SIAM J. NUMER. ANAL.
Vol. 38, No. 5, pp. 1439–1453
© 2000 Society for Industrial and Applied Mathematics

SPECTRAL VISCOSITY APPROXIMATIONS TO HAMILTON–JACOBI SOLUTIONS*

OLGA LEPSKY†

Abstract. The spectral viscosity approximate solution of convex Hamilton–Jacobi equations with periodic boundary conditions is studied. It is proved in this paper that the approximation and its gradient remain uniformly bounded, formally spectral accurate, and converge to the unique viscosity solution. The L^1-convergence rate of the order $1-\varepsilon \forall \varepsilon > 0$ is obtained.

Key words. Hamilton–Jacobi equation, viscosity solution, spectral viscosity method, vanishing viscosity method, convergence rate, error estimate

AMS subject classifications. 35L65, 65M70, 65M15

PII. S0036142999353582

1. Introduction.

In this paper we consider the nonlinear Hamilton–Jacobi equations

$$\begin{aligned} \partial_t u(x,t) + H(x,t,\nabla_x u(x,t)) = 0 \text{ in } \Omega\times[0,T], \quad 0<T<\infty, \\ u(x,0)=\varphi(x) \ \text{ in } \Omega, \ \ \varphi \in L^\infty(\Omega), \end{aligned} \tag{1}$$

with Hamiltonian $H(x,t,p)$ strictly convex with respect to x and p:

$$\alpha I \le \begin{pmatrix} H_{xx} & H_{xp} \\ H_{px} & H_{pp} \end{pmatrix} \le \beta I \quad (0<\alpha,\beta<\infty), \tag{2}$$

where I denotes a unit $2d\times 2d$ matrix. We also assume that $\Omega = [0,2\pi]^d, d = 1,2,3,\ldots$, the initial condition $\varphi(x)$, and the Hamiltonian $H(x,t,p)$ are 2π-periodic in x.

Equations of Hamilton–Jacobi type arise in many areas of application, including the calculus of variations, control theory, differential games, image processing (see, e.g., [17], [21]). The generalized solutions to (1) are Lipschitz continuous (hence differentiable almost everywhere), but may have discontinuous derivatives, regardless of the smoothness of the initial condition $\varphi(x)$ [17]. Solutions with such discontinuities are not unique. The definition of the viscosity solution to (1), its well-posedness (in L^∞) and other properties were formulated and systematically studied by Kruzhkov, Lions, Crandall, Evans, Souganidis, and others [12], [17], [6], [8], [23]. Following these results, first and higher order numerical methods were developed for Hamilton–Jacobi equations: finite difference methods ([7], [23], [20], [10], [15], [16]), finite volume methods ([1], [13]), finite element methods ([3], [9], [14]). Recently, Lin and Tadmor provided the convergence framework for general approximate solutions of multidimensional Hamilton–Jacobi equations [15], which we use in the paper.

In this paper we suggest the numerical solution of the 2π-periodic initial value problem (1) in R^d $(d\ge 1)$ by a spectral viscosity method. We prove (see Theorem 5.2) that the numerical solution converges to the exact unique viscosity solution of (1) and obtain the L^1-convergence rate of the order $1-\varepsilon$ $\forall\varepsilon > 0$. The spectral

*Received by the editors March 10, 1999; accepted for publication (in revised form) June 13, 2000; published electronically November 17, 2000.
http://www.siam.org/journals/sinum/38-5/35358.html

†Department of Mathematics, Brown University, Providence, RI 02906 (olga@math.brown.edu).

viscosity method was introduced and analyzed for scalar conservation laws and certain hyperbolic systems of conservation laws by Tadmor and his coworkers (see, e.g., [24], [22], [25], [5], [19], [18]). In the one-dimensional case the derivative of the viscosity solution of convex Hamilton–Jacobi equations (1) is equivalent to the entropy solution of scalar conservation laws. However, in the multidimensional case the gradient of the viscosity solution of (1) satisfies a weakly hyperbolic system of conservation laws [12], [17], [11], for which the convergence of the spectral viscosity method was not proven.

The following abbreviations are used throughout this paper:

$$\partial_t = \frac{\partial}{\partial t},\ \partial_j^s = \frac{\partial^s}{\partial x_j^s},\ \partial_{jk}^{2s} = \frac{\partial^{2s}}{\partial x_j^s \partial x_k^s},\ \partial_x^s = (\partial_1^s, \partial_2^s, \ldots, \partial_d^s)\,,$$
$$\partial_x = \nabla_x,\quad D_x^2 \text{ is a matrix with entries } \partial_{jk}^2.$$

To solve the (1) by a spectral viscosity method we approximate the spectral (pseudospectral) projection of the viscosity solution $P_N u$ by an N-trigonometric polynomial

$$u_N = \sum_{|\xi_i| \le N} \widehat{u}_\xi(t) e^{i\xi\cdot x}, \tag{3}$$

which is governed by the semidiscrete approximation

$$\partial_t u_N(x,t) + P_N H\left(x,t,\nabla_x u_N(x,t)\right) = \varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t). \tag{4}$$

Together with one's favorite ODE solver, (4) gives a fully discrete method for the approximate solutions of (1).

Following [5], we use the spectral viscosity $\varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t)$. We show that the spectral viscosity is small enough to retain the formal spectral accuracy of the overall approximation, while sufficiently large to enforce the uniform stability and L^1-convergence of the approximation $u_N(x,t)$ to the unique viscosity solution (see Lemma 3.1, Lemma 3.2, and Theorem 5.2). It consists of the following three ingredients:

(i) a vanishing viscosity amplitude ε_N,

$$\varepsilon_N \sim N^{-\theta}, \theta \in (0,1)\,; \tag{5}$$

(ii) a viscosity-free spectrum of size $m_N \gg 1$.

For the proof of $W^{1,\infty}$-stability of u_N and convergence of the truncation error to zero, it is enough to take $m_N \sim N^\delta,\ 0 < \delta < \theta$, while for the proof of semiconcave stability, and hence, convergence we use

$$m_N \sim \frac{N^{\theta/4}}{(\log N)^{1/4}}; \tag{6}$$

(iii) a family of viscosity kernels $Q_N^j(x,t)$, activated only on high wave numbers $\|\xi\| = \left(\sum_{i=1}^{d} \xi_i^2\right)^{1/2} \ge m_N$, such that

$$Q_N^j(x,t) = \sum_{|\xi_i| \le N,\ \|\xi\| \ge m_N} \widehat{Q}_\xi^j(t) e^{i\xi\cdot x},$$

$$\varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t) = -\varepsilon_N \sum_{|\xi_i| \le N,\ \|\xi\| \ge m_N} \left(\sum_{j=1}^{d} \widehat{Q}_\xi^j(t) \xi_j^2 \right) \widehat{u}_\xi(t) e^{i\xi\cdot x}. \tag{7}$$

Here we take the Fourier coefficients $\widehat{Q}^j_\xi(t)$ such that $\widehat{Q}^j_\xi(t) = \widehat{Q}^j_p(t)$, when $\|\xi\| = \left(\sum_{i=1}^d \xi_i^2\right)^{1/2} = p$,

$$\begin{aligned} \widehat{Q}^j_p(t) &= 0 \;\forall p < m_N, \\ 1 - \mathrm{Const}\frac{m_N^4}{p^4} \le \widehat{Q}^j_p(t) &\le 1 \;\forall p \ge m_N, \\ \widehat{Q}^j_p(t) &\le \widehat{Q}^j_{p+1}(t). \end{aligned} \tag{8}$$

Define also

$$\sum_{j=1}^d \partial_j^2 R^j_N(x,t)* \; = \Delta - \sum_{j=1}^d \partial_j^2 Q^j_N(x,t) * . \tag{9}$$

Then we may rewrite the semidiscrete approximation (4) as

$$\begin{aligned} &\partial_t u_N(x,t) + H\left(x,t,\nabla_x u_N(x,t)\right) - \varepsilon_N \Delta u_N(x,t) \\ &= \left(I - P_N\right) H\left(x,t,\nabla_x u_N(x,t)\right) - \varepsilon_N \sum_{j=1}^d \partial_j^2 R^j_N(x,t) * u_N(x,t). \end{aligned} \tag{10}$$

The convergence analysis of the spectral viscosity method is based on its close resemblance to the vanishing viscosity approximation

$$\partial_t u_\varepsilon(x,t) + H\left(x,t,\nabla_x u_\varepsilon(x,t)\right) - \varepsilon \Delta u_\varepsilon(x,t) = 0. \tag{11}$$

Here $u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t)$ in $W^{1,\infty}$ is the unique viscosity solution.

The paper is organized as follows. In section 2 we obtain the L^p-estimates $(1 \le p \le \infty)$ of the terms on the right of (10). These estimates enable us to show in section 3 that the spectral viscosity approximation $u_N(x,t)$ and its gradient remain uniformly bounded with the growth of N. In section 4 we prove the semiconcave stability of the approximate solution. Combining this result with the decay rate of the truncation error, obtained in section 5, and applying Lin–Tadmor convergence theorem [15], we conclude the convergence of the method and obtain its L^1-convergence rate (Theorem 5.2).

2. Preliminary estimates. In this section we will estimate the two terms on the right of (10).

In view of Theorem 9.1 of [26] $\forall p \ge 1$ we have

$$\left\|\varepsilon_N \sum_{j=1}^d \partial_j^2 R^j_N * u_N\right\|_{L^p(x)} \le \left\|\varepsilon_N \sum_{j=1}^d \partial_j^2 R^j_N\right\|_{L^1(x)} \|u_N\|_{L^p(x)}, \tag{12}$$

$$\left\|\varepsilon_N \sum_{j=1}^d \partial_j^2 R^j_N * u_N\right\|_{L^p(x)} \le \left\|\varepsilon_N \sum_{j=1}^d \partial_j^2 R^j_N\right\|_{L^p(x)} \|u_N\|_{L^1(x)}. \tag{13}$$

LEMMA 2.1.

$$\sum_{j=1}^d \left\|-\varepsilon_N \partial_j^{2s} R^j_N\left(x,t\right)\right\|_{L^1(x)} \le C_0 \varepsilon_N\, m_N^s \log N \le \begin{cases} C_1 \frac{\log N}{N^\theta}, & s = 0, \\ C_1 \left(\frac{\log N}{N^\theta}\right)^{3/4}, & s = 1, \end{cases}$$

where $s \in \{0,1\}$; C_0, C_1 *do not depend on* N.

Proof. This lemma corresponds to Lemma 3.1 in [5]. $\square$

LEMMA 2.2.

$$\sum_{j=1}^{d} \left\| -\varepsilon_N \partial_j^{2s} R_N^j(x,t) \right\|_{L^\infty(x)} \le C_0 \varepsilon_N m_N^{2s+2} \log N \le \begin{cases} C_1 \left(\frac{\log N}{N^\theta}\right)^{1/2}, & s=0, \\ C_1, & s=1, \end{cases}$$

where $s \in \{0,1\}$ *and* C_0, C_1 *do not depend on* N.

Remark. Lemma 2.2 is a multidimensional generalization of Lemma 3.1 in [25].

Proof. By (9) and (8) for $q=2$ and $|\xi| = p$

$$\widehat{R}_p^j(t) = \begin{cases} 1 \text{ if } p < m_N, \\ 1 - \widehat{Q}_p^j(t) \le \text{Const} \left(\frac{m_N^2}{p^2}\right)^q \text{ if } p \ge m_N. \end{cases}$$

Then for $s \ge 0$

$$\begin{aligned}
&\left\| -\varepsilon_N \partial_j^{2s} R_N^j(x,t) \right\|_{L^\infty(x)} \\
&= \varepsilon_N \left\| \partial_j^{2s} \sum_{p=0}^{N\sqrt{d}} \widehat{R}_p^j(t) \sum_{|\xi_i| \le N,\ \|\xi\| = p} e^{i\xi\cdot x} \right\|_{L^\infty(x)} \\
&\le \varepsilon_N \left\| \partial_j^{2s} \sum_{p=0}^{m_N - 1} \sum_{|\xi_i| \le N,\ \|\xi\| = p} e^{i\xi\cdot x} \right\|_{L^\infty(x)} \\
&\quad + \text{Const } \varepsilon_N \left\| \partial_j^{2s} \sum_{p=m_N}^{N\sqrt{d}} \left(\frac{m_N^2}{p^2}\right)^q \sum_{|\xi_i| \le N,\ \|\xi\| = p} e^{i\xi\cdot x} \right\|_{L^\infty(x)} \\
&\le \text{Const } \varepsilon_N \left(\left\| \sum_{p=0}^{m_N - 1} p^{2s} \sum_{|\xi_i| \le N,\ \|\xi\| = p} e^{i\xi\cdot x} \right\|_{L^\infty(x)} \right) \\
&+ \text{Const } \varepsilon_N \left(\left\| \sum_{p=m_N}^{N\sqrt{d}} \frac{m_N^{2q}}{p^{2q-2s}} \sum_{|\xi_i| \le N,\ \|\xi\| = p} e^{i\xi\cdot x} \right\|_{L^\infty(x)} \right) \\
&\le \text{Const } d\varepsilon_N \left(\left\| \sum_{p=0}^{m_N - 1} p^{2s+1} \right\|_{L^\infty(x)} + \left\| \sum_{p=m_N}^{N\sqrt{d}} \frac{m_N^{2q}}{p^{2q-2s-1}} \right\|_{L^\infty(x)} \right) \\
&\le \text{Const } d\varepsilon_N \left(m_N^{2s+2} + \left\{ \begin{array}{l} m_N^{2s+2} \text{ if } 2q-2s-1 > 1, \\ m_N^{2s+2} \log\left(N\sqrt{d}\right) \text{ if } 2q-2s-1 = 1 \end{array} \right\} \right) \\
&\le \text{Const } d\varepsilon_N m_N^{2s+2} \log N, \text{ since } q = 2 \ge 1+s.
\end{aligned}$$

Here d is the space dimension. Now, using (5) and (6) completes the proof of the lemma. $\square$

Note. For the simplicity of the proofs from now on we will assume that Hamiltonian depends only on ∇u, i.e., $H = H(\nabla u)$, such that

$$\alpha I \le H_{pp}(p) \le \beta I \quad (0 < \alpha, \beta < \infty), \qquad (14)$$
$$p = (p_1, \ldots, p_d), \ I \text{ is a unit } d \times d \text{ matrix.}$$

However, all the results and proofs of this paper are easily extended to the case $H = H(x, t, \nabla u)$ satisfying (2).

LEMMA 2.3. *Denote*

$$\begin{aligned} \partial_p^\alpha H(p) &:= \partial_{p_1}^{\alpha_1} \cdots \partial_{p_d}^{\alpha_d} H(p), \\ \|H\|_{C^k(x)} &:= \max_{|\alpha|=k} \left\|\partial_p^\alpha H(p)\right\|_{L^\infty(B_N)}, \\ B_N &:= \left\{ p : |p| \le \|\nabla_x u_N\|_{L^\infty(x)} \right\}. \end{aligned}$$

Then $\forall s \ge 1$

$$\|\partial_x^s H(\nabla_x u_N(x,t))\|_{L^2(x)} \le K_s \left\|\partial_x^{s+1} u_N\right\|_{L^2(x)}, \tag{15}$$

where K_s *depends on* $\|\nabla_x u_N\|_{L^2(x)}$, H, *and* s :

$$K_s \sim \sum_{r=1}^{s} \|H\|_{C^r(x)} \|\nabla_x u_N\|_{L^2(x)}^{r-1}. \tag{16}$$

Proof. Define a multi-index $\alpha = (\alpha_1, \dots, \alpha_r)$, $\forall \alpha_i \in \{1, 2, \dots, s\}$, $|\alpha| = \sum_{l=1}^r \alpha_l$. By the chain rule we have

$$\partial_j^s H(\nabla_x u_N(x,t)) = \sum_{r=1}^{s} \sum_{|\alpha|=s} \sum_{i_1=1}^{d} \cdots \sum_{i_r=1}^{d} \left(\frac{\partial^r H}{\partial p_{i_1} \cdots \partial p_{i_r}} \right) \left(\partial_j^{\alpha_1} \partial_{i_1} u_N\right) \cdots \left(\partial_j^{\alpha_r} \partial_{i_r} u_N\right). \tag{17}$$

The Hölder inequality followed by the Gagliardo–Nirenberg inequality implies that

$$\begin{aligned} &\left\|\left(\partial_j^{\alpha_1} \partial_{i_1} u_N\right) \cdots \left(\partial_j^{\alpha_r} \partial_{i_r} u_N\right)\right\|_{L^2(x)} \\ &\le \prod_{k=1}^{r} \left\|\partial_j^{\alpha_k} \partial_{i_k} u_N\right\|_{L^{2s/\alpha_k}(x)} \le \sum_{m=1}^{d} \prod_{k=1}^{r} \left\|\partial_j^{\alpha_k} \partial_m u_N\right\|_{L^{2s/\alpha_k}(x)} \\ &\le C_0 \sum_{m=1}^{d} \prod_{k=1}^{r} \left\|\partial_j^s \partial_m u_N\right\|_{L^2(x)}^{\alpha_k/s} \|\partial_m u_N\|_{L^\infty(x)}^{1-\alpha_k/s} \\ &\le C_0 \sum_{m=1}^{d} \left\|\partial_j^s \partial_m u_N\right\|_{L^2(x)} \|\partial_m u_N\|_{L^\infty(x)}^{r-1}. \end{aligned} \tag{18}$$

By Parseval's identity followed by the Young inequality

$$\begin{aligned} \left\|\partial_j^s \partial_m u_N\right\|_{L^2(x)} &= \left(\sum_{|\xi| \le N} \xi_j^{2s} \xi_m^2 \left(\widehat{u}_\xi\right)^2 \right)^{1/2} \le \left(\sum_{|\xi| \le N} \xi_j^{2s} \xi_m^2 \left(\widehat{u}_\xi\right)^2 \right)^{1/2} \\ &\le \left(\sum_{|\xi| \le N} \xi_j^{2s} \xi_m^2 \left(\widehat{u}_\xi\right)^2 \right)^{1/2} \le \left(\sum_{|\xi| \le N} \left(\frac{\xi_j^{s+1}}{\frac{s+1}{s}} + \frac{\xi_m^{s+1}}{\frac{1}{s+1}} \right)^2 \left(\widehat{u}_\xi\right)^2 \right)^{1/2} \\ &\le 2(s+1) \left(\sum_{|\xi| \le N} \left(|\xi|^{s+1}\right)^2 \left(\widehat{u}_\xi\right)^2 \right)^{1/2} \\ &\le 2(s+1) \left\|\partial_x^{s+1} u_N\right\|_{L^2(x)}. \end{aligned} \tag{19}$$

Combining (17), (18), and (19) we obtain

$$\left\|\partial_x^s H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ \leq C\left\|\partial_x^{s+1} u_N\right\|_{L^2(x)} \sum_{r=1}^{s} \|H\|_{C^r(x)} \|\nabla u_N\|_{L^2(x)}^{r-1}. \qquad \square$$

3. Uniform boundedness of the approximate solution and its gradient. We begin with the following facts:

(i) In view of the approximation error estimate (see section 9.7 of [4]) $\forall s \geq r \geq 0$ there holds

$$\left\|\partial_x^r (I-P_N) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ \leq \frac{C}{N^{s-r}} \left\|\partial_x^s H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)}. \tag{20}$$

(ii) The Sobolev imbedding theorem (see, e.g., [2]) combined with (i) implies

$$\left\|\partial_x^r (I-P_N) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^\infty(x)} \\ \leq \left\|\partial_x^{r+1+[d/2]} (I-P_N) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ \leq \frac{C}{N^{s-r-1-[d/2]}} \left\|\partial_x^s H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)}. \tag{21}$$

Here $[d/2] = \max\{z \in Z,\ z \leq d/2\}$; C is a constant independent of N.

Assumptions. From now on we assume that our data (the initial conditions $u_N(x,0)$ and the flux $H\left(\nabla_x u(x,t)\right)$) satisfy the following conditions:

(a) $\left\|\partial_x^s u_N(x,0)\right\|_{L^\infty} \leq$Const for $s = 0, 1$;

(b) $\left\|\partial_x^s u_N(x,0)\right\|_{L^2} \leq$Const$/\varepsilon_N^{s-1/2}$ for $s \geq 2$;

(c) The flux H is sufficiently smooth; that is, $\|\nabla_x u\|_{L^\infty(x,t)} < \infty$ implies

$$\left\|H\left(\nabla_x u\right)\right\|_{C^s}(x,t) < \infty$$

for sufficiently large s.

Lemma 3.1. *Consider the spectral viscosity approximation* (4). *Then for any $s \geq 2$ holds*

$$\|\partial_x^s u_N\|_{L^2(x,t)} \leq \frac{C(\nabla_x u_N)}{\varepsilon_N^{s-1}} \prod_{p=1}^{s-2} \left(\|K_p\|_{L^\infty(t)} + 1\right), \tag{22}$$

$$\|\partial_x^s u_N\|_{L^2(x)} \leq \frac{C(\nabla_x u_N)}{\varepsilon_N^{s-1/2}} \prod_{p=1}^{s-1} \left(\|K_p\|_{L^\infty(t)} + 1\right), \tag{23}$$

$$\|\partial_x^s u_N\|_{L^\infty(x)} \leq \frac{C(\nabla_x u_N)}{\varepsilon_N^{s_d}} \prod_{p=1}^{s-1} \left(\|K_p\|_{L^\infty(t)} + 1\right). \tag{24}$$

Here

$$C(\nabla_x u_N) = \left(\left\|H\left(\nabla_x u_N(x,t)\right)\right\|_{C^0(x)} + \|\nabla_x u_N\|_{L^2(x)} + 1\right),$$

K_s *is defined by* (16) *for* $s \geq 1$, $s_d = s + 1/2 + [d/2]$.

Proof. Multiplying (10) by $\partial_x^{2s} u_N(x,t)$, $s \geq 1$ and integrating in x yields

$$\begin{aligned}(25)\qquad &\frac{1}{2}\frac{d}{dt}\left\|\partial_x^s u_N\right\|_{L^2(x)}^2 + \varepsilon_N \left\|\partial_x^{s+1} u_N\right\|_{L^2(x)}^2 \\ &\leq \left\|\partial_x^{s+1} u_N\right\|_{L^2(x)} \left\|\partial_x^{s-1} P_N H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ &\quad + \left\|\partial_x^s u_N\right\|_{L^2(x)} \left\|\varepsilon_N \sum_{j=1}^{d} \partial_j^2 R_N^j(x,t) * \partial_x^s u_N\right\|_{L^2(x)}.\end{aligned}$$

Using (12) and Lemma 2.2 for the first term on the right of (25) and the Cauchy–Schwarz inequality for the second term, we end up with

$$\begin{aligned}(26)\qquad &\frac{1}{2}\frac{d}{dt}\left\|\partial_x^s u_N\right\|_{L^2(x)}^2 + \frac{\varepsilon_N}{2}\left\|\partial_x^{s+1} u_N\right\|_{L^2(x)}^2 \\ &\leq C_1\left\|\partial_x^s u_N\right\|_{L^2(x)}^2 + \frac{1}{2\varepsilon_N}\left\|\partial_x^{s-1} P_N H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)}^2.\end{aligned}$$

It remains to estimate the last term. In view of the interpolation error estimate (see section 9 of [4]) and (15)

$$\begin{aligned}&\left\|\partial_x^{s-1} P_N H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ &\leq \left\|\partial_x^{s-1} H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} + \left\|\partial_x^{s-1}\left(I - P_N\right) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ &\leq C\left\|\partial_x^{s-1} H\left(\nabla_x u_N(x,t)\right)\right\|_{L^2(x)} \\ &\leq C\begin{cases} K_{s-1}\left\|\partial_x^s u_N\right\|_{L^2(x)} & \text{for } s \geq 2, \\ K_0 & \text{for } s = 1.\end{cases}\end{aligned}$$

Substituting this estimate to (26) yields

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla_x u_N\right\|_{L^2(x)}^2 + \varepsilon_N\left\|\partial_x^2 u_N\right\|_{L^2(x)}^2 \leq 2C_1\left\|\nabla_x u_N\right\|_{L^2(x)}^2 + \frac{C_2 K_0^2}{\varepsilon_N}, \\ &\frac{d}{dt}\left\|\partial_x^s u_N\right\|_{L^2(x)}^2 + \varepsilon_N\left\|\partial_x^{s+1} u_N\right\|_{L^2(x)}^2 \leq \left(2C_1 + \frac{C_2 K_{s-1}^2}{\varepsilon_N}\right)\left\|\partial_x^s u_N\right\|_{L^2(x)}^2 \quad \text{for } s \geq 2.\end{aligned}$$

The temporal integration implies

$$\begin{aligned}&\left\|\nabla_x u_N\right\|_{L^2(x)}^2 + \varepsilon_N\left\|\partial_x^2 u_N\right\|_{L^2(x,t)}^2 \\ &\leq \left\|\nabla_x u_N(x,0)\right\|_{L^2}^2 + 2C_1\left\|\nabla_x u_N\right\|_{L^2(x,t)}^2 + \frac{C_2\left\|K_0\right\|_{L^\infty(t)}^2}{\varepsilon_N}; \\ &\left\|\partial_x^s u_N(x,t)\right\|_{L^2(x)}^2 + \varepsilon_N\left\|\partial_x^{s+1} u_N\right\|_{L^2(x,t)}^2 \\ &\leq \left\|\partial_x^s u_N(x,0)\right\|_{L^2}^2 + \left(2C_1 + \frac{C_2\left\|K_{s-1}\right\|_{L^\infty(t)}^2}{\varepsilon_N}\right)\left\|\partial_x^s u_N\right\|_{L^2(x,t)},\end{aligned}$$

and eliminating the squares gives

$$
\begin{aligned}
&\|\nabla_x u_N\|_{L^2(x)} + \varepsilon_N^{1/2} \left\|\partial_x^2 u_N\right\|_{L^2(x,t)} \\
&\le \sqrt{2}\left(\|\nabla_x u_N(x,0)\|_{L^2} + \sqrt{2C_1}\,\|\nabla_x u_N\|_{L^2(x,t)} + \frac{\|K_0\|_{L^\infty(t)}}{\varepsilon_N^{1/2}}\sqrt{C_2}\right), \\
&\|\partial_x^s u_N(x,t)\|_{L^2(x)} + \varepsilon_N^{1/2}\left\|\partial_x^{s+1} u_N\right\|_{L^2(x,t)} \\
&\le \sqrt{2}\left(\|\partial_x^s u_N(x,0)\|_{L^2(x)} + \left(\sqrt{2C_1} + \frac{\sqrt{C_2}\,\|K_{s-1}\|_{L^\infty(t)}}{\varepsilon_N^{1/2}}\right)\|\partial_x^s u_N\|_{L^2(x,t)}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left\|\partial_x^2 u_N\right\|_{L^2(x,t)} &\le \mathrm{Const}\left(\frac{\|K_0\|_{L^\infty(t)}}{\varepsilon_N} + \frac{1}{\varepsilon_N^{1/2}}\|\nabla_x u_N\|_{L^2(x,t)} + \frac{1}{\varepsilon_N^{1/2}}\right), \\
\left\|\partial_x^{s+1} u_N\right\|_{L^2(x,t)} &\le \mathrm{Const}\left(\left(\frac{\|K_{s-1}\|_{L^\infty(t)}}{\varepsilon_N} + \frac{1}{\varepsilon_N^{1/2}}\right)\|\partial_x^s u_N\|_{L^2(x,t)} + \frac{1}{\varepsilon_N^{1/2}}\right) \\
\|\partial_x^s u_N(x,t)\|_{L^2(x)} &\le \mathrm{Const}\left(\left(\frac{\|K_{s-1}\|_{L^\infty(t)}}{\varepsilon_N^{1/2}} + 1\right)\|\partial_x^s u_N\|_{L^2(x,t)} + 1\right),
\end{aligned}
$$

which in turn yields the (22) and (23) for ε_N sufficiently small. We apply the Sobolev inequality $\|\partial_x^s u_N\|_{L^\infty(x)} \le \|\partial_x^{s+1+[d/2]} u_N\|_{L^2(x)}$ to obtain (24). ∎

LEMMA 3.2 (uniform boundedness of the approximation and its gradient). *Consider the spectral viscosity approximation* (4). *Then* $\forall N$ *we have*

$$
\|u_N(x,t)\|_{L^\infty(x)} + \|\nabla_x u_N(x,t)\|_{L^\infty(x)} \le C \;\; \forall t \in [0,T] \;\; \textit{for } T < \infty,
$$

where C does not depend on N.

Proof. Differentiating (10) in x we obtain

$$
\begin{aligned}
&\partial_t \nabla_x u_N(x,t) + \nabla_x H\left(\nabla_x u_N(x,t)\right) - \varepsilon_N \Delta\left(\nabla_x u_N(x,t)\right) \\
&= \nabla_x\left(I - P_N\right) H\left(\nabla_x u_N(x,t)\right) - \varepsilon_N \sum_{j=1}^{d} \partial_j^2 R_N^j(x,t) * \nabla_x u_N(x,t).
\end{aligned} \tag{27}
$$

Notice that

$$
\frac{1}{d}\|\nabla_x u_N(x,t)\|_{L^\infty(x)} \le \max_{j,x}|\partial_j u_N(x,t)| = |\partial_k u_N(x_0,t)|. \tag{28}
$$

First assume that $\partial_k u_N(x_0,t) > 0 \;\forall t \in [0,T]$. Then $\partial_k H\left(\nabla_x u_N(x_0,t)\right) = \sum_{j=1}^d \partial_{p_j} H(p)\, \partial_j \partial_k u_N = 0$ and $-\varepsilon_N \Delta \partial_k u_N(x_0,t) \ge 0$. Denote $v_N(t) := \partial_k u_N(x_0,t)$; here k and x_0 depend on t. It is easy to show that $v_N(t)$ is a Lipschitz continuous function. Therefore $v_N(t)$ is differentiable almost everywhere (a.e.) (by Rademacher's theorem). It follows from (27) that for a.e. t

$$
\begin{aligned}
\frac{dv_N(t)}{dt} &\le \left\|\nabla_x\left(I - P_N\right) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^\infty(x)} \\
&\quad + \left\|\varepsilon_N \nabla_x \sum_{j=1}^{d} \partial_j^2 R_N^j(x,t) * u_N(x,t)\right\|_{L^\infty(x)}.
\end{aligned}
$$

We use (21), (15), and (23) to estimate the first term on the right:

$$\begin{aligned}&\left\|\nabla_x\left(I-P_N\right)H\left(\nabla_x u_N(x,t)\right)\right\|_{L^\infty(x)}\\ &\leq \frac{K_s}{N^{s-2-[d/2]}}\left\|\partial_x^{s+1}u_N\right\|_{L^2(x)}\\ &\leq \frac{C_0}{N^{s-2-[d/2]-\theta(s+1/2)}}\Big(\|K_0\|_{L^2(t)}+\|\nabla_x u_N\|_{L^2(x)}+1\Big)K_s\prod_{p=1}^{s}\Big(\|K_p\|_{L^\infty(t)}+1\Big);\end{aligned}$$

while (12) combined with Lemma 2.1 provides the estimate for the second term:

$$\left\|\varepsilon_N\nabla_x\sum_{j=1}^{d}\partial_j^2 R_N^j(x,t)*u_N(x,t)\right\|_{L^\infty(x)}\leq C\left(\frac{\log N}{N^\theta}\right)^{3/4}\|\nabla_x u_N\|_{L^\infty(x)}.$$

Finally, applying (28) we obtain for a.e. t

$$\begin{aligned}\frac{dv_N(t)}{dt}&\leq \frac{C}{N^{s-2-[d/2]-\theta(s+1/2)}}K_s\prod_{p=0}^{s}\Big(\|K_p\|_{L^\infty(t)}+1\Big)\\ &\quad+\left(\frac{CK_s\prod_{p=1}^{s}\Big(\|K_p\|_{L^\infty(t)}+1\Big)}{N^{s-2-[d/2]-\theta(s+1/2)}}+C\left(\frac{\log N}{N^\theta}\right)^{3/4}\right)v_N(t).\end{aligned}\tag{29}$$

Take $s\geq\frac{2+[d/2]+5\theta/4}{1-\theta}$; then we have

$$s-2-[d/2]-\theta\left(s+1/2\right)\geq 3\theta/4>0.$$

Now fix $\varepsilon>0$ and assume that $\exists t_N\leq T$ such that $\forall N\geq N_0$:

$$\begin{aligned}&\varepsilon/2<v_N(t_N)-v_N(0)\leq\varepsilon,\\ &0\leq v_N(t)-v_N(0)\leq\varepsilon/2\ \forall t\leq t_N.\end{aligned}\tag{30}$$

Then integrating (29) in time and applying the definition of K_s (in the statement of Lemma 3.1), we get

$$v_N(t_N)-v_N(0)\leq\frac{C_0\left(\varepsilon,s\right)}{N^{s-2-[d/2]-\theta(s+1/2)}}+C_1\left(\varepsilon\right)\left(\frac{\log N}{N^\theta}\right)^{3/4},\tag{31}$$

where $C_0\left(\varepsilon,s\right)$ and $C_1\left(\varepsilon\right)$ depend only on ε, s, and T. It follows that

$$v_N(t_N)-v_N(0)\leq\varepsilon/2\ \text{ for big } N,$$

which contradicts the assumption (30).

From this contradiction and from the continuity of $v_N(t)$ it follows that for any $C>0$ there exist N_0 such that $v_N(t)-v_N(0)\leq C\ \forall\ N\geq N_0$ and $\forall\ t\in[0,T]$. Since $v_N(t)=\|\nabla_x u_N(x,t)\|_{L^\infty(x)}$, and $v_N(t)$ is continuous on $[0,T]$, we obtain that in the considered case

$$\|\nabla_x u_N(x,t)\|_{L^\infty(x)}\leq\|\nabla_x u_N(x,0)\|_{L^\infty}+C.\tag{32}$$

Next, assume that for k and x_0 satisfying (28) $\partial_k u_N(x_0,t) < 0 \; \forall t \in [0,T]$. Define $w_N(t) := \partial_k u_N(x_0,t) = \min_x \partial_k u_N(x,t)$. Then

$$\partial_k H\left(\nabla_x u_N(x_0,t)\right) = \sum_{j=1}^{d} \partial_{p_j} H\left(p\right) \partial_j \partial_k u_N = 0,$$

$$-\varepsilon_N \Delta \partial_k u_N(x_0,t) \le 0,$$

$w_N(t)$ is a Lipschitz continuous function, and we obtain from (27)

$$\begin{aligned} &\frac{d}{dt} w_N(t) \\ &\ge -\left\| \nabla_x \left(I - P_N\right) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^\infty(x)} \\ &- \left\| \varepsilon_N \nabla_x \sum_{j=1}^{d} \partial_j^2 R_N^j(x,t) * u_N(x,t) \right\|_{L^\infty(x)} \end{aligned}$$

for a.e. t. Applying the same estimates as in the case $\partial_k u_N(x_0,t) > 0$ we prove (32) for the case when $\partial_k u_N(x_0,t) < 0 \; \forall t \in [0,T]$. Finally, the case when $\partial_k u_N(x_0,t)$, satisfying (28), changes sign on $[0,T]$ may be reduced to the previous two cases.

Now we will use the same strategy to prove the uniform boundedness of $u_N(x,t)$. If

$$\|u_N(x,t)\|_{L^\infty(x)} = \max_x u_N(x,t) \; \forall t \in [0,T],$$

then $w_N(t) := \max_x u_N(x,t)$ is a Lipschitz continuous function, and (10) yields for a.e. t

$$\begin{aligned} &\frac{d}{dt} w_N^k(t) + H(0) \\ &\le -\left\| \left(I - P_N\right) H\left(\nabla_x u_N(x,t)\right)\right\|_{L^\infty(x)} - \left\| \varepsilon_N \sum_{j=1}^{d} \partial_j^2 R_N^j(x,t) * u_N(x,t) \right\|_{L^\infty(x)}. \end{aligned}$$

Applying (21), (15), (23), and (12), Lemma 2.1 for the estimation of the first and second term, respectively, and using the uniform boundedness of the gradient we obtain for a.e. t

$$\frac{d}{dt} w_N(t) \le -H(0) + \frac{C_s}{N^{s-1-[d/2]-\theta(s+1/2)}} + C\left(\frac{\log N}{N^\theta}\right)^{3/4} w_N(t).$$

It follows that

$$\frac{d}{dt} w_N(t) \le -H(0) + \frac{C_s}{N^{s-1-[d/2]-\theta(s+1/2)}} + C w_N(t).$$

By the Gronwall lemma the last inequality yields

$$0 \le w_N(t) \le \frac{\left(e^{tC} - 1\right)}{C}\left(-H(0) + \frac{C_s}{N^{s-1-[d/2]-\theta(s+1/2)}}\right) + w_N(0)\, e^{tC} \tag{33}$$

$\forall\ t \in [0,T]$.

Next, if

$$\|u_N(x,t)\|_{L^\infty(x)} \le -\min_x u_N(x,t)\ \forall t \in [0,T],$$

then let $w_N(t) := \min_x u_N(x,t)$ and the similar arguments imply

$$0 \ge w_N(t) \ge \frac{\left(1-e^{tC}\right)}{C}\left(-H(0) + \frac{C_s}{N^{s-1-[d/2]-\theta(s+1/2)}}\right) + w_N(0)\, e^{tC}. \tag{34}$$

Combining (33) and (34) we conclude that

$$\|u_N(x,t)\|_{L^\infty(x)} \le \frac{\left(e^{tC}-1\right)}{C}\left(-H(0) + \frac{C_s}{N^{s-1-[d/2]-\theta(s+1/2)}}\right) + \|u_N(x,0)\|_{L^\infty} e^{tC}. \qquad \square$$

Corollary 3.3. *For any $s \ge 2$ and for any N holds*

$$\|\partial_x^s u_N\|_{L^2(x)} = C_s N^{\theta(s-1/2)},$$
$$\|\partial_x^s u_N\|_{L^2(x,t)} = C_s N^{\theta(s-1)},$$
$$\|\partial_x^s u_N\|_{L^\infty(x)} = C_s N^{\theta(s+1/2+[d/2])}.$$

Proof. The corollary follows from Lemma 3.1 combined with Lemma 3.2. $\square$

Corollary 3.4. *For any $s \ge 2$ and for any N holds*

$$\begin{aligned} &\|\partial_x^r (I-P_N) H(\nabla_x u_N(x,t))\|_{L^2(x)} \le C_s N^{-p_s}, && p_s = s - r - \theta(s+1/2),\\ &\|\partial_x^r (I-P_N) H(\nabla_x u_N(x,t))\|_{L^2(x,t)} \le C_s N^{-\gamma_s}, && \gamma_s = s - r - \theta s,\\ &\|\partial_x^r (I-P_N) H(\nabla_x u_N(x,t))\|_{L^\infty(x)} \le C_s N^{-q_s}, && q_s = p_s - 1 - [d/2]. \end{aligned} \tag{35}$$

Proof. The corollary follows from Lemma 3.2, Corollary 3.3, and estimates (15), (20), and (21). $\square$

4. Semiconcave stability of the approximate solution. In order to prove the main result of this section we need the following auxiliary lemma.

Lemma 4.1. *Let $w_N(x,t) = \frac{\partial^2}{\partial\xi^2} u_N(x,t) = \left\langle \xi, D_x^2 u_N(x,t)\xi \right\rangle$, where $u_N(x,t)$ is a trigonometric polynomial, 2π-periodic in x. Then $\|w_N\|_{L^1(x)} = \left\|2w_N^+\right\|_{L^1(x)}$.*

Proof. $|w_N(x,t)| \equiv 2w_N(x,t)^+ - w_N(x,t)$. By the Green theorem $\int_\Omega w_N(x,t)\,dx$ depends solely on the boundary data (which in our case of a periodic Cauchy problem vanishes). Therefore $\int_\Omega |w_N(x,t)|\,dx = \int_\Omega 2w_N(x,t)^+\,dx$. $\square$

Lemma 4.2 (semiconcave stability of the approximation). *The spectral viscosity approximation $u_N(x,t)$ is semiconcave stable, i.e., $\exists k(t) \in L^1(0,T)$, $T < \infty$, such that $\sup_{x\in\Omega,|\xi|=1} \left\langle \xi, D_x^2 u_N(x,t)\xi \right\rangle^+ \le k(t)$ for $t \in [0,T]$ and $\forall\ N$.*

Proof. Let $w_N := \frac{\partial^2}{\partial\xi^2} u_N(x,t) = \left\langle \xi, D_x^2 u_N(x,t)\xi \right\rangle$. Differentiating (10) with respect to x_j, x_k, and taking the inner product with a constant unit vector ξ yields

$$\partial_t w_N(x,t) + \left\langle \xi, D_x^2 u_N(x,t)\cdot D_p^2 H \cdot D_x^2 u_N(x,t)\xi \right\rangle + \langle \nabla_p H, \nabla_x w\rangle \tag{36}$$
$$= \varepsilon_N \Delta_x w_N - \varepsilon_N \sum_{j=1}^d \partial_j^2 R_N^j(x,t) * w_N + \left\langle \xi, D_x^2 (I-P_N) H(\nabla_x u_N(x,t))\,\xi \right\rangle.$$

The strict convexity of H (14) and the Cauchy–Schwarz inequality imply

$$\left\langle \xi, D_x^2 u_N(x,t) \cdot D_p^2 H \cdot D_x^2 u_N(x,t)\xi \right\rangle \geq \alpha \left\| D_x^2 u_N(x,t)\xi \right\|^2 \geq \alpha w_N^2. \tag{37}$$

Let $W_N(t) = w_N(x_0,t) = \max_{x\in\Omega} w_N(x,t)$. It is a Lipschitz continuous and therefore a.e. differentiable. Without loss of generality assume that $W_N(t) > 0$ for $t \in [0,T]$. Then $\nabla_x w_N(x_0,t) = 0$, $\Delta_x w_N(x_0,t) \leq 0$, and by (36) and (37) we have for a.e. t

$$\begin{aligned} &\frac{d}{dt} W_N + \alpha W_N^2 \\ &\leq -\varepsilon_N \left(\sum_{j=1}^{d} \partial_j^2 R_N^j * w_N \right)(x_0,t) + \left\langle \xi, D_x^2 \left(I - P_N\right) H\left(\nabla_x u_N(x_0,t)\right) \xi \right\rangle. \end{aligned}$$

Equation (13), Lemma 2.2, and Lemma 4.1 yield

$$\begin{aligned} I_N &= \left| -\varepsilon_N \left(\sum_{j=1}^{d} \partial_j^2 R_N^j * w_N \right)(x_0,t) \right| \leq \left\| -\varepsilon_N \sum_{j=1}^{d} \partial_j^2 R_N^j * w_N \right\|_{L^\infty(x)} \\ &\leq \left\| -\varepsilon_N \sum_{j=1}^{d} \partial_j^2 R_N^j \right\|_{L^\infty(x)} \|w_N\|_{L^1(x)} = 2C_1 \left\| w_N^+ \right\|_{L^1(x)} \leq 2\left|\Omega\right| C_1 W_N. \end{aligned}$$

By (35)

$$\begin{aligned} II_N &= \left| \left\langle \xi, D_x^2 \left(I - P_N\right) H\left(\nabla_x u_N(x_0,t)\right) \xi \right\rangle \right| \\ &\leq \left\| D_x^2 \left(I - P_N\right) H\left(\nabla_x u_N(x,t)\right) \right\|_{L^\infty(x)} \\ &\leq \frac{C_s}{N^{s-3-[d/2]-\theta(s+1/2)}} \leq \text{Const, for } s \geq \frac{3 + [d/2] + \theta/2}{1-\theta}. \end{aligned}$$

Thus, for a.e. t

$$\frac{d}{dt} W_N + \alpha W_N^2 \leq I_N + II_N \leq C_2 W_N + C_3, \ \alpha > 0.$$

Integration in time of the last inequality yields

$$\begin{aligned} W_N(t) \leq k(t) &= W_N(0)\, e^{C_2 t} + \frac{C_3\left(e^{C_2 t} - 1\right)}{C_2}, \\ k(t) &\in L^1(0,T) \ \text{ for } T < \infty. \qquad \square \end{aligned}$$

Lemma 4.1 and Lemma 4.2 imply the following corollary.

COROLLARY 4.3. *For any vector* $|\xi| = 1$, $\left\| \left\langle \xi, D_x^2 u_N(x,t)\xi \right\rangle \right\|_{L^1(x)} \leq 2\left|\Omega\right| Ck(t)$.

5. Convergence of the method and the error estimate. To prove convergence we get the following truncation error estimate.

LEMMA 5.1. *Define the truncation error* $F(u_N) = \partial_t u_N + H(\nabla_x u_N)$. *Then* $\forall$ N *holds*

$$\left\| F(u_N) \right\|_{L^1(x,t)} = CN^{-\theta} \log N + C_s N^{-s+s\theta}. \tag{38}$$

Moreover, if $\|\partial^s u_N\|_{L^2(x,t)} \leq$ *Const for some* $s \geq 2$ *and* $\forall\ N \geq N_0$*, then*

$$\|F(u_N)\|_{L^1(x,t)} \leq C_s N^{-(s/4+1/2)\theta} \log N + C_s N^{-(s+1)}. \tag{39}$$

Proof. From (4) it follows that

$$\begin{aligned}
&\|F(u_N)\|_{L^1(x,t)} \\
&\leq \left\| \varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t) \right\|_{L^1(x,t)} + \|(I - P_N) H (\nabla_x u_N(x,t))\|_{L^1(x,t)}.
\end{aligned}$$

In view of (9), Corollary 4.3, (12), and Lemma 2.1 we have

$$\begin{aligned}
I_N &= \left\| \varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t) \right\|_{L^1(x,t)} \\
&= \left\| \Delta u_N(x,t) - \varepsilon_N \sum_{j=1}^{d} R_N^j(x,t) * \partial_j^2 u_N(x,t) \right\|_{L^1(x,t)} \\
&\leq \varepsilon_N \|\Delta u_N(x,t)\|_{L^1(x,t)} + \sum_{j=1}^{d} \left\| \varepsilon_N R_N^j(x,t) \right\|_{L^1(x,t)} \left\| \partial_j^2 u_N(x,t) \right\|_{L^1(x,t)} \\
&\leq C\varepsilon_N + CN^{-\theta} \log N \leq CN^{-\theta} \log N.
\end{aligned}$$

By (35)

$$II_N = \|(I - P_N) H (\nabla_x u_N(x,t))\|_{L^1(x,t)} \leq C_s N^{-s+s\theta} \quad \forall s \geq 2. \tag{40}$$

It follows that $II_N \leq$Const $N^{-\theta}$ $\forall s \geq \frac{\theta}{1-\theta}$. Now (38) follows from estimates for I_N and II_N.

Next, assume $\|\partial^s u_N\|_{L^2(x,t)} \leq$Const $(\forall s \geq 2)$. It follows from (7), (6), and the approximation error estimate [4] that for $s \geq 2$

$$\begin{aligned}
I_N &= \left\| \varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t) \right\|_{L^1(x,t)} \\
&\leq CN^{-s\theta/4} \left\| \varepsilon_N \partial_x^s \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * u_N(x,t) \right\|_{L^2(x,t)} \\
&= C\, N^{-s\theta/4} \left\| \varepsilon_N \sum_{j=1}^{d} \partial_j^2 Q_N^j(x,t) * \partial_x^s u_N(x,t) \right\|_{L^2(x,t)}.
\end{aligned}$$

Using (9), (12), and Lemma 2.1 we obtain

$$\begin{aligned}
I_N &\leq C\, N^{-s\theta/4} \sum_{j=1}^{d} \left(\varepsilon_N + \left\| \varepsilon_N R_N^j(x,t) \right\|_{L^1(x,t)} \right) \left\| \partial_j^2 \partial_x^s u_N(x,t) \right\|_{L^2(x,t)} \\
&\leq CN^{-(1+s/4)\theta} \log N \left\| \partial_j^2 \partial_x^s u_N(x,t) \right\|_{L^2(x,t)}.
\end{aligned}$$

Finally, by Parseval's identity and Young's inequality

$$\sum_{j=1}^{d} \left\|\partial_j^2 \partial_x^s u_N(x,t)\right\|_{L^2(x,t)} \leq C_s \left\|\partial_x^{s+2} u_N(x,t)\right\|_{L^2(x,t)};$$

hencefore,

$$I_N \leq C_s N^{-(1+s/4)\theta} \log N \left\|\partial_x^{s+2} u_N(x,t)\right\|_{L^2(x,t)}.$$

In order to estimate II_N we use (20), (15), and (16) :

$$II_N = \|(I-P_N)\, H\left(\nabla_x u_N(x,t)\right)\|_{L^1(x,t)} \leq C_s N^{-(s+1)} \left\|\partial_x^{s+2} u_N(x,t)\right\|_{L^2(x,t)}.$$

Since the two last inequalities hold for any integer $s \geq 0$, the second estimate of the lemma (39) follows. The proof of the lemma is complete. ∎

THEOREM 5.2 (convergence of the spectral viscosity method). *Let $u(x,t)$ be the unique viscosity solution of the 2π-periodic initial value problem* (1) *and let $u_N(x,t)$ be the spectral viscosity approximation, i.e., N-trigonometric polynomial satisfying* (4). *Then for $t \in [0,T]$ and $\forall\ N$ we have the bound*

$$\begin{aligned}\|u(x,t)-u_N(x,t)\|_{L^1(x)} \quad &\leq C\,(T)\,\|u(x,0)-u_N(x,0)\|_{L^1(x)} \\ &\quad + C\,(T)\,N^{-\theta}\log N + C(s,T) N^{-s(1-\theta)}.\end{aligned}$$

Moreover, if $\|\partial^s u_N\|_{L^2(x,t)} \leq$Const for some $s \geq 2$ and $\forall\ N \geq N_0$, then

$$\begin{aligned}\|u(x,t)-u_N(x,t)\|_{L^1(x)} \quad &\leq C\,(T)\,\|u(x,0)-u_N(x,0)\|_{L^1(x)} \\ &\quad + C\,(s,T)\,N^{-(s/4+1/2)\theta}\log N + C\,(s,T)\,N^{-(s+1)}.\end{aligned}$$

Proof. The theorem follows directly from Theorem 2.1 in [15], semiconcave stability (Lemma 4.2), and the truncation error estimate (Lemma 5.1). ∎

Remark. The work is underway on the spectral viscosity method for the initial-boundary Hamilton–Jacobi problem with nonperiodic boundary conditions.

Acknowledgments. I would like to express my deep gratitude to Chi-Wang Shu for helpful suggestions during the preparation of this paper. I am also greatly indebted to Eitan Tadmor and to unknown referees for comments on a draft of this paper. I wish to thank Jan Hesthaven and Jing Shi for stimulating conversations, which motivated the present study.

REFERENCES

[1] R. ABGRALL, *Numerical discretization of the first-order Hamilton–Jacobi equation on triangular meshes*, Comm. Pure Appl. Math., 49 (1996), pp. 1339–1373.
[2] R. A. ADAMS, *Sobolev Spaces*, Academic Press, San Diego, 1978.
[3] T. BARTH AND J. SETHIAN, *Numerical schemes for the Hamilton–Jacobi and level set equations on triangulated domains*, J. Comput. Phys., 145 (1998), pp. 1–40.
[4] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI, AND T. ZANG, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, 1988.
[5] G.-Q. CHEN, Q. DU, AND E. TADMOR, *Spectral viscosity approximations to multidimensional scalar conservation laws*, Math. Comp., 61 (1993), pp. 629–643.
[6] M. G. CRANDALL AND P. L. LIONS, *Viscosity solutions of Hamilton–Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), pp. 1–42.

[7] M. G. Crandall and P. L. Lions, *Two approximations of solutions of Hamilton–Jacobi equations*, Math. Comp., 43 (1984), pp. 1–19.

[8] M. G. Crandall, L. C. Evans, and P. L. Lions, *Properties of viscosity solutions of Hamilton–Jacobi equations,* Trans. Amer. Math. Soc., 282 (1984), pp. 487–502.

[9] C. Hu and C.-W. Shu, *A discontinuous Galerkin finite element method for Hamilton–Jacobi equations,* SIAM J. Sci. Comput., 21 (1999), pp. 666–690.

[10] G.-S. Jiang and D. Peng, *Weighted ENO schemes for Hamilton–Jacobi equations*, SIAM J. Sci. Comput., 21 (2000), pp. 2126–2143.

[11] S. Jin and Z. Xin, *Numerical passage from systems of conservation laws to Hamilton–Jacobi equations and relaxation schemes*, SIAM J. Numer. Anal., 35 (1998), pp. 2385–2404.

[12] S. N. Kruzhkov, *The Cauchy problem in the large for nonlinear equations and for certain quasilinear systems of the first order with several variables*, Soviet Math. Dokl., 5 (1964), pp. 493–496.

[13] F. Lafon and S. Osher, *High order two dimensional nonoscillatory methods for solving Hamilton–Jacobi scalar equations*, J. Comput. Phys., 123 (1996), pp. 235–253.

[14] O. Lepsky, C. Hu, and C.-W. Shu, *Analysis of the discontinuous Galerkin method for Hamilton–Jacobi equations*, Appl. Numer. Math., 33 (2000), pp. 423–434.

[15] C.-T. Lin and E. Tadmor, *L^1-stability and error estimates for approximate Hamilton–Jacobi solutions*, Numer. Math., to appear.

[16] C.-T. Lin and E. Tadmor, *High-resolution nonoscillatory central schemes for Hamilton–Jacobi equations*, SIAM J. Sci. Comput., 21 (2000), pp. 2163–2186.

[17] P. L. Lions, *Generalized Solutions of Hamilton–Jacobi Equations*, Pitman, Boston, 1982.

[18] H. Ma, *Chebyshev–Legendre spectral viscosity method for nonlinear conservation laws*, SIAM J. Numer. Anal., 35 (1998), pp. 869–892.

[19] Y. Maday, S. M. Ould Kaber, and E. Tadmor, *Legendre pseudospectral viscosity method for nonlinear conservation laws*, SIAM J. Numer. Anal., 30 (1993), pp. 321–342.

[20] S. Osher and C.-W. Shu, *High-order essentially nonoscillatory schemes for Hamilton–Jacobi equations*, SIAM J. Numer. Anal., 28 (1991), pp. 907–922.

[21] E. Rouy and A. Tourin, *A viscosity solutions approach to shape-from-shading*, SIAM J. Numer. Anal., 29 (1992), pp. 867–884.

[22] S. Schochet, *The rate of convergence of spectral-viscosity methods for periodic scalar conservation laws*, SIAM J. Numer. Anal., 27 (1990), pp. 1142–1159.

[23] P. E. Souganidis, *Approximation schemes for viscosity solutions of Hamilton–Jacobi equations*, J. Differential Equations, 59 (1985), pp. 1–43.

[24] E. Tadmor, *Convergence of spectral methods for nonlinear conservation laws,* SIAM J. Numer. Anal., 26 (1989), pp. 30–44.

[25] E. Tadmor, *Total variation and error estimates for spectral viscosity approximations*, Math. Comp., 60 (1993), pp. 245–256.

[26] R. L. Wheeden and A. Zygmund, *Measure and Integral. An Introduction to Real Analysis*, Marcel Dekker, New York, 1977.