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CRITICAL THRESHOLDS IN A QUASILINEAR HYPERBOLIC MODEL OF BLOOD FLOW

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Abstract. Critical threshold phenomena in a one dimensional quasi-linear hyperbolic model of blood flow with viscous damping are investigated. We prove global in time regularity and finite time singularity formation of solutions simultaneously by showing the critical threshold phenomena associated with the blood flow model. New results are obtained showing that the class of data that leads to global smooth solutions includes the data with negative initial Riemann invariant slopes and that the magnitude of the negative slope is not necessarily small, but it is determined by the magnitude of the viscous damping. For the data that leads to shock formation, we show that shock formation is delayed due to viscous damping.

1. **Introduction.** We consider a one-dimensional, reduced model of a viscous, incompressible, Newtonian fluid flow in a cylindrical tube. This model can be derived from the Navier-Stokes equations assuming axially symmetric flow in a cylindrical tube with elastic walls and with small aspect ratio $\epsilon = R/L$. Here R is the tube radius and L is the tube length. The model has been used by many authors to simulate blood flow through cylindrical sections of the cardiovascular system or through the network of blood vessel [2, 4, 21, 22, 20] and as such presents a benchmark for one-dimensional cardiovascular flow studies.

The model equations describe conservation of mass and momentum given in terms of the cross-sectional area $A(t,x)$ and the flow rate $m(t,x) = A(t,x)U(t,x)$. Here U denotes the averaged axial velocity $V_x(x,r,t)$ across the cross-section of the vessel of radius $R(t,x)$:

$$U(t,x) = \frac{1}{R(t,x)^2}\int_0^{R(t,x)} 2rV_x(x,r,t)dr.$$

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The averaged leading-order equations describing conservation of mass and balance of axial momentum read:

$$\frac{\partial A}{\partial t} + \frac{\partial m}{\partial x} = 0, \tag{1.1}$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}(\alpha \frac{m^2}{A}) + \frac{A}{\rho}\frac{\partial}{\partial x}p(A) = -\mu \frac{m}{A}, \tag{1.2}$$

$t > 0$, $x \in \mathbb{R}$. Here $\rho > 0$ denotes fluid density. $\alpha > 1$ and $\mu > 0$ are constants defined by

$$\alpha = \frac{1}{R^2U^2}\int_0^R 2rV_x^2 dr \geq 1 \quad \text{and} \quad \mu = \frac{2\alpha}{\alpha - 1}\nu,$$

where $\nu > 0$ is fluid viscosity. Fluid pressure is denoted by $p(A)$. It is through this term that modeling of vessel wall mechanics comes into play. We will be assuming

$$p(A) = G_0\left(\left(\frac{A}{A_r}\right)^{\frac{\beta}{2}} - 1\right)$$

where $A_r > 0$ is the reference cross-sectional area, G_0 describes the stiffness of the vessel wall and $\beta > 0$ captures the linearity/nonlinearity of the stress-strain response. For $\beta = 1$ one gets the well-known Law of Laplace. In this case $G_0 = Eh\left((1-\sigma^2)A_r^{1/2}\right)^{-1}$, where E is the Youngs modulus, h is the vessel wall thickness, and σ is the Poisson ratio. In this manuscript we will consider $\beta = 2$ since this captures well the nonlinear pressure-radius relationship observed in experiments [21]. Extensions of the ideas in this paper to the case $\beta = 1$ is straight forward. A detailed analysis of the conditions under which system (1.1) (1.2) is a good approximation of the full, three-dimensional model can be found in [3].

Regularity, existence and stability of solutions to system (1.1) (1.2) was studied in [2, 1]. In both works it was assumed that the viscous damping term given by $\mu m/A$ is negligible. The main contribution of this manuscript is in studying the influence of the viscous damping on the solutions to system (1.1) (1.2). We are concerned with both global in time regularity and finite time singularity development as in [6, 13, 14, 19, 24]. We obtain two new results: (1) For $\mu > 0$ the class of data that gives rise to smooth solutions is richer than that for the case $\mu = 0$; and (2) For the physiologically relevant data that give rise to shock formation, shock formation is delayed in time for the case when $\mu > 0$. The precise estimates on the data are given in Theorem 2.1 for the case when $\alpha = 1$ and $\mu > 0$, and a generalization to the case $\alpha > 1$, $\alpha - 1$ small, and $\mu > 0$ is given in Section 5.

From the analysis point of view, this work presents a generalization of the classical results in hyperbolic conservation laws in the following sense. It is well known that solutions to systems of quasi-linear hyperbolic conservation laws break down in finite time. Lax [8] studied a development of singularities in homogeneous, genuinely nonlinear 2×2 systems of equations. It was shown that solutions blow up it finite time if a Riemann invariant has a negative initial slope. John [7] extended the above results to systems of n equations. Nishida [18] proved that global smooth solutions to the p-system with damping exist if the initial Riemann invariant slopes are small. In this work we show that the initial Riemann invariant slopes that lead to global smooth solutions to system (1.1) (1.2) can be negative and not necessarily small, but of order of the damping coefficient $\mu > 0$. We show this by using an approach similar to that of Li and Liu presented in [10, 11, 12] where a large-time regularity and a finite-time breakdown of solutions was studied for a 2×2 hyperbolic relaxation system describing traffic flow. We use the special features of system (1.1) (1.2) to track the slope dynamics of the Riemann invariants and to track the dynamics of the system effectively.

2. Statement of the main result for $\alpha = 1$ and $\mu > 0$.

We consider system (1.1) (1.2) defined on an infinitely long cylindrical domain $x \in R$ for $t > 0$ with Cauchy data

$$(A, m)(0, x) = (A_0, m_0)(x), \quad x \in R. \tag{2.1}$$

The eigenvalues of system (1.1) (1.2) are given by

$$\lambda_1 = \frac{m}{A} - \sqrt{\frac{G_0 A}{\rho A_r}} \leq \frac{m}{A} + \sqrt{\frac{G_0 A}{\rho A_r}} = \lambda_2. \tag{2.2}$$

System (1.1) (1.2) is strictly hyperbolic if $A > 0$. Canic and Kim [2] showed that if the system is strictly hyperbolic initially and if the velocity prescribed on the left boundary satisfies the sub-characteristic condition, $\lambda_1 < x_1'(t) < \lambda_2$, then it will stay strictly hyperbolic in the domain of the existence of a smooth solution. The sub-characteristic condition for a class of 2×2 relaxation systems is required in [5, 9, 15, 16, 17, 23] for linear and nonlinear stability of shock waves.

To state our main results, we introduce the following notation:

$$r^{\pm} = A^{-\frac{1}{4}} \left(\frac{m}{A} \pm 2\sqrt{\frac{G_0 A}{\rho A_r}} \right)_x, \tag{2.3}$$

$$g(A) = \int_{A_{\min}}^{A} \frac{\mu}{2\xi^{\frac{9}{4}}} (1 + \delta) d\xi, \tag{2.4}$$

and

$$G^{\pm}(A, m) = -\frac{6\mu}{A^{\frac{5}{4}}} \left(1 \mp \frac{m}{A} \sqrt{\frac{A_r \rho}{G_0 A}} \right) - g(A), \tag{2.5}$$

where $(\cdot)_x$ denotes the partial derivative with respect to x, $\delta > 0$ is defined in (3.10) and $A_{\min}$ is the minimum of A as defined below. Results in the following theorem will be stated in terms of the derivatives of the Riemann invariants at time $t = 0$ via the expression (2.3).

Theorem 2.1. *Consider the system (1.1) (1.2) with $\alpha = 1$, $\rho / G_0 << 1$, $\mu > 0$ and $\beta = 2$, subject to C^1 bounded initial data $(A_0, m_0)(x)$.*

There are constants $0 < A_{\min} < A_{\max}$, $m_{\min} < m_{\max}$ depending only on the bounds of initial data (A_0, m_0) such that for all $x \in \mathbb{R}$,

$$(A(t, x), \ m(t, x)) \in D = [A_{\min}, \ A_{\max}] \times [m_{\min}, \ m_{\max}].$$

Furthermore,

i) if initially for at least one point $x \in \mathbb{R}$ either

$$r^{+}(0, x) < g(A_0(x)) + \inf_{(A,m) \in D} G^{+}(A, m)$$

or

$$r^{-}(0, x) < g(A_0(x)) + \inf_{(A,m) \in D} G^{-}(A, m)$$

holds, then the solution must develop a finite time singularity where either r^+ or r^- goes to $-\infty$;
ii) if the initial data (m_0, A_0) is such that

$$\inf_{(A,\, m)\in D}(-g(A) - G^{\pm}(A,m)) \geq \sup_{A\in I} g(A) - \inf_{A\in I} g(A) + C\mu\delta, \tag{2.6}$$

where $\delta > 0$, defined in (3.10), and $C > 0$, depending only on initial data $(A_0, m_0)(x)$, and $I = [A_{\min},\ \ A_{\max}]$, then the solution remains smooth for all time, provided that for all $x \in \mathbb{R}$ the following holds

$$r^{\pm}(0,x) \geq g(A_0(x)) + \sup_{(A,m)\in D} G^{\pm}(A,m) + C\mu\delta. \tag{2.7}$$

Remark 2.1. Under condition (2.6), the lower thresholds on the right hand side of (2.7) is nonpositive and is proportional to μ. Thus the set of initial data leading to global regularity is rich. In particular, it allows initial Riemann invariant slopes to be negative. This is in sharp contrast with the generic breakdown in the homogeneous hyperbolic systems [8].

Now we outline the plan for the remaining part of the paper. In Section 3, we reformulate the problem in terms of its Riemann invariants. We then derive a closed dynamical system for two nonlinear quantities involving solution derivatives and state the critical threshold results. In Section 4, we establish both lower and upper thresholds for the corresponding system of ordinary differential equations, which, when applied to the derived slope dynamics, leads to the claimed threshold results. In Section 5, we extend the results to the case $\alpha > 1$ and $\alpha - 1$ small. Concluding remarks are provided in Section 6.

3. **The reformulated problem.**

System (1.1) (1.2) has the following Riemann invariants:

$$R^{\pm} = U \pm k(A), \quad k(A) := 2\sqrt{\frac{G_0 A}{\rho A_r}}. \tag{3.1}$$

$$U = \frac{1}{2}(R^- + R^+), \quad A = k^{-1}\left(\frac{1}{2}(R^+ - R^-)\right). \tag{3.2}$$

They satisfy

$$R^-_t + \lambda_1 R^-_x = -\frac{\mu U}{A}, \tag{3.3}$$

$$R^+_t + \lambda_2 R^+_x = -\frac{\mu U}{A}, \tag{3.4}$$

$t > 0$, $x \in \mathbb{R}$, subject to the corresponding initial data

$$R^{\pm}(0,x) = R^{\pm}_0(x) = \frac{m_0(x)}{A_0(x)} \pm k(A_0(x)), \quad x \in \mathbb{R}. \tag{3.5}$$

Through this reformulated system, there exists a uniform invariant region for the system (1.1) (1.2), see [16]. Thus there exist constants $0 < A_{\min} < A_{\max}$ and $m_{\min} < m_{\max}$, depending only on the initial data (A_0, m_0), such that

$$(A(t,x),\ \ m(t,x)) \in D = [A_{\min},\ \ A_{\max}] \times [m_{\min},\ \ m_{\max}],\ \ \forall\, x \in \mathbb{R},$$

for $t \geq 0$.

We now estimate the derivatives of the solution through

$$r^{\pm} = A^{-\frac{1}{4}} R_x^{\pm}. \tag{3.6}$$

It is clear that the boundedness of (A_x, m_x) is equivalent to the boundedness of $r^{\pm}$ and $A \geq A_{\min} > 0$.

In order to estimate the quantities $r^{\pm}$, we first derive the dynamical systems satisfied by $r^{\pm}$. Define function h via

$$h = -\frac{1}{4} \ln A, \tag{3.7}$$

and denote

$$a = \frac{3}{4} e^{-h} = \frac{3}{4} A^{\frac{1}{4}} > 0. \tag{3.8}$$

We further set

$$b^{\pm} = \frac{\mu}{2A}\left(1 \mp U\sqrt{\frac{A_r \rho}{G_0 A}}\right), \quad g = \int_{A_{\min}}^{A} \frac{\mu}{2\xi^{\frac{9}{4}}}(1+\delta) d\xi. \tag{3.9}$$

Since in the abdominal aorta typically G_0 is at the order of $10^4 N/m^2 - 10^5 N/m^2$, $\rho = 1050 kg/m^3$, the cross sectional area A is bounded and of the same order of magnitude as the unstressed cross sectional area A_r, the characteristic axial velocity is bounded and of order $0.1 m/s$, we choose initial data (A_0, m_0) in the above range so that

$$\left| |U|_{\max} \sqrt{\frac{A_r \rho}{G_0 A_{\min}}} \right| < \delta \ll 1 \tag{3.10}$$

for some $0 < \delta \ll 1$, where $|U|_{\max}$ is the maximum of $|U|$ which can be obtained through (3.2) and the boundedness of $R^{\pm}$.

Thus (3.9) and (3.10) imply that

$$b^{\pm} > \frac{\mu}{2A}(1-\delta) > 0$$

and

$$b^{\pm} = \frac{\mu}{2A}(1+\delta) - w^{\pm} \tag{3.11}$$

where

$$0 < w^{\pm} = \frac{\mu}{2A}\left(\delta \pm U\sqrt{\frac{A_r \rho}{G_0 A}}\right) < \frac{\mu}{A}\delta \ll 1. \tag{3.12}$$

We now derive the dynamical systems satisfied by $r^{\pm}$.

Lemma 3.1. *The dynamical systems for $r^{\pm}$ are given by the following*

$$(\partial_t + \lambda_1 \partial_x)(r^- - g) + a(r^-)^2 + b^- r^- = w^+ r^+, \tag{3.13}$$

$$(\partial_t + \lambda_2 \partial_x)(r^+ - g) + a(r^+)^2 + b^+ r^+ = w^- r^-, \tag{3.14}$$

$t > 0$, $x \in \mathbb{R}$.

Proof. Set $s^{\pm} = R_x^{\pm}$, and differentiate (3.3) and (3.4) w.r.t. x, to obtain

$$s_t^- + \lambda_1 s_x^- + \left(\frac{\partial \lambda_1}{\partial R^-} s^- + \frac{\partial \lambda_1}{\partial R^+} s^+\right) s^- = \left(-\frac{\mu U}{A}\right)_x, \tag{3.15}$$

$$s_t^+ + \lambda_2 s_x^+ + \left(\frac{\partial \lambda_2}{\partial R^-} s^- + \frac{\partial \lambda_2}{\partial R^+} s^+\right) s^+ = \left(-\frac{\mu U}{A}\right)_x, \tag{3.16}$$

$t > 0$, $x \in \mathbb{R}$.

From (2.2) and (3.1), we obtain

$$\lambda_1 = \frac{1}{4}R^+ + \frac{3}{4}R^-, \ \lambda_2 = \frac{1}{4}R^- + \frac{3}{4}R^+. \tag{3.17}$$

This implies that both characteristic families are genuinely nonlinear.

Differentiate function h along the first characteristic curve $x_1'(t) = \lambda_1$ and use (3.1), (3.3), (3.4) and (3.17) to derive

$$h' = -\frac{A'}{4A} = \frac{(R^- - R^+)'(A)}{8Ak'(A)} = \frac{(\lambda_2 - \lambda_1)R_x^+}{4k(A)} = \frac{1}{4}R_x^+ = \frac{1}{4}s^+. \tag{3.18}$$

Substitution of (3.1), (3.11) and (3.18) into (3.15), yields

$$(s^-)' + h's^- + \frac{3}{4}(s^-)^2 + b^-s^- = -\frac{\mu}{2A}(1+\delta)s^+ + w^+s^+. \tag{3.19}$$

Multiplying the above equation by e^h, we obtain

$$(r^-)' + a(r^-)^2 + b^-r^- = g' + w^+e^hs^+,$$

where r^-, a, b^- and w^+ are defined in (3.6), (3.8), (3.9) and (3.12) respectively. From (3.9) and (3.18), we see that the first term on the right hand side of (3.19) is nothing but g'. Thus we derived equation (3.13).

Equation (3.14) is derived in a similar way. Thus Lemma 3.1 is proved. □

4. **Critical thresholds.** The reformulated dynamical systems (3.13) (3.14) enable us to identify both upper and lower thresholds for $r^\pm$. This is stated in Theorem 4.1. The main result of this paper, namely, Theorem 2.1, will follow directly from Theorem 4.1.

Theorem 4.1. *Assume that the initial data (A_0, m_0) is such that (3.10) holds.*

i) If at least at one point $x \in \mathbb{R}$, either

$$r^+(0,x) < g(A_0(x)) + \inf_{(A,\, m)\in D}\left(-g(A) - \frac{b^+(A,m)}{a(A)}\right) \tag{4.1}$$

or

$$r^-(0,x) < g(A_0(x)) + \inf_{(A,\, m)\in D}\left(-g(A) - \frac{b^-(A,m)}{a(A)}\right) \tag{4.2}$$

holds, then the solution of system (3.13) (3.14) must develop a singularity at finite time.

ii) If

$$\inf_{(A,\, m)\in D}\left(\frac{b^\pm(A,m)}{a(A)}\right) \geq \sup_{A\in I} g(A) - \inf_{A\in I} g(A) + C\mu\delta, \tag{4.3}$$

where $\delta > 0$, defined in (3.10), and $C > 0$, depending only on initial data $(A_0, m_0)(x)$, and $I = [A_{\min},\ A_{\max}]$, then the solution of system (3.13) (3.14) remains smooth for all time, provided that for all $x \in \mathbb{R}$

$$r^\pm(0,x) \geq g(A_0(x)) + \sup_{(A,\, m)\in D}\left(-g(A) - \frac{b^\pm(A,m)}{a(A)}\right) + C\mu\delta. \tag{4.4}$$

Moreover, the right hand side of (4.4) is nonpositive.

Proof. We first prove that under condition (4.3), the right hand side of (4.4) is nonpositive. Indeed,

$$
\begin{aligned}
&g(A_0(x)) + \sup_{(A,m)\in D}\left(-g(A) - \frac{b^{\pm}(A,m)}{a(A)}\right) + C\mu\delta \\
\le & g(A_0(x)) + \sup_{A\in I}(-g(A)) + \sup_{(A,m)\in D}\left(-\frac{b^{\pm}(A,m)}{a(A)}\right) + C\mu\delta \\
\le & \sup_{A\in I} g(A) - \inf_{A\in I} g(A) - \inf_{(A,m)\in D}\frac{b^{\pm}(A,m)}{a(A)} + C\mu\delta \\
\le & 0.
\end{aligned}
$$

Along each characteristic field, under (3.10) (3.12), equations (3.13) (3.14) for $r^{\pm}$ are perturbations of ordinary differential equations of the form

$$\frac{d}{dt}(r-g) + ar^2 + br = 0, \quad r(0) = r_0.$$

The above equation can be written as

$$\frac{d}{dt}B + a(t)(B - b_1(t))(B - b_2(t)) = 0, \quad B(0) = B_0 \tag{4.5}$$

where

$$B = r - g, \quad b_2 = -g, \quad b_1 = -g - \frac{b}{a}, \quad B_0 = r_0 - g(0).$$

Threshold results for ordinary differential equation (4.5) are given in Lemma 3.1 in [12], which will be stated below. Applying this lemma to equations (3.13) (3.14), we see that

$$B = r^{\pm} - g, \quad b_2 = -g, \quad b_1 = -g - \frac{b^{\pm}}{a}.$$

It is easy to check that the conditions in the lemma, namely, $a > 0$, $b_1 \le b_2$ and a, b_1, b_2 are uniformly bounded for all time, are satisfied. Under condition(3.10), (3.12) holds. Modifying the proof of Lemma 3.1 in [12], we obtain the desired threshold conditions (4.1)-(4.4). ☐

Lemma 4.1. *Consider equation (4.5) with* $\inf a > 0$, $b_1 \le b_2$ *and such that* a, b_1, b_2 *are uniformly bounded. We have*
(i)If initial data B_0 *is such that* $B_0 < \min b_1$*, then solution to (4.5) will experience a finite time blow up at* $0 < t_* \le t^* < +\infty$

$$\lim_{t\to t_*} B(t) = -\infty,$$

where t^* *satisfies*

$$\int_0^{t^*} a(s)ds = \frac{1}{\min b_2 - \min b_1}\ln\left(1 + \frac{\min b_2 - \min b_1}{\min b_1 - B_0}\right)$$

which equals to $\frac{1}{\min b_2 - B_0}$ *if* $\min b_2 = \min b_1$.
(ii) If there exists a constant $\bar{b}$ *such that*

$$b_1(t) \le \bar{b} \le b_2(t),$$

then (4.5) admits a unique global bounded solution satisfying

$$\bar{b} \le B(t) \le \max\{B_0, \max b_2\}$$

provided that $B_0 \ge \bar{b}$.

5. **The case $\alpha > 1$, $\alpha - 1$ small and $\mu > 0$.** Now we consider the case $\alpha > 1$, $\alpha - 1$ small and $\mu > 0$. The aim is to derive the dynamical system for Riemann invariants as in the previous case and see that the resulting system can be written as a perturbation of system (3.15) and (3.16) in terms of the parameter $\alpha - 1$. Then, for small enough $\alpha - 1$ the same results will hold as for the unperturbed system corresponding to $\alpha = 1$.

The eigenvalues of system (1.1) (1.2) for $\alpha > 1$ and $\alpha - 1$ small can be written as:

$$\lambda_1(\alpha) = \frac{\alpha m}{A} - \sqrt{\alpha(\alpha-1)(\frac{m}{A})^2 + \frac{G_0 A}{\rho A_r}} = \lambda_1(1) + O(\alpha - 1)\frac{m}{A}, \tag{5.1}$$

$$\lambda_2(\alpha) = \frac{\alpha m}{A} + \sqrt{\alpha(\alpha-1)(\frac{m}{A})^2 + \frac{G_0 A}{\rho A_r}} = \lambda_2(1) + O(\alpha - 1)\frac{m}{A}, \tag{5.2}$$

where $\lambda_1(1)$ and $\lambda_2(1)$ are the eigenvalues of system (1.1) (1.2) when $\alpha = 1$ and they are defined in (2.2).

For systems of two hyperbolic equations, there exist the Riemann invariants $R^-(\alpha)$ and $R^+(\alpha)$ such that

$$\nabla R^-(\alpha) \cdot r_2 = 0,\ \nabla R^+(\alpha) \cdot r_1 = 0,$$

where r_1 and r_2 are the right eigenvectors corresponding to $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ respectively and

$$R_t^-(\alpha) + \lambda_1(\alpha) R_x^-(\alpha) = -\frac{\mu}{A} U, \tag{5.3}$$

$$R_t^+(\alpha) + \lambda_2(\alpha) R_x^+(\alpha) = -\frac{\mu}{A} U, \tag{5.4}$$

$t > 0$, $x \in \mathbb{R}$.

From (5.1) (5.2) we derive

$$R^{\pm}(\alpha) = R^{\pm}(1) + O(\alpha - 1)\frac{m}{A^2} = R^{\pm}(1) + O(\alpha - 1)\frac{U}{A}.$$

Thus (5.3) (5.4) imply

$$R_t^-(\alpha) + \lambda_1(\alpha) R_x^-(\alpha) = -\frac{\mu}{2A}(R^-(\alpha) + R^+(\alpha)) + O(\alpha - 1)\frac{U}{A},$$

$$R_t^+(\alpha) + \lambda_2(\alpha) R_x^+(\alpha) = -\frac{\mu}{2A}(R^-(\alpha) + R^+(\alpha)) + O(\alpha - 1)\frac{U}{A},$$

$x \in \mathbb{R}$, $t > 0$.

Differentiate (5.3) (5.4) w.r.t. x and use

$$r^{\pm} = R_x^{\pm}$$

to obtain

$$\begin{aligned} r_t^- + \lambda_1 r_x^- + \frac{\partial \lambda_1}{\partial R^-}(r^-)^2 + (\frac{\mu}{2A} + \frac{\mu U}{2A}\sqrt{\frac{A_r \rho}{G_0 A}} + \frac{\partial \lambda_1}{\partial R^+} r^+) r^- \\ = -\frac{\mu}{2A}(1 - U\sqrt{\frac{A_r \rho}{G_0 A}}) r^+ + O(\alpha - 1)(r^-, r^+) \\ r_t^+ + \lambda_2 r_x^+ + \frac{\partial \lambda_2}{\partial R^+}(r^+)^2 + (\frac{\mu}{2A} - \frac{\mu U}{2A}\sqrt{\frac{A_r \rho}{G_0 A}} + \frac{\partial \lambda_2}{\partial R^-} r^-) r^+ \\ = -\frac{\mu}{2A}(1 + U\sqrt{\frac{A_r \rho}{G_0 A}}) r^- + O(\alpha - 1)(r^-, r^+), \end{aligned}$$

$x \in \mathbb{R}$, $t > 0$.

These equations are small perturbations of their counterparts (3.15) and (3.16) when $\alpha = 1$.

Therefore global smooth solution global in time regularity and finite time singularity formation of solutions of (1.1) (1.2) (2.1) under conditions similar to those stated in Theorem 2.1, provided that $\alpha - 1$ is small enough, namely

$$0 < \alpha - 1 \leq C\mu$$

which is equivalent to

$$\frac{(\alpha-1)^2}{\alpha} \leq C\nu$$

for some constant $C > 0$.

6. **Concluding remarks.** We proved global in time regularity and finite time singularity formation of solutions by showing the critical threshold phenomena for a hyperbolic model of blood flow. In particular, we identified lower thresholds for finite time singularities in solutions and upper thresholds for the global existence of the smooth solutions. The thresholds are represented in terms of the initial slopes of the Riemann invariants and the initial cross-section. Two new results were obtained. The first says that shock formation due to the viscous damping term is delayed compared with the shock formation without viscous damping, which is as expected physically. The second says that the class of initial data for which global smooth solutions exist is rich er than the one predicted by inviscid theory. Namely, the slope of the Riemann invariants associated with the initial data can be negative, and the magnitude of the negative slope is proportional to the magnitude of the viscous damping term.

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