{\frac{1}{6}} \text{ and } \lambda_{20} + \lambda_{10} < 0.$$

(iii)  $0 < \lambda_{20} - \lambda_{10} < \left(\frac{k^3 \rho_0^4}{4c_b}\right)^{\frac{1}{6}}$  and either  $\alpha > 1$  or  $\alpha \leq 1$  with

$$\lambda_{20} + \lambda_{10} < \operatorname{sgn}(1 - \beta) \times \sqrt{(\beta + 1)(\lambda_{20} - \lambda_{10})^2 - \frac{4k}{3} \left(2\rho_0 + c_b - \frac{3c_b}{\beta}\right)},$$

where  $\alpha$  and  $\beta$  with  $\alpha < \beta$  are given by

$$\frac{3}{4kc_b}(\lambda_{20} - \lambda_{10})^2 = -\frac{1}{\xi^2} + \frac{\rho_0}{c_b} \frac{1}{\sqrt{\xi}}, \quad \beta = \max\{\xi\}$$

and

$$\frac{3}{4kc_b}(\lambda_{20} - \lambda_{10})^2 = -\frac{1}{\alpha\beta} + \frac{2\rho_0}{c_b} \frac{1}{\sqrt{\alpha} + \sqrt{\beta}}.$$

**Remark 2.5.** Some remarks are in order at this point.

(i) In Theorems 2.1 and 2.2, the threshold bound denoted by  $\Lambda_3(k, \rho_0)$  is well defined for  $k < 0$  since the quantity under the square root is non-negative; i.e.,

$$k \left( c_b^{\frac{1}{3}} \rho_0^{\frac{2}{3}} - \frac{2}{3} \rho_0 - \frac{1}{3} c_b \right) = -\frac{k}{3} \left( 2\rho_0^{\frac{1}{3}} + c_b^{\frac{1}{3}} \right) \left( \rho_0^{\frac{1}{3}} - c_b^{\frac{1}{3}} \right)^2 \geq 0.$$

(ii) From Theorems 2.1 and 2.2, we see that, for each fixed  $\rho_0$ , the lower bound in (2.1) for global existence and the upper bound in (2.2) for finite-time blow up are identical. Thus, the obtained thresholds are optimal. This is in the sense that, if  $\lambda_{10} = \lambda_{30}$ , then Theorems 2.1 and 2.2 can be combined into one theorem with an “if and only if” statement; otherwise, if the bound  $\Lambda_3(k, \rho_0)$  lies between  $\lambda_{10}$  and  $\lambda_{30}$ , i.e.,

$$\lambda_{10} < \Lambda_3(k, \rho_0) \leq \lambda_{30},$$

it is unclear whether the  $C^1$  solution regularity persists for all time.

(iii) The set of initial configurations which give rise to global bounded solutions is very rich in phase space  $(\rho, \lambda_1, \lambda_2, \lambda_3)$ , which can be visualized through a qualitative diagram in the subspace  $(\lambda_1, \lambda_3 - \lambda_1, \rho)$  (Fig. 1). From the figure, one may also see that a critical threshold surface should lie somewhere between the two shaded surfaces.

(iv) The condition for global regularity in Theorem 2.3 is obtained using only a global invariant, which is a set of measure zero in the space of eigenvalues with  $\rho_0 > 0$ . This global existence result, though starting from a thin initial set, when combined with Theorem 2.4 does suggest the existence of a critical threshold for the case  $k > 0$ . It would be interesting to identify a larger set of initial data than that in Theorem 2.3 for the global existence.

(v) For the  $k > 0$  case, the spectral gap  $\lambda_2 - \lambda_1$  as described in Theorem 2.4 plays an important role. This fact is consistent with the known result in the 2D case (Theorem 1.2 in [27]).

(vi) The results in Theorems 2.1–2.4 may suggest the critical threshold phenomenon for the full Euler–Poisson equations.

For the proof of each theorem, we need the following lemma.

**Lemma 2.6** (Spectral Dynamics [26, Lemma 3.1]). Consider the nonlinear transport equation  $\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} = \vec{F}$ . Let  $\lambda := \lambda(\nabla_x \mathbf{u})(t, x)$  denote an eigenvalue of  $\nabla_x \mathbf{u}$  with corresponding left and right normalized eigenpair,  $\langle l, r \rangle = 1$ . Then  $\lambda$  is governed by the forced Riccati equation

$$\lambda' + \lambda^2 := \partial_t \lambda + \mathbf{u} \cdot \nabla_x \lambda + \lambda^2 = \langle l, \nabla_x \vec{F} r \rangle.$$

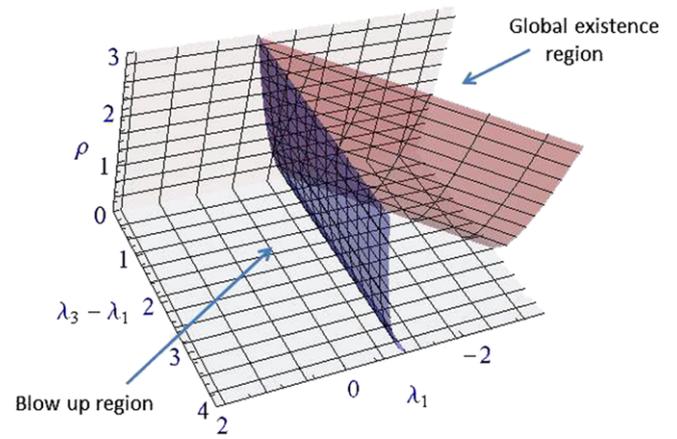


Fig. 1. The subthreshold and the superthreshold are shaded surfaces when  $k < 0$ .

This lemma when applied to (1.3) gives

$$\lambda_i' + \lambda_i^2 = \frac{k}{n}(\rho - c_b), \quad i \in \{1, \dots, n\}, \tag{2.3a}$$

$$\rho' + \rho(\lambda_1 + \dots + \lambda_n) = 0. \tag{2.3b}$$

From (2.3a) it follows that

$$(\lambda_i - \lambda_j)' = -(\lambda_i - \lambda_j)(\lambda_i + \lambda_j), \quad i, j \in \{1, \dots, n\}.$$

This shows that, if  $\lambda(M_0) \in \mathbb{R}$ , then  $\lambda(M) \in \mathbb{R}$ . Moreover, if  $\lambda(M_0) \in \mathbb{R}$ , then the order of  $\{\lambda_i\}_{i=1}^n$  is preserved in time; i.e.,

$$\text{If } \lambda_1(0) \leq \dots \leq \lambda_n(0), \quad \text{then } \lambda_1(t) \leq \dots \leq \lambda_n(t) \quad \text{for } t \geq 0. \tag{2.4}$$

This monotonicity-preserving property remains valid in a strict sense because  $\lambda_i - \lambda_j \neq 0$  as long as  $\lambda_{i0} - \lambda_{j0} \neq 0$ . Note that (2.3) is an  $(n + 1)$ -by- $(n + 1)$  ordinary differential equation (ODE) system; when  $n \geq 3$ , it is no longer possible to employ the precise phase plane analysis as carried out in [25] for  $(u_x, \rho)$  and in [27] for  $(\beta, \rho)$ , with  $\beta$  being a combined quantity of two eigenvalues through a global invariant. The key argument in our proofs here is to use the order-preserving property of generic eigenvalues.

In the proof of Theorems 2.1–2.2 with  $k < 0$ , the order-preserving property of  $\lambda(M)$  together with non-negativity of the density enables us to obtain the following:

$$-n\rho\lambda_n \leq \rho' \leq -n\rho\lambda_1.$$

This two-sided differential inequality leads to the desired thresholds for both global existence and finite-time blow up. In the presence of complex eigenvalues, we also need the following order-preserving property to prove the global boundedness of the imaginary part of the eigenvalues. If  $\lambda_1(0)$  is real and

$$\lambda_1(0) \leq \operatorname{Re}(\lambda_j(0)), \quad \text{then } \lambda_1(t) \leq \operatorname{Re}(\lambda_j(t))$$

$$\text{for } t \geq 0, \quad j \in \{2, \dots, n\}.$$

In the proof of Theorem 2.4 with  $k > 0$ , we use the order-preserving property to deduce some 2-by-2 ODE systems with controllable time-dependent coefficients, which when combined with some comparison argument lead to the desired blow up results.

From system (2.3), it follows that

$$(\lambda_i - \lambda_j) = (\lambda_{i0} - \lambda_{j0}) e^{-\int_0^t (\lambda_i + \lambda_j) ds}, \quad i \in \{1, 2, 3\},$$

and

$$\rho = \rho_0 e^{-\int_0^t (\lambda_1 + \lambda_2 + \lambda_3) ds}.$$

These combined lead to the spectral invariant as obtained in [26]; i.e.,

$$S(t) := \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\rho^2} = S(0). \tag{2.5}$$

Our results are obtained from a comparison of eigenvalues of the original system to solutions of dominated systems, which implicitly follow the order indicated by this spectral invariant; therefore our results are consistent with (2.5). This comment also applies to the higher-dimensional case.

We point out that, for the  $k < 0$  case, only the density and one eigenvalue need to be controlled for proving the global existence or the finite-time blow up. Hence, for the  $k < 0$  case, the key arguments summarized above work equally well for arbitrary  $n$ -dimensional REP equations ( $n > 3$ ). For the  $k > 0$  case, our argument for solution blow up extends only to 4-dimensional REP equations. For completeness, we also state  $n$ -dimensional results for the  $k < 0$  case, the 4-dimensional blow up result, and the  $n$ -dimensional global existence result for the  $k > 0$  case below, and we outline some main arguments of their proofs in Section 4.

**Theorem 2.7** (Extension of Theorem 2.1). Consider the  $nD$  attractive REP system (1.3) with  $k < 0$  and with non-zero background  $c_b > 0$ . Assume that  $\lambda_{10} \in \mathbb{R}$  and  $\lambda_{10} \leq \text{Re}(\lambda_{i0})$ ,  $i = 2, 3, \dots, n$ . The solution of the  $nD$  REP system remains bounded for all time if  $\rho_0 > 0$  and

$$\lambda_{10} > \text{sgn}(\rho_0 - c_b) \sqrt{k \left( \frac{c_b^{\frac{n-2}{n}}}{n-2} \rho_0^{\frac{2}{n}} - \frac{2}{n(n-2)} \rho_0 - \frac{c_b}{n} \right)}.$$

**Theorem 2.8** (Extension of Theorem 2.2). Consider the  $nD$  attractive REP system (1.3) with  $k < 0$  and with non-zero background  $c_b > 0$ . Assume that  $\lambda(M_0) \in \mathbb{R}$ . The solution of the  $nD$  REP system will blow up in finite time if  $\rho_0 > 0$  and

$$\lambda_{n0} < \text{sgn}(\rho_0 - c_b) \sqrt{k \left( \frac{c_b^{\frac{n-2}{n}}}{n-2} \rho_0^{\frac{2}{n}} - \frac{2}{n(n-2)} \rho_0 - \frac{c_b}{n} \right)}. \quad (2.6)$$

**Theorem 2.9** (Extension of Theorem 2.3). Consider the  $nD$  repulsive REP system (1.3) with  $k > 0$  and with non-zero background  $c_b > 0$ . The solution of the  $nD$  REP system remains bounded for all time if all eigenvalues are initially real and identical.

**Theorem 2.10** (Finite-Time Blow Up for 4D REP with  $k > 0$ ). Consider the 4D repulsive REP system (1.3) with  $k > 0$  and with non-zero background  $c_b > 0$ . Assume that  $\lambda(M_0) \in \mathbb{R}$ . The solution of the 4D REP system will blow up in finite time provided that  $\rho_0 > 0$  and

$$\lambda_{20} - \lambda_{10} \geq \sqrt{k\rho_0}.$$

**Remark 2.11.** Two remarks are in order at this point.

- (i) The bound  $A_n(k, \rho_0)$  in Theorems 2.7 and 2.8 is well defined since the quantity under the square root is non-negative:

$$\begin{cases} -\frac{k}{n} \frac{1}{\left(\frac{n}{2}-1\right)} \left(c_b^{\frac{2}{n}} - \rho_0^{\frac{2}{n}}\right)^2 \\ \times \left(\sum_{j=1}^{\frac{n}{2}-1} j c_b^{\frac{j-1}{(n/2)}} \rho_0^{\frac{(n/2)-1-j}{(n/2)}}\right) \geq 0, & n\text{-even,} \\ -\frac{k}{n} \frac{1}{(n-2)} \left(c_b^{\frac{1}{n}} - \rho_0^{\frac{1}{n}}\right)^2 \\ \times \left((n-2)c_b^{\frac{n-2}{n}} + \sum_{j=1}^{n-2} 2j c_b^{\frac{j-1}{n}} \rho_0^{\frac{n-1-j}{n}}\right) \geq 0, & n\text{-odd.} \end{cases}$$

- (ii) Under the assumptions of Theorem 2.8, we may use the trace of  $M$  to derive a different threshold for the finite-time blow up. In fact, taking the trace of (1.3a), we obtain

$$(\text{tr}(M))' + \text{tr}(M^2) = k(\rho - c_b),$$

which holds for both full Euler–Poisson equations and restricted Euler–Poisson equations. When  $\lambda(M) \in \mathbb{R}$ , we have

$$\text{tr}(M^2) \geq \frac{1}{n} (\text{tr}(M))^2. \quad (2.7)$$

Hence the trace  $d = \text{tr}(M) = \sum_{i=1}^n \lambda_i$  satisfies

$$d' \leq -\frac{d^2}{n} + k(\rho - c_b). \quad (2.8)$$

This, when combined with  $\rho' = -\rho d$ , leads to the following blow up condition.

The solution of the  $nD$  REP system will blow up in finite time if  $\rho_0 > 0$  and

$$\frac{d_0}{n} < \text{sgn}(\rho_0 - c_b) \sqrt{k \left( \frac{c_b^{\frac{n-2}{n}}}{n-2} \rho_0^{\frac{2}{n}} - \frac{2}{n(n-2)} \rho_0 - \frac{c_b}{n} \right)}. \quad (2.9)$$

This threshold condition is slightly sharper than (2.6).

We note that the same threshold condition (2.9) for finite-time blow up is obtained in [33] for the full EP system by assuming that  $\nabla \times \mathbf{u}_0 = \mathbf{0}$ , with which  $M_0$  is symmetric, and so is  $M(t)$  for  $t > 0$ . This ensures that  $\lambda(M)$  is real for all time. In contrast, for the REP system  $\lambda(M)$  remains real as long as it is real at  $t = 0$ .

### 3. Attractive case, $k < 0$

We start this section with a proposition which compares the following two ODE systems:

$$\begin{cases} \rho' = \alpha\rho\lambda + \rho f(t), \\ \lambda' = \beta\lambda^2 + k\rho + \gamma \end{cases} \quad (3.1)$$

and

$$\begin{cases} a' = \alpha ab, \\ b' = \beta b^2 + ka + \gamma. \end{cases} \quad (3.2)$$

Here,  $\alpha, \beta, \gamma$ , and  $k$  are fixed constants, and  $f(t)$  is a continuous function.

**Proposition 3.1.** Let  $\alpha, k < 0$ .

If  $f(t) \geq 0, \forall t \geq 0$ , then

$$\begin{cases} a(0) < \rho(0) \\ \lambda(0) < b(0) \end{cases} \text{ implies that } \begin{cases} a(t) < \rho(t), \\ \lambda(t) < b(t). \end{cases}$$

If  $f(t) \leq 0, \forall t \geq 0$ , then

$$\begin{cases} \rho(0) < a(0) \\ b(0) < \lambda(0) \end{cases} \text{ implies that } \begin{cases} \rho(t) < a(t), \\ b(t) < \lambda(t). \end{cases}$$

**Proof.** This proposition can be proved by contradiction. Let  $f(t) \geq 0$ , and suppose that  $t_1$  is the earliest time when the above proposition is violated. Then

$$\begin{aligned} a(t_1) &= a(0)e^{\int_0^{t_1} \alpha b ds} \\ &< \rho(0)e^{\int_0^{t_1} \alpha \lambda ds} \\ &\leq \rho(0)e^{\int_0^{t_1} \alpha \lambda ds} e^{\int_0^{t_1} f(s) ds} \\ &= \rho(t_1). \end{aligned} \quad (3.3)$$

Therefore, we are left with only one possibility:  $\lambda(t_1) = y(t_1)$ . Consider

$$(b - \lambda)' = \beta(b^2 - \lambda^2) + k(a - \rho). \tag{3.4}$$

Since  $b(t) - \lambda(t) > 0$  for  $t < t_1$  and  $b(t_1) - \lambda(t_1) = 0$ , at  $t = t_1$ , we have

$$(b(t_1) - \lambda(t_1))' \leq 0.$$

But the right-hand side of (3.4) when it is evaluated at  $t = t_1$ ,

$$k(a(t_1) - \rho(t_1)) > 0,$$

which leads to the contradiction, as needed. The proof of the  $f(t) \leq 0$  case is similar.  $\square$

### 3.1. Proof of Theorem 2.1

As we remarked above, the spectral dynamics lemma tells us that the velocity gradient equation yields

$$\lambda'_1 + \lambda_1^2 = \frac{k(\rho - c_b)}{3}, \tag{3.5a}$$

$$\lambda'_2 + \lambda_2^2 = \frac{k(\rho - c_b)}{3}, \tag{3.5b}$$

$$\lambda'_3 + \lambda_3^2 = \frac{k(\rho - c_b)}{3}, \tag{3.5c}$$

$$\rho' + \rho(\lambda_1 + \lambda_2 + \lambda_3) = 0. \tag{3.5d}$$

We first show the order-preserving property of eigenvalues. As we showed above, if  $\lambda(M_0) \in \mathbb{R}$ , then  $\lambda(M) \in \mathbb{R}$  and

$$\lambda_{10} \leq \lambda_{20} \leq \lambda_{30} \text{ implies that } \lambda_1(t) \leq \lambda_2(t) \leq \lambda_3(t) \text{ for } t \geq 0.$$

Note that the gradient velocity matrix  $M(t)$  is a real matrix; therefore, its eigenvalues are generically in complex conjugate pairs. In case  $\lambda(M_0) \in \mathbb{C}$ , the above property also holds in the following sense.

**Lemma 3.2.** Assume that  $\lambda_{10} \in \mathbb{R}$ . Then, for  $j \in \{2, 3\}$ ,

$$\text{Re}(\lambda_{j0}) - \lambda_{10} \geq 0 \text{ implies that } \text{Re}(\lambda_j(t)) - \lambda_1(t) \geq 0,$$

as long as they remain finite.

**Proof.** Let  $\lambda_j = \alpha + \beta i$ . Then the real part of (3.5b) (or (3.5c)) leads to

$$\alpha' = -(\alpha^2 - \beta^2) + \frac{k(\rho - c_b)}{3}. \tag{3.6}$$

By subtracting (3.5a) from the above equation, we get

$$(\alpha - \lambda_1)' = -(\alpha^2 - \lambda_1^2) + \beta^2 \geq -(\alpha - \lambda_1)(\alpha + \lambda_1).$$

Thus,  $(\alpha_0 - \lambda_{10}) \geq 0$  implies that  $(\alpha - \lambda_1)(t) \geq 0$ .  $\square$

In order to show the global existence, we rewrite (3.5a) and (3.5d) as

$$\begin{cases} \rho' = -\rho(\lambda_1 + \lambda_2 + \lambda_3) \\ = -\rho(3\lambda_1) - \rho(\lambda_2 + \lambda_3 - 2\lambda_1), \\ \lambda'_1 = -\lambda_1^2 + \frac{k(\rho - c_b)}{3}. \end{cases} \tag{3.7}$$

Comparing this with the following ODE,

$$\begin{cases} a' = -3ab, \\ b' = -b^2 + \frac{k(a - c_b)}{3}, \end{cases} \tag{3.8}$$

we find the following monotonicity relation between (3.7) and (3.8).

**Lemma 3.3.** Assume that  $k < 0$ ,  $\lambda_{10} \in \mathbb{R}$ , and  $\lambda_{10} \leq \text{Re}(\lambda_{20}) \leq \text{Re}(\lambda_{30})$ . Then

$$\begin{cases} \rho(0) < a(0) \\ b(0) < \lambda_1(0) \end{cases} \text{ implies that } \begin{cases} \rho(t) < a(t) \\ b(t) < \lambda_1(t) \end{cases} \text{ for } t \geq 0,$$

as long as they remain finite.

**Proof.** The order-preserving property in Lemma 3.2 gives  $-\rho(\lambda_2 + \lambda_3 - 2\lambda_1) \leq 0$ . Hence, by Proposition 3.1, the above lemma follows.  $\square$

Note that the modified ODE system (3.8) admits three critical points:

$$(0, b_{\pm}) := \left(0, \pm \sqrt{\frac{-kc_b}{3}}\right) \text{ and } (c_b, 0).$$

One can verify that  $(0, b_+)$  is a nodal sink,  $(0, b_-)$  is a nodal source, and  $(c_b, 0)$  is a saddle point. We now use these facts to construct the threshold via phase plane analysis. Following the same  $q$ -transformation as that employed in [26], we set  $q = b^2$  to obtain

$$\frac{dq}{da} = 2b \frac{b'}{a'} = \frac{2q}{3a} - \frac{2k(a - c_b)}{9a},$$

which yields

$$\frac{d}{da} \left( a^{-\frac{2}{3}} q \right) = -\frac{2}{9} k a^{-\frac{2}{3}} (1 - c_b a^{-1}). \tag{3.9}$$

Upon integration, a global invariant of system (3.8) is given by

$$\frac{b^2 + \frac{2}{3}ka + \frac{1}{3}kc_b}{a^{\frac{2}{3}}} = \text{const}. \tag{3.10}$$

Therefore, the separatrix at  $(c_b, 0)$  is given by zero-level set

$$\frac{b^2 + \frac{2}{3}ka + \frac{1}{3}kc_b}{a^{\frac{2}{3}}} - kc_b^{\frac{1}{3}} = 0.$$

This gives the following lemma.

**Lemma 3.4.** Consider system (3.8), subject to initial data  $(a_0, b_0)$ . If

$$(a_0, b_0) \in \Omega_1, \text{ then } \lim_{t \rightarrow \infty} (a(t), b(t)) = (0, b_+) = \left(0, \sqrt{\frac{-kc_b}{3}}\right).$$

Here,

$$\Omega_1 := \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0, y > \text{sgn}(x - c_b) \times \sqrt{k \left( c^{\frac{1}{3}} b x^{\frac{2}{3}} - \frac{2}{3}x - \frac{1}{3}c_b \right)} \right\}.$$

**Proof.** Note that  $\Omega_1$  is an invariant region for modified system (3.8) and that  $(0, b_+)$  is the nodal sink. From these facts, the lemma follows.  $\square$

Since  $\Omega_1$  is an open set and an invariant region of modified system (3.8), if  $(\rho_0, \lambda_{10}) \in \Omega_1$  and  $\lambda_{10} \leq \text{Re}(\lambda_{20}) \leq \text{Re}(\lambda_{30})$ , then Lemmas 3.3 and 3.4 gives the lower bound of  $\lambda_1$ ; i.e.,

$$\lambda^* \leq \lambda_1(t) \leq \text{Re}(\lambda_2(t)) \leq \text{Re}(\lambda_3(t))$$

where  $\lambda^* = \min \left\{ \lambda_1(0), \sqrt{\frac{-kc_b}{3}} \right\}$ .  $\tag{3.11}$

If  $\lambda(M_0) \in \mathbb{R}$ , then it suffices to show that  $\lambda_3$  is bounded from above. From (3.5c),

$$\lambda'_3 = -\lambda_3^2 + \frac{k(\rho - c_b)}{3} \leq -\lambda_3^2 - \frac{kc_b}{3}, \tag{3.12}$$

and we have

$$\lambda'_3 < - \left( \lambda_3 + \sqrt{-\frac{kc_b}{3}} \right) \left( \lambda_3 - \sqrt{-\frac{kc_b}{3}} \right).$$

Thus,  $\lambda_3(t) \leq \max \left\{ \lambda_3(0), \sqrt{-\frac{kc_b}{3}} \right\}$ . Together with (3.11), this proves Theorem 2.1 when  $\text{Im}(\lambda_{j0}) = 0, j \in \{2, 3\}$ .

If  $\text{Im}(\lambda_{j0}) \neq 0$  for some  $j \in \{2, 3\}$ , then we need to bound both  $\alpha(t) := \text{Re}(\lambda_j(t))$  and  $\beta(t) := \text{Im}(\lambda_j(t))$ . We show that there exist uniform upper bounds of  $\alpha(t)$  and  $|\beta(t)|$ .

**Lemma 3.5.** Assume that  $\lambda_{10} \in \mathbb{R}$  and  $\text{Im}(\lambda_{j0}) \neq 0$ . If  $(\rho_0, \lambda_{10}) \in \Omega_1$  and  $\lambda_{10} \leq \text{Re}(\lambda_{j0})$ , then

$$\alpha(t) \leq \max \left\{ \text{Re}(\lambda_{j0}), \sqrt{\{\text{Im}(\lambda_{j0})K^*\}^2 - \frac{kc_b}{3}} \right\} \quad \text{and}$$

$$|\beta(t)| \leq |\text{Im}(\lambda_{j0})|K^*,$$

where  $K^*$  is a constant independent of  $t$ .

**Proof.** From the imaginary part of (3.5b), we have  $\beta' = -2\alpha\beta$ . Hence

$$\begin{aligned} |\beta(t)| &= |\beta(0)e^{-\int_0^t 2\alpha(s) ds}| \\ &\leq |\beta(0)|e^{-\int_0^t 2\lambda_1(s) ds}, \end{aligned} \quad (3.13)$$

where the inequality comes from Lemma 3.2. Note that  $\Omega_1$  is an open set and that, given any initial data  $(\rho_0, \lambda_{10}) \in \Omega_1$  for system (3.7), we can find  $\epsilon > 0$  and initial data  $(a(0), b(0)) := (\rho_0 + \epsilon, \lambda_{10} - \epsilon) \in \Omega_1$  for modified system (3.8). Therefore, by Lemma 3.3 and the fact that there exists time  $T^* \geq 0$  such that  $b(t) > 0$  for all  $t \geq T^*$ , we have

$$e^{-\int_0^t 2\lambda_1(s) ds} \leq e^{-\int_0^t 2b(s) ds} \leq \max_{0 \leq t \leq T^*} \left\{ e^{-\int_0^t 2b(s) ds} \right\} =: K^*.$$

This gives  $|\beta(t)| \leq |\beta(0)|K^*$ .

Also, by (3.5a) and the upper bound of  $|\beta(t)|$ , we have

$$\begin{aligned} \alpha'(t) &< -\alpha^2(t) + (|\beta(0)|K^*)^2 - \frac{kc_b}{3} \\ &= - \left( \alpha(t) + \sqrt{(|\beta(0)|K^*)^2 - \frac{kc_b}{3}} \right) \\ &\quad \times \left( \alpha(t) - \sqrt{(|\beta(0)|K^*)^2 - \frac{kc_b}{3}} \right). \end{aligned} \quad (3.14)$$

$$\text{Thus, } \alpha(t) \leq \max \left\{ \alpha(0), \sqrt{(|\beta(0)|K^*)^2 - \frac{kc_b}{3}} \right\}. \quad \square$$

Together with (3.11), this completes the proof of Theorem 2.1.

### 3.2. Proof of Theorem 2.2

For the blow up condition, we rewrite (3.5c) and (3.5d):

$$\begin{cases} \rho' = -\rho(\lambda_1 + \lambda_2 + \lambda_3) \\ \quad = -\rho(3\lambda_3) - \rho(\lambda_1 - \lambda_3) - \rho(\lambda_2 - \lambda_3), \\ \lambda'_3 = -\lambda_3^2 + \frac{k(\rho - c_b)}{3}. \end{cases} \quad (3.15)$$

Similarly, we shall compare the above system with the following modified system:

$$\begin{cases} a' = -3ab, \\ b' = -b^2 + \frac{k(a - c_b)}{3}. \end{cases} \quad (3.16)$$

Following a similar proof to that of Lemma 3.3, we find the monotonicity relation between (3.15) and (3.16).

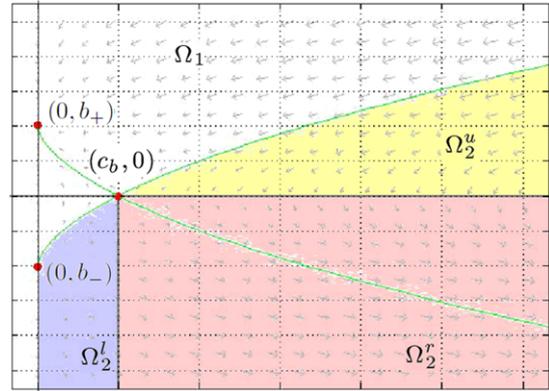


Fig. 2.  $\Omega_1$  and  $\Omega_2$  for  $k < 0$ .

**Lemma 3.6.** Assume that  $\lambda(M_0) \in \mathbb{R}$  and  $\lambda_{10} \leq \lambda_{20} \leq \lambda_{30}$ . Then

$$\begin{cases} a(0) < \rho(0) \\ \lambda_3(0) < b(0) \end{cases} \quad \text{implies that} \quad \begin{cases} a(t) < \rho(t) \\ \lambda_3(t) < b(t) \end{cases} \quad \text{for } t \geq 0.$$

We shall prove the blow up of solutions to modified system (3.16), i.e.,  $b(t) \rightarrow -\infty$  in finite time, which in turn, by Lemma 3.6, implies that  $\lambda_3(t) \rightarrow -\infty$  in finite time.

Note that system (3.16) is the same as (3.8). We thus have the same global invariant as (3.10). Hence, from the separatrix curve given by

$$\frac{b^2 + \frac{2}{3}ka + \frac{1}{3}kc_b}{a^{\frac{2}{3}}} - kc_b^{\frac{1}{3}} = 0,$$

we can show the blow up region of system (3.16).

**Lemma 3.7.** Consider the modified system (3.16), subject to initial data  $(a_0, b_0)$ . If  $(a_0, b_0) \in \Omega_2$ , then  $b \rightarrow -\infty, a \rightarrow \infty$  at a finite time. Here,

$$\begin{aligned} \Omega_2 := & \left\{ (x, y) \mid x > 0, y < \text{sgn}(x - c_b) \right. \\ & \left. \times \sqrt{k \left( c^{\frac{1}{3}} b x^{\frac{2}{3}} - \frac{2}{3}x - \frac{1}{3}c_b \right)} \right\}. \end{aligned}$$

**Proof.** Note that  $\Omega_2$  is an invariant region, which is decomposed as  $\Omega_2^l \cap \Omega_2^r \cap \Omega_2^u$ , with

$$\begin{aligned} \Omega_2^l &:= \Omega_2 \cap \{(x, y) \mid x \leq c_b\}, \\ \Omega_2^r &:= \Omega_2 \cap \{(x, y) \mid x > c_b, y < 0\} \end{aligned}$$

and  $\Omega_2^u := \Omega_2 \cap \{(x, y) \mid x > c_b, y \geq 0\}$  (see Fig. 2 in Section 3.1). It is straightforward to verify that, if  $(a_0, b_0) \in \Omega_2^l \cup \Omega_2^u$ , then  $(a(t), b(t)) \in \Omega_2^r$  in finite time. Note that, if  $(a_0, b_0) \in \Omega_2^r$ , then  $a(t)$  is increasing in  $t$ . Thus,  $a(t) > c_b, \forall t$ . This implies that  $b' < -b^2$ , which upon integration yields

$$b(t) < \frac{b_0}{tb_0 + 1}.$$

Hence, the blow up time  $t^B$  of  $b(t)$  must satisfy

$$t^B < -\frac{1}{b_0}.$$

Also  $a(t)$  approaches  $\infty$  in finite time due to the global invariant (3.10).  $\square$

The last step of proving Theorem 2.2 is to combine the comparison principle in Lemma 3.6 with Lemma 3.7. We notice that  $\Omega_2$  is an open set and that, for any given initial data  $(\rho_0, \lambda_{30}) \in \Omega_2$  for original system (3.15), we can always find  $\epsilon > 0$  such that the initial data  $(\rho_0 - \epsilon, \lambda_{30} + \epsilon) \in \Omega_2$  for modified system (3.16). This latter initial data will lead to finite-time blow up of the modified system and thus the initial data  $(\rho_0, \lambda_{30}) \in \Omega_2$  will lead to finite-time blow up of the original system.

**4. Repulsive case,  $k > 0$**

*4.1. Proof of Theorem 2.3*

This subsection is devoted to the proof of global existence for REP equations with  $k > 0$ . The spectral dynamics lemma tells us that the velocity gradient equation yields

$$\begin{cases} \lambda'_i = -\lambda_i^2 + \frac{k(\rho - c_b)}{3}, & i = 1, 2, 3, \\ \rho' = -\rho(\lambda_1 + \lambda_2 + \lambda_3). \end{cases} \quad (4.1)$$

Since  $\lambda_{10} = \lambda_{20} = \lambda_{30}$ , by the first equation of (4.1), we have  $\lambda_1(t) = \lambda_2(t) = \lambda_3(t), \forall t \geq 0$ . Let  $\lambda := \lambda_i$ ; then, by (4.1), we have

$$\begin{cases} \lambda' = -\lambda^2 + \frac{k(\rho - c_b)}{3}, \\ \rho' = -3\rho\lambda. \end{cases} \quad (4.2)$$

To obtain a global invariant we set  $q := \lambda^2$ ; then, from (4.2) we deduce that

$$\frac{dq}{d\rho} = 2\lambda \frac{\lambda'}{\rho'} = -\frac{2}{3\rho} \left( -q + \frac{k(\rho - c_b)}{3} \right).$$

Against the integrating factor of  $\rho^{-\frac{2}{3}}$ , we have

$$\frac{d}{d\rho} \left( \rho^{-\frac{2}{3}} q \right) = -\frac{2}{9} k \rho^{-\frac{2}{3}} + \frac{2kc_b}{9} \rho^{-\frac{5}{3}}.$$

Integrations with  $q = \lambda^2$  give

$$\rho^{-\frac{2}{3}} \lambda^2 = -\frac{2k}{3} \rho^{\frac{1}{3}} - \frac{kc_b}{3} \rho^{-\frac{2}{3}} + Const$$

or

$$\frac{\lambda^2 + \frac{2k}{3}\rho + \frac{kc_b}{3}}{\rho^{2/3}} = Const.$$

From this it follows that  $\rho$  is bounded from above and away from zero, which in turn gives the boundedness of  $\lambda$  for all  $t \geq 0$ . This complete the proof of Theorem 2.3.

*4.2. Proof of Theorem 2.4*

This section is devoted to the proof of finite-time blow up for REP equations with  $k > 0$ . From (4.1), it follows that

$$\begin{aligned} (\lambda_2 - \lambda_1)' &= -(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_1), \\ (\lambda_2 + \lambda_1)' &= -\lambda_1^2 - \lambda_2^2 - \frac{2kc_b}{3} + \frac{2k\rho}{3}. \end{aligned} \quad (4.3)$$

Let  $x := \lambda_2 - \lambda_1, y := \lambda_2 + \lambda_1$  and  $g(t) := \frac{2k}{3}\rho x^{-\frac{3}{2}}$ . Then (4.3) becomes

$$x' = -xy, \quad (4.4a)$$

$$y' = -\frac{y^2}{2} + G(x, g(t)), \quad (4.4b)$$

where we have used the following:

$$G(x, g(t)) := -\frac{x^2}{2} - \frac{2kc_b}{3} + g(t)x^{\frac{3}{2}}.$$

From (4.4a), we have

$$x(t) = x(0)e^{-\int_0^t y(s) ds},$$

and hence  $x(t) \equiv 0$  is an invariant. We thus consider only the  $x(0) = \lambda_{20} - \lambda_{10} > 0$  case. A simple calculation gives

$$\begin{aligned} g'(t) &= \left( \frac{2k}{3} \rho x^{-\frac{3}{2}} \right)' \\ &= \frac{2k}{3} x^{-\frac{3}{2}} \left( \rho' - \frac{3}{2} x^{-1} x' \rho \right) \\ &= \frac{2k}{3} \rho x^{-\frac{3}{2}} \left\{ -(\lambda_1 + \lambda_2 + \lambda_3) + \frac{3}{2} y \right\} \\ &= \frac{2k}{3} \rho x^{-\frac{3}{2}} \left\{ \frac{1}{2} (\lambda_1 + \lambda_2) - \lambda_3 \right\} \\ &\leq 0. \end{aligned}$$

Here, the last inequality comes from the order-preserving property of  $\lambda(M)$  and  $x(t) > 0, \forall t \geq 0$ . Therefore  $g(t)$  is non-increasing in time. This fact gives the bound of  $g(t)$ ,

$$0 < g(t) \leq g(0) = \frac{2k}{3} \frac{\rho_0}{x_0^{3/2}}.$$

Using the upper bound of  $g(t)$ , we arrive at the following observation.

**Lemma 4.1.** *The solution of (4.4) will blow up in finite time if one of the following two conditions is fulfilled:*

- (i)  $x_0 > \left( \frac{k^3 \rho_0^4}{4c_b} \right)^{\frac{1}{6}}$ ,
- (ii)  $x_0 = \left( \frac{k^3 \rho_0^4}{4c_b} \right)^{\frac{1}{6}}, y_0 < 0$ .

**Proof.** Since  $0 < g(t) \leq g(0), \forall t \geq 0$ , we have

$$G(x, g(t)) \leq G(x, g(0)), \quad \forall x > 0.$$

Also, a simple calculation gives

$$\max_{x>0} \{G(x, g(0))\} = \frac{1}{6} \frac{k^4 \rho_0^4}{x_0^6} - \frac{2kc_b}{3}.$$

Therefore, from (4.4b), it follows that

$$y' \leq -\frac{y^2}{2} + \frac{1}{6} \frac{k^4 \rho_0^4}{x_0^6} - \frac{2kc_b}{3},$$

which gives the desired results.  $\square$

Using the given initial data  $x_0$  and  $\rho_0$ , we replace the time-dependent coefficient  $g(t)$  of (4.4b) by

$$N := g(0) = \frac{2k}{3} \frac{\rho_0}{x_0^{\frac{3}{2}}}$$

and construct a corresponding new system. That is, finding the other blow up region of system (4.4) is carried out by comparison with the following system:

$$\begin{cases} a' = -ab, \\ b' = -\frac{b^2}{2} + G(a, N). \end{cases} \quad (4.5)$$

From now on, we assume that  $x_0 < \left( \frac{k^3 \rho_0^4}{4c_b} \right)^{\frac{1}{6}}$  so that the system (4.5) has two equilibrium points  $(a_i^*, 0), i = 1, 2$ , with

$$0 < a_2^* < \frac{9N^2}{4} < a_1^*.$$

Indeed,  $G(a, N)$  has its local maximum at  $a = \frac{9}{4}N^2$  and  $G(\frac{9}{4}N^2, N) > 0$ . Further calculation shows that  $(a_1^*, 0)$  is a saddle point and  $(a_2^*, 0)$  is a spiral of ODE system (4.5).

We first show the monotonicity relation between (4.4) and (4.5).

**Lemma 4.2.**

$$\begin{cases} 0 < a_0 < x_0 \\ y_0 < b_0, \end{cases} \text{ implies that } \begin{cases} a(t) < x(t) \\ y(t) < b(t), \end{cases} \quad \forall t \geq 0,$$

as long as  $a(t) > \frac{9}{4}N^2, \forall t \geq 0$ .

**Proof.** It can be proved by contradiction. Suppose that  $t_1$  is the earliest time when the above lemma is violated, then

$$a(t_1) = a_0 e^{-\int_0^{t_1} b(s) ds} < x_0 e^{-\int_0^{t_1} b(s) ds} < x_0 e^{-\int_0^{t_1} y(s) ds} = x(t_1).$$

Therefore, we are left with only one possibility:  $y(t_1) - b(t_1) = 0$ . From (4.4b) and the second equation of (4.5),

$$(b - y)' = -\frac{1}{2}(b - y)(b + y) + G(a, N) - G(x, g(t)). \quad (4.6)$$

At  $t = t_1$ , we have

$$(b - y)'(t_1) \leq 0.$$

But the right-hand side of (4.6) is positive. In fact, when it is evaluated at  $t = t_1$ ,

$$\begin{aligned} \text{RHS} &= G(a(t_1), N) - G(x(t_1), g(t_1)) \\ &\geq G(a(t_1), g(t_1)) - G(x(t_1), g(t_1)) \\ &= G_x(\eta, g(t_1))(a(t_1) - x(t_1)). \end{aligned}$$

The last equality comes from the mean value theorem with  $\eta \in (a(t_1), x(t_1))$ . Also,

$$\begin{aligned} G_x(\eta, g(t_1)) &= \eta^{\frac{1}{2}} \left( \frac{3}{2}g(t_1) - \eta^{\frac{1}{2}} \right) \\ &\leq \eta^{\frac{1}{2}} \left( \frac{3}{2}N - \eta^{\frac{1}{2}} \right) \\ &< 0, \end{aligned}$$

since  $\eta \geq a(t_1) > \frac{9}{4}N^2$ . Therefore, the right-hand side of (4.6) is positive. This leads to a contradiction, as needed.  $\square$

Now we want to find finite-time blow up conditions for system (4.5), which, in turn, by Lemma 4.2, imply the finite-time blow up of the original system (4.4). To this end, we set  $q := b^2$ . Then, from (4.5), we deduce that

$$\frac{dq}{da} = 2b \cdot \frac{b'}{a'} = \frac{q}{a} + a + \frac{4kc_b}{3a} - 2N\sqrt{a}.$$

So,

$$\frac{d}{da} \left( \frac{q}{a} \right) = 1 + \frac{4kc_b}{3a^2} - \frac{2N}{\sqrt{a}} = -\frac{2}{a^2}G(a, N).$$

Integration leads to a global invariant:

$$\frac{b^2}{a} = -2 \int_c^a \frac{G(a, N)}{a^2} da, \quad \text{where } c \text{ is some constant.} \quad (4.7)$$

By setting  $(a, b) = (a_1^*, 0)$ , we find the separatrix curve passing  $(a_1^*, 0)$ ,

$$\frac{b^2}{a} = -2 \int_{a_1^*}^a \frac{G(s, N)}{s^2} ds. \quad (4.8)$$

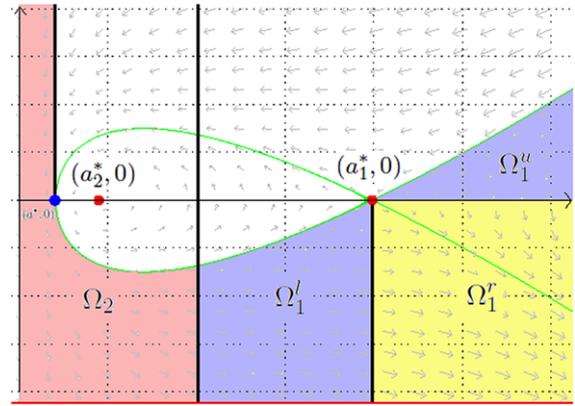


Fig. 3.  $\Omega_1$  and  $\Omega_2$  for  $k > 0$ .

The above curve has two  $x$  intercepts. One is  $(a_1^*, 0)$  and the other is denoted by  $(a^*, 0)$  with  $0 < a^* < a_2^*$ . In fact, consider

$$\int_a^{a_1^*} \frac{G(s, N)}{s^2} ds = \int_a^{a_2^*} \frac{G(s, N)}{s^2} ds + \int_{a_2^*}^{a_1^*} \frac{G(s, N)}{s^2} ds.$$

Note that  $G(a, N) \geq 0, \forall a \in [a_2^*, a_1^*]$  and  $\lim_{a \rightarrow 0^+} \int_a^{a_2^*} \frac{G(s, N)}{s^2} ds \rightarrow -\infty$ . This proves the existence of intercept  $(a^*, 0)$  and the following identity:

$$\int_{a^*}^{a_1^*} \frac{G(s, N)}{s^2} ds = 0. \quad (4.9)$$

Together with the comparison lemma, (4.8) gives the following results.

**Lemma 4.3.** The solution of (4.4) will blow up in finite time if

$$(x_0, y_0) \in \Omega_1,$$

where

$$\Omega_1 := \left\{ (x, y) \mid x > \frac{9}{4}N^2 \text{ and } y < \text{sgn}(x - a_1^*) \sqrt{2x \int_x^{a_1^*} \frac{G(s, N)}{s^2} ds} \right\}.$$

**Proof.** Since we have the comparison between two systems (4.4) and (4.5), it suffices to show the finite-time blow up of the solution for modified system (4.5). From (4.8), we know that  $\Omega_1$  is an invariant region, which is decomposed as  $\Omega_1^l \cap \Omega_1^r \cap \Omega_1^u$ , with

$$\Omega_1^l := \Omega_1 \cap \left\{ (x, y) \mid x \leq \frac{9}{4}N^2 \right\},$$

$$\Omega_1^r := \Omega_1 \cap \left\{ (x, y) \mid x > \frac{9}{4}N^2, y < 0 \right\},$$

and  $\Omega_1^u := \Omega_1 \cap \{(x, y) \mid y \geq 0\}$  (see Fig. 3). It is straightforward to verify that, if  $(a_0, b_0) \in \Omega_1^l \cup \Omega_1^u$ , then  $(a(t), b(t)) \in \Omega_1^r$  in finite time. Therefore, without loss of generality, we let  $(a_0, b_0) \in \Omega_1^r$ , then  $a(t) > a_1^*$  and  $b(t) < 0$  for all  $t \geq 0$ . This implies that

$$b' = -\frac{b^2}{2} + G(a, N) < -\frac{b^2}{2},$$

which upon integration yields

$$b(t) < \frac{2b_0}{tb_0 + 2}.$$

Hence, the blow up time  $t^B$  of  $b(t)$  must satisfy

$$t^B < -\frac{2}{b_0}.$$

Also,  $a(t)$  approaches  $\infty$  in finite time due to the global invariant in (4.7).  $\square$

The blow up condition in the above lemma was obtained by comparison with system (4.5) as long as  $a(t) > \frac{9}{4}N^2$ . In the region where  $a(t) \leq \frac{9}{4}N^2$ , we obtain blow up results by a different argument.

**Lemma 4.4.** *The solution of (4.4) will blow up in finite time if*

$$(x_0, y_0) \in \Omega_2^l \cup \Omega_2^r,$$

where

$$\Omega_2^l := \{(x, y) \mid 0 < x < a^*, \forall y\} \cup \{(x, y) \mid x = a^*, y \neq 0\},$$

and

$$\Omega_2^r := \left\{ (x, y) \in \mathbb{R}^2 \mid a^* < x \leq \frac{9}{4}N^2 \text{ and } y < -\sqrt{2x \int_x^{a_1^*} \frac{G(s, N)}{s^2} ds} \right\}.$$

**Proof.** In Lemma 4.1, we showed that  $G(x, g(t)) \leq G(x, N), \forall x > 0$ . This gives the following ODI.

$$\begin{cases} x' = -xy, \\ y' \leq -\frac{y^2}{2} + G(x, N). \end{cases} \quad (4.10)$$

If  $(x_0, y_0) \in \Omega_2^l$  with  $y_0 \geq 0$ , then, from  $x' = -xy$ , we have  $x(t) < a^*, \forall t > 0$  as long as  $y \geq 0$ . Hence

$$y' \leq G(x, N) \leq G(a^*, N) < 0.$$

Therefore,  $y(t)$  will be negative after  $t^* = -\frac{y_0}{G(a^*, N)}$ .

We now consider  $(x_0, y_0) \in \Omega_2^l$  with  $y_0 < 0$ ; if  $x(t) \leq a^*$  for all  $t > 0$ , we have

$$y' \leq -\frac{y^2}{2} + G(x, N) < -\frac{y^2}{2}.$$

This leads to the finite-time blow up, unless  $(x(t), y(t))$  enters  $\Omega^r$  in finite time.

In such a case with  $(x_0, y_0) \in \Omega_2^r$ , we deduce that

$$\frac{dy^2}{dx} = 2y \cdot \frac{y'}{x'} = -\frac{2}{x}y' \geq -\frac{2}{x} \left( -\frac{y^2}{2} + G(x, N) \right).$$

Therefore,

$$\frac{d}{dx} \left( \frac{y^2}{x} \right) \geq -\frac{2}{x^2} G(x, N).$$

Integration gives

$$\frac{y^2}{x} - \frac{y_0^2}{x_0} \geq -2 \int_{x_0}^x \frac{G(s, N)}{s^2} ds. \quad (4.11)$$

Now, consider a point  $(x_0, y_*)$  on separatrix curve (4.8) which is above  $(x_0, y_0)$ ; i.e.,

$$\frac{y_*^2}{x_0} = -2 \int_{a_1^*}^{x_0} \frac{G(s, N)}{s^2} ds. \quad (4.12)$$

Since  $y_0^2 > y_*^2$  and (4.11), we obtain

$$\frac{y^2}{x} + 2 \int_{x_0}^x \frac{G(s, N)}{s^2} ds \geq \frac{y_0^2}{x_0} > \frac{y_*^2}{x_0} = -2 \int_{a_1^*}^{x_0} \frac{G(s, N)}{s^2} ds.$$

We thus have

$$\frac{y^2}{x} > -2 \int_{a_1^*}^x \frac{G(s, N)}{s^2} ds. \quad (4.13)$$

This relation shows that, if  $(x_0, y_0) \in \Omega_2^r$ , then no  $(x(t), y(t))$  crosses separatrix curve (4.8). Therefore, if  $(x_0, y_0) \in \Omega_2^r$  with  $x_0 \leq a_2^*$ , then

$$(x(t), y(t)) \in \Omega^r \cap \{(x, y) \mid x > a_2^*\}$$

in finite time.

It is left to consider  $(x_0, y_0) \in \Omega_2^r \cap \{(x, y) \mid x > a_2^*\}$ . This set ensures that  $\exists \delta > 0$  such that

$$y_0^2 = \delta + 2x_0 \int_{x_0}^{a_1^*} \frac{G(s, N)}{s^2} ds.$$

Therefore, from (4.11),

$$\begin{aligned} \frac{y^2}{x} &\geq \frac{y_0^2}{x_0} - 2 \int_{x_0}^x \frac{G(s, N)}{s^2} ds \\ &> \frac{\delta}{2x_0} + 2 \int_{x_0}^{a_1^*} \frac{G(s, N)}{s^2} ds - 2 \int_{x_0}^x \frac{G(s, N)}{s^2} ds \\ &\geq \frac{\delta}{2x_0} + 2 \int_x^{a_1^*} \frac{G(s, N)}{s^2} ds \\ &\geq \frac{\delta}{2x_0}, \end{aligned} \quad (4.14)$$

where the last inequality comes from the fact that  $a_2^* < x_0 < x(t), \forall t > 0$ ,

$$G(x, N) \geq 0, x \in [a_2^*, a_1^*] \text{ and } G(x, N) \leq 0, x \in [a_1^*, \infty].$$

By substituting the inequality in (4.14) into the first equation in (4.10), we obtain

$$\begin{aligned} x' &= -xy \\ &\geq x^{\frac{3}{2}} \sqrt{\frac{\delta}{2x_0}}, \quad \delta > 0. \end{aligned} \quad (4.15)$$

Therefore,  $x(t) \rightarrow \infty$  in finite time. This gives the desired result.  $\square$

By combining the blow up conditions in Lemmas 3.3 and 3.4, we can get the following blow up condition. Either

$$0 < x_0 < a^* \quad (4.16)$$

or

$$\begin{aligned} x_0 \geq a^* \text{ with } y_0 < \text{sgn}(x_0 - a_1^*) \\ \times \sqrt{x_0^2 + \left( a_1^* + \frac{4kc_b}{a_1^*} \right) x_0 - 4Nx_0^{\frac{3}{2}} - \frac{4kc_b}{3}}. \end{aligned} \quad (4.17)$$

The last step of proving Theorem 2.4 is to convert the blow up conditions in (4.16) and (4.17) into conditions which involve the original variables  $\rho_0$  and  $\lambda_i$ .

Let  $\beta := \frac{a_1^*}{x_0}$ . Since  $G(x, N) = -\frac{x^2}{2} - \frac{2kc_b}{3} + Nx^{\frac{3}{2}}$  and  $G(a_i^*, N) = 0, i = 1, 2$ , we have

$$-\frac{(\beta x_0)^2}{2} - \frac{2kc_b}{3} + \frac{2k\rho_0}{3x_0^{3/2}} (\beta x_0)^{\frac{3}{2}} = 0.$$

This is equivalent to

$$\frac{3}{4kc_b}x_0^2 = -\frac{1}{\beta^2} + \frac{\rho_0}{c_b} \cdot \frac{1}{\sqrt{\beta}}. \tag{4.18}$$

Also, let  $\alpha := \frac{a^*}{x_0}$ . Since the separatrix curve (4.8) passes through  $(a^*, 0)$ , we have

$$\begin{aligned} 0 &= \left( a^* - \frac{4kc_b}{3a^*} - 4N\sqrt{a^*} \right) - \left( a_1^* - \frac{4kc_b}{3a_1^*} - 4N\sqrt{a_1^*} \right) \\ &= \left( \alpha x_0 - \frac{4kc_b}{3\alpha x_0} - \frac{8k\rho_0}{3x_0^{3/2}}\sqrt{\alpha x_0} \right) \\ &\quad - \left( \beta x_0 - \frac{4kc_b}{3\beta x_0} - \frac{8k\rho_0}{3x_0^{3/2}}\sqrt{\beta x_0} \right), \end{aligned} \tag{4.19}$$

or

$$(\alpha - \beta)x_0^2 - \frac{4kc_b}{3} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) - \frac{8k\rho_0}{3} (\sqrt{\alpha} - \sqrt{\beta}) = 0.$$

This is equivalent to

$$\frac{3}{4kc_b}x_0^2 = -\frac{1}{\alpha\beta} + \frac{2\rho_0}{c_b} \frac{1}{\sqrt{\alpha} + \sqrt{\beta}}. \tag{4.20}$$

With  $\alpha, \beta$  introduced above, the blow up conditions in (4.16) and (4.17) can be written as

$$\alpha > 1$$

and

$$\alpha \leq 1 \quad \text{with } \lambda_{20} + \lambda_{10} < \text{sgn}(1 - \beta) \times \sqrt{(\beta + 1)(\lambda_{20} - \lambda_{10})^2 - \frac{4k}{3} \left( 2\rho_0 + c_b - \frac{3c_b}{\beta} \right)},$$

respectively. This, together with Lemma 4.1, completes the proof of Theorem 2.4.

### 5. Extension to $n$ dimensions

In this section, we outline the proofs of the  $n$ -dimensional theorems. We also prove the 4-dimensional theorem for the  $k > 0$  case.

**Proof of Theorem 2.7.** From (2.3b) it follows that

$$\begin{aligned} \rho' &= -\rho \left( \sum_{i=1}^n \lambda_i \right) \\ &= -n\rho\lambda_1 - \rho \left\{ \sum_{i=2}^n \lambda_i - (n-1)\lambda_1 \right\} \\ &= -n\rho\lambda_1 - \rho \left\{ \sum_{i=2}^n \text{Re}(\lambda_i) - (n-1)\lambda_1 \right\}. \end{aligned} \tag{5.1}$$

Consider any  $\lambda_j, j \in \{2, \dots, n\}$ . If  $\lambda_j \in \mathbb{R}$ , then the order-preserving property of real eigenvalues gives  $\lambda_j - \lambda_1 \geq 0$ . If  $\text{Im}(\lambda_j) \neq 0$ , then Lemma 3.2 implies that  $\text{Re}(\lambda_j) - \lambda_1 \geq 0$ . Thus,

$$-\rho \left\{ \sum_{i=2}^n \text{Re}(\lambda_i) - (n-1)\lambda_1 \right\} \leq 0.$$

Therefore, ODE system

$$\begin{cases} \rho' = -n\rho\lambda_1 - \rho \left\{ \sum_{i=2}^n \text{Re}(\lambda_i) - (n-1)\lambda_1 \right\}, \\ \lambda_1' = -\lambda_1^2 + \frac{k(\rho - c_b)}{n}, \end{cases}$$

can be compared with

$$\begin{cases} a' = -nab, \\ b' = -b^2 + \frac{k(a - c_b)}{n}. \end{cases} \tag{5.2}$$

This gives

$$\frac{db^2}{da} = \frac{2b^2}{na} - \frac{2k}{n^2} + \frac{2kc_b}{n^2a};$$

that is,

$$\frac{d}{da} \left( a^{-\frac{2}{n}} b^2 \right) = -\frac{2k}{n^2} a^{-\frac{2}{n}} + \frac{2kc_b}{n^2} a^{-1-\frac{2}{n}}.$$

Upon integration, the separatrix passing through  $(c_b, 0)$  is obtained and expressed by

$$b^2 = k \left( \frac{c_b^{\frac{n-2}{n}} a^{\frac{2}{n}}}{n-2} - \frac{2a}{n(n-2)} - \frac{c_b}{n} \right). \tag{5.3}$$

Using (5.3), define an invariant region of (5.2) by

$$\begin{aligned} \Omega'_1 &= \left\{ (x, y) \mid x > 0, y > \text{sgn}(x - c_b) \right. \\ &\quad \left. \times \sqrt{k \left( \frac{c_b^{\frac{n-2}{n}} x^{\frac{2}{n}}}{n-2} - \frac{2x}{n(n-2)} - \frac{c_b}{n} \right)} \right\}. \end{aligned}$$

Since  $\Omega'_1$  is an open set and an invariant region of system (5.2), for any given  $(\rho_0, \lambda_{10}) \in \Omega'_1$ , we can always find  $\epsilon > 0$  and initial data  $(a(0), b(0)) := (\rho_0 + \epsilon, \lambda_{10} - \epsilon) \in \Omega'_1$  for system (5.2). Therefore, Proposition 3.1 gives

$$\lambda^* \leq \lambda_1(t) \leq \text{Re}(\lambda_2(t)) \leq \dots \leq \text{Re}(\lambda_n(t)),$$

$$\text{where } \lambda^* := \min \left\{ \lambda_1(0), \sqrt{\frac{-kc_b}{n}} \right\}.$$

We now turn to finding an upper bound of  $\max_i \{\text{Re}(\lambda_i(t))\}$  and  $\max_i \{|\text{Im}(\lambda_i(t))|\}$ . For any  $j \in \{1, \dots, n\}$ , let  $\alpha = \text{Re}(\lambda_j)$  and  $\beta = \text{Im}(\lambda_j)$ . Then, by (3.5),

$$\alpha' = -\alpha^2 + \beta^2 + \frac{k}{n}(\rho - c_b).$$

If  $\beta(0) = 0$ , then  $\beta(t) = 0$ , and

$$\alpha' \leq -\alpha^2 - \frac{kc_b}{n} = -\left( \alpha + \sqrt{\frac{-kc_b}{n}} \right) \left( \alpha - \sqrt{\frac{-kc_b}{n}} \right).$$

This gives  $\lambda_j(t) \leq \max \left\{ \lambda_{j0}, \sqrt{\frac{-kc_b}{n}} \right\}$ .

If  $\beta(0) \neq 0$ , then Lemma 3.5 gives the upper bounds of  $\alpha(t)$  and  $|\beta(t)|$ .

Therefore, we complete the proof of Theorem 2.7.  $\square$

**Proof of Theorem 2.8.** From (2.3b),

$$\begin{aligned} \rho' &= -\rho \left( \sum_{i=1}^n \lambda_i \right) \\ &= -n\rho\lambda_n - \rho \left\{ \sum_{i=1}^{n-1} \lambda_i + (1-n)\lambda_n \right\}. \end{aligned} \tag{5.4}$$

The order-preserving property of real eigenvalues gives

$$-\rho \left\{ \sum_{i=1}^{n-1} \lambda_i + (1-n)\lambda_n \right\} \geq 0.$$

Therefore, ODE system

$$\begin{cases} \rho' = -n\rho\lambda_n - \rho \left\{ \sum_{i=1}^{n-1} \lambda_i + (1-n)\lambda_n \right\}, \\ \lambda_n' = -\lambda_n^2 + \frac{k(\rho - c_b)}{n}, \end{cases} \quad (5.5)$$

can be compared with the same system in (5.2). Using the global invariant in (5.3), we define the blow up region of (5.2) by

$$\Omega'_2 = \left\{ (x, y) \mid x > 0, y < \text{sgn}(x - c_b) \times \sqrt{k \left( \frac{c_b^{\frac{n-2}{n}} x^{\frac{2}{n}}}{n-2} - \frac{2x}{n(n-2)} - \frac{c_b}{n} \right)} \right\}.$$

For any given initial data  $(\rho_0, \lambda_{n0}) \in \Omega'_2$  for original system (5.5), we can find  $\epsilon > 0$  such that the initial data  $(a(0), b(0)) := (\rho_0 - \epsilon, \lambda_{n0} + \epsilon) \in \Omega'_2$  for system (5.2). We know that  $a(t) \rightarrow \infty$  and  $b(t) \rightarrow -\infty$  at a finite time. Therefore, by Proposition 3.1, the initial data  $(\rho_0, \lambda_{n0}) \in \Omega'_2$  will lead to finite-time blow up of the original system.  $\square$

**Proof of Theorem 2.9.** Since  $\lambda_{10} = \lambda_{20} = \dots = \lambda_{n0}$ , we have  $\lambda_1(t) = \lambda_2(t) = \dots = \lambda_n(t), \forall t \geq 0$ . Let  $\lambda := \lambda_i$ . Then (2.3) leads to

$$\begin{cases} \lambda' = -\lambda^2 + \frac{k(\rho - c_b)}{n}, \\ \rho' = -n\rho\lambda. \end{cases} \quad (5.6)$$

Using the same  $q = \lambda^2$  transform as employed in the proof of Theorem 2.3 gives the following global invariant:

$$\frac{\lambda^2 + \frac{2k}{n(n-2)}\rho + \frac{kc_b}{n}}{\rho^{\frac{2}{n}}} = \text{Const.}$$

This ensures the boundedness of both  $\lambda$  and  $\rho$ , hence completing the proof of Theorem 2.9.  $\square$

**Proof of Theorem 2.10.** Let  $x := \lambda_2 - \lambda_1$  and  $y := \lambda_2 + \lambda_1$ . Then (2.3) leads to

$$\begin{cases} x' = -xy, \\ y' = -\frac{y^2}{2} - \frac{x^2}{2} + \frac{k\rho}{2} - \frac{kc_b}{2}. \end{cases} \quad (5.7)$$

Suppose that  $x_0 = \lambda_{20} - \lambda_{10} > 0$ . Then  $x(t) \neq 0, \forall t \geq 0$ , and the second equation of (5.7) leads to

$$y' = -\frac{y^2}{2} + \left( \frac{k\rho}{x^2} - 1 \right) \frac{x^2}{2} - \frac{kc_b}{2}.$$

From  $\frac{k\rho_0}{x_0^2} - 1 \leq 0$ , we can show that  $\frac{k\rho}{x^2} - 1 \leq 0, \forall t \geq 0$ . In fact,

$$\begin{aligned} \left( \frac{k\rho}{x^2} - 1 \right)' &= k \left( \frac{\rho'}{x^2} - 2 \cdot \frac{\rho}{x^3} \cdot x' \right) \\ &= \frac{k\rho}{x^2} \{ -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + 2y \} \end{aligned}$$

$$\begin{aligned} &= \frac{k\rho}{x^2} \{ (\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4) \} \\ &\leq 0. \end{aligned}$$

Therefore,

$$y' \leq -\frac{y^2}{2} - \frac{kc_b}{2} < -\frac{y^2}{2},$$

which ensures a finite-time blow up for any  $y_0 \in \mathbb{R}$ . This proves Theorem 2.10.  $\square$

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