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CRITICAL THRESHOLDS IN A RELAXATION SYSTEM WITH RESONANCE OF CHARACTERISTIC SPEEDS

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Abstract. We study critical threshold phenomena in a dynamic continuum traffic flow model known as the Payne and Whitham (PW) model. This model is a quasi-linear hyperbolic relaxation system, and when equilibrium velocity is specifically associated with pressure, the equilibrium characteristic speed resonates with one characteristic speed of the full relaxation system. For a scenario of physical interest we identify a lower threshold for finite time singularity in solutions and an upper threshold for the global existence of the smooth solution. The set of initial data leading to global smooth solutions is large, in particular allowing initial velocity of negative slope.

1. **Introduction.** We are concerned with both global in time regularity and finite time singularity formation in solutions for hyperbolic relaxation systems as in [24].

Let us begin with the following 2×2 hyperbolic relaxation system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x + \frac{p(\rho)_x}{\rho} = \frac{1}{\tau}(v_e(\rho) - u), \end{cases} \qquad x \in R,\ t > 0 \tag{1}$$

subject to the initial data

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in R. \tag{2}$$

This system arises from a continuum model of traffic flows, see [40, 43]. The first equation in (1) is a conservation law, while the second one describes drivers' acceleration behavior. Here $\tau > 0$ is the relaxation time, $p(\rho)$ is the pressure with $p'(\rho) > 0$ and $v_e(\rho)$ is the equilibrium velocity with $v_e'(\rho) < 0$. In some physical situations [40, 44], these two profiles are related in a special way such as

$$p'(\rho) = (\rho v_e'(\rho))^2 > 0. \tag{3}$$

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System (1) is a strictly hyperbolic balance law, with characteristic speeds

$$\lambda_1(\rho, u) = u + \rho v_e'(\rho) < u - \rho v_e'(\rho) = \lambda_2(\rho, u). \tag{4}$$

But with (3) the equilibrium equation,

$$\rho_t + (q(\rho))_x = 0, \quad q(\rho) = \rho v_e(\rho), \tag{5}$$

has a characteristic speed $\lambda_*(\rho) = q'(\rho) = v_e(\rho) + \rho v_e'(\rho) = \lambda_1$, implying that the usual sub-characteristic condition [35, 43] is only satisfied marginally. Hence the conventional large-time stability analysis based on the dissipation mechanism provided by non-vanishing speed gaps cannot be applied. Our approach is to study the critical threshold phenomena: the global existence of the smooth solution or the finite time singularity depends on whether the initial data cross such critical thresholds.

Two physical scenarios are of our interest:

i) The Payne [40] and Whitham [43] (PW) model: (1) with

$$p(\rho) = c_0^2 \rho. \tag{6}$$

This model, a classical dynamic continuum description of traffic flow, has been adopted in the study of traffic jam dynamics, see e.g. [12, 15, 21, 36]. The data from the Lincoln Tunnel, New York obtained by Greenberg in 1959 [9] suggest

$$v_e(\rho) = c_0 ln \frac{1}{\rho}, \;\; 0 < \rho \le 1, \tag{7}$$

in which the maximum density has been normalized to 1.

ii) The Zhang model [44]: (1) with

$$p(\rho) = \frac{v_f^2}{3}\rho^3, v_e(\rho) = v_f(1-\rho) \tag{8}$$

where v_f is the free flow speed, and the equilibrium velocity has been rescaled from an actual measurement taken from [10]. For this model existence of weak solutions as well as L^1 stability theory have been established in [19, 20].

Scenario ii) has recently been investigated in [24], where the authors identified five upper thresholds for finite time singularity in solutions and three lower thresholds for global existence of smooth solutions. The purpose of this paper is to confirm the critical threshold phenomena for scenario i). A similar set of both upper and lower thresholds for this case can still be identified by fully using the feature of the underlying scenario. For the sake of brevity, we present herein only one upper and one lower thresholds and the associated results.

The first result tells us the upper threshold for the global smoothness of (1) with (6) and (7).

Theorem 1.1. *(Global in time regularity) Consider the relaxation system (1) with (6) and (7), subject to initial data (2) satisfying* $(\rho_0, u_0) \in C^1(R) \times C^1(R)$ *and*

$$0 < \nu \le \rho_0(x) \le 1$$

for all $x \in R$ *and for some* $\nu > 0$*. Denote*

$$m = \frac{1}{\nu} e^{\frac{\|u_0\|_\infty}{c_0}}. \tag{9}$$

If both

$$-\frac{2}{\tau\sqrt{m}} \le \frac{u_0'(x)}{\sqrt{\rho_0(x)}} + c_0 \frac{\rho_0'(x)}{\rho_0(x)^{\frac{3}{2}}} \le 0$$

and

$$\frac{u_0'(x)}{\sqrt{\rho_0(x)}} - c_0\frac{\rho_0'(x)}{\rho_0(x)^{\frac{3}{2}}} \geq 0$$

hold for all $x \in R$, then the Cauchy problem (1) (2) admits a global smooth solution satisfying

$$\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty + c_0|ln\nu|, \quad t \in R^+$$

and

$$m^{-1} \leq \rho(x,t) \leq m, \quad (x,t) \in R \times R^+.$$

Furthermore, we have the following estimates on the first derivatives

$$\min_{x\in R} A^+(x) \leq \frac{u_x(x,t)}{\sqrt{\rho(x,t)}} + c_0\frac{\rho_x(x,t)}{\rho^{3/2}(x,t)} \leq 0$$

and

$$\min_{x\in R}\frac{A^-(x)}{\sqrt{\rho_0(x)}+\sqrt{m}A^-(x)t/2} \leq \frac{u_x(x,t)}{\sqrt{\rho(x,t)}} - c_0\frac{\rho_x(x,t)}{\rho^{3/2}(x,t)} \leq \max_{x\in R} A^-(x)$$

where

$$A^\pm(x) = \frac{u_0'(x)}{\sqrt{\rho_0(x)}} \pm c_0\frac{\rho_0'(x)}{\rho_0^{3/2}(x)}. \tag{10}$$

The result on lower threshold for finite time singularity is summarized below.

Theorem 1.2. *(Finite-time singularity) Consider the same problem as stated in Theorem 1.1. If*

$$\frac{u_0'(x)}{\sqrt{\rho_0(x)}} + c_0\frac{\rho_0'(x)}{\rho_0(x)^{\frac{3}{2}}} \geq -\frac{2\sqrt{m}}{\tau}$$

fails to hold at any point $x \in R$, then the solution of Cauchy problem (1) (2) must develop singularity in a finite time T^*. Moreover,

$$\lim_{t\to T^*}\left(\min_{x\in R}\left\{\frac{u_x(x,t)}{\sqrt{\rho(x,t)}} + c_0\frac{\rho_x(x,t)}{\rho^{3/2}(x,t)}\right\}\right) = -\infty$$

and

$$T^* < \tau \min_{x\in R} ln\left|\frac{A^+(x)}{A^+(x)+\frac{2\sqrt{m}}{\tau}}\right|$$

where m and $A^\pm$ are given in (9) and (10), respectively.

Concerning these theorems, several remarks are in order.

Remark 1. The set of initial data leading to global regularity is rich. In particular, it allows the initial Riemann invariant of negative slope. This is in sharp contrast to the generic breakdown in homogeneous hyperbolic systems [18].

Remark 2. No smallness of data is assumed for the global existence of the smooth solution. The critical thresholds we identified reveal the genuine nonlinear phenomena hidden in the system.

Remark 3. Note that the bounds for the derivatives of the initial Riemann invariants are of order $\frac{1}{\tau}$. This implies that the smaller the relaxation time τ, the larger the set of initial data leads to global smooth solutions. This means that the shorter the drivers' reaction time, the larger the set of initial conditions leads to global smooth traffic flows.

The issues of global in time regularity and finite time singularity formation are fundamental for hyperbolic balance laws, and have been investigated by many authors, see, e.g., [18, 16, 34, 39, 5, 37, 42, 7, 24]. Hyperbolic relaxation systems belong to a special class of balance laws, for which a sub-characteristic type condition is always necessary for even linear stability [43]. An abundant research on nonlinear stability theory for various relaxation systems has appeared in past decades, see e.g., [1, 3, 14, 17, 28, 27, 22, 23, 35, 38], relying on some sub-characteristic type structure conditions [35].

Our results rely on tracking nonlinear dynamics of the slopes of the Riemann invariants. As such, we believe that the arguments entertained here and those in [24] for special cases will be helpful for more general relaxation systems with large data, in which classical stability analysis is difficult to apply directly. For hyperbolic balance laws such as (1), the coupling of different characteristic fields makes it difficult to detect a sharp critical threshold. Nevertheless, for the relaxation system (1) with (6) (7), we are able to decouple the ratio of the slope of one Riemann invariant and the half power of the density from the system and to track its dynamics.

The concept of critical threshold seems a right idea to go beyond the stability regime for nonlinear evolution equations. The critical threshold phenomenon was first observed and studied in [7] for a class of Euler-poisson equations; and further extended to other problems of various types [29, 25, 8, 41, 2]. The study of multi-D critical threshold phenomena becomes more challenging, and a new tool of *spectral dynamics* has been first introduced in [30] to estimate the velocity gradient. Using spectral dynamics as a crucial tool in the study of the critical threshold phenomena has been justified for several interesting models [32, 33, 26].

We now conclude this section by outlining the rest of this paper. In Section 2 we reformulate corresponding results in terms of the Riemann invariants. Section 3 contains *a priori* estimates of solutions in L^∞ norm for (1). Section 4 is devoted to identifying the upper threshold for global existence of smooth solutions, as well as the lower threshold for the finite time singularity formation. This is done by deriving the a priori estimate of the solution derivatives through some nonlinear quantities.

2. **Reformulation of the Problem.**

We start reformulation of (1) with (6) and (7). Set $w = ln\rho$. Multiplying the first equation of (1) by $\frac{1}{\rho}$, we have

$$\begin{cases} w_t + uw_x + u_x = 0, \\ u_t + uu_x + c_0^2 w_x = \frac{1}{\tau}(-c_0 w - u). \end{cases} \tag{11}$$

Multiplying system (11) by the left eigenvectors of the Jacobian of the flux

$$l_i(w,u) = ((-1)^i c_0, 1), \quad i = 1,\ 2,$$

we have

$$\begin{cases} R_t^- + \lambda_1 R_x^- = -\frac{1}{\tau} R^+, \\ R_t^+ + \lambda_2 R_x^+ = -\frac{1}{\tau} R^+ \end{cases} \tag{12}$$

where

$$\lambda_1 = \frac{R^- + R^+}{2} - c_0, \quad \lambda_2 = \frac{R^- + R^+}{2} + c_0 \tag{13}$$

and the Riemann invariants

$$\begin{cases} R^-(w,u) = u - c_0 w \\ R^+(w,u) = u + c_0 w \end{cases} \tag{14}$$

define a one-to-one mapping from (ρ, u), $\rho > 0$, to (R^-, R^+) in the phase space.

The corresponding initial data is

$$(R^-, R^+)(x,0) = (R_0^-, R_0^+)(x) = (u_0 - c_0 ln\rho_0, u_0 + c_0 ln\rho_0)(x). \tag{15}$$

In order to prove Theorem 1.1 and Theorem 1.2, it suffices to establish the following for Cauchy problem (12), (15).

Theorem 2.1. *Consider the system (12) subject to C^1 bounded initial data (15). Let m be defined in (9). If*

$$0 \geq \frac{R^+_{0,x}(x)}{\sqrt{\rho_0(x)}} \geq -\frac{2}{\tau\sqrt{m}}, \quad x \in R$$

and

$$R^-_{0,x}(x) \geq 0, \quad x \in R,$$

then the Cauchy problem (12) (15) has a unique smooth solution for all time $t > 0$.

Moreover, we have

$$0 \geq \frac{R^+_x(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R^+_{0,x}(x)}{\sqrt{\rho_0(x)}}, \quad (x,t) \in R \times R^+$$

and

$$\max_{x\in R} \frac{R^-_{0,x}(x)}{\sqrt{\rho_0(x)}} \geq \frac{R^-_x(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R^-_{0,x}(x)}{\sqrt{\rho_0(x)} + \frac{\sqrt{m}}{2} R^-_{0,x}(x)t}, \quad (x,t) \in R \times R^+.$$

Theorem 2.2. *Assume that $R_0^\pm(x) \in C^1(R)$ and $\|R_0^\pm\|_\infty$ are bounded. If*

$$\frac{R^+_{0,x}(x)}{\sqrt{\rho_0(x)}} \geq -\frac{2\sqrt{m}}{\tau}$$

fails to hold at any point $x \in R$, then the C^1 solution of the Cauchy problem (12) (15) will develop a finite time singularity. Moreover,

$$\lim_{t\to T^*} \left(\min_{x\in R} \left\{ \frac{R^+_x(x,t)}{\sqrt{\rho(x,t)}} \right\} \right) = -\infty$$

for

$$T^* < \tau \min_{x\in R} ln \left(\frac{\tau R^+_{0,x}(x)}{2\sqrt{m\rho_0(x)} + \tau R^+_{0,x}(x)} \right).$$

The local existence of smooth solutions of hyperbolic problem is classical, see e.g. Douglis [6] and Hartman and Wintner [11]. According to the theory of first order quasilinear hyperbolic equations [4], solutions to initial value problems exist as long as one can place an *a priori* limitation on the magnitude of their first derivatives.

Equipped with the classical local existence results in [6] and [11], we need only to establish the *a priori* estimates, which will be presented in Lemma 3.1 and Lemma 4.1. The finite time singularity formation follows from the proof of Lemma 3.1.

Using expressions of the Riemann invariants (14) to convert back to variables u and ρ, we prove our main results as stated in Theorem 1.1- Theorem 1.2. Note that in Lemma 3.2, we showed that ρ is bounded away from zero if ρ_0 is. Therefore finite time blow up of $\frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}}$ implies the finite time blow up of $u_x(x,t)$ or $\rho_x(x,t)$. Theorem 1.2 follows directly from Theorem 2.2.

3. **Bounds for smooth solutions.** We give *a priori* estimates of solutions in L^∞ norm in this section.

We first establish the uniform bounds for the Riemann invariants (R^-, R^+) of (1) with (6) (7).

Lemma 3.1. *Assume that $R_0^\pm \in C^1(R)$ and that*

$$\|R_0^-\|_\infty + \|R_0^+\|_\infty \le M$$

for some $M > 0$. Then the C^1 solution of the Cauchy problem (12) (15) satisfies the a priori estimates

$$\|R^+(\cdot,t)\|_\infty \le \|R_0^+\|_\infty e^{-\frac{t}{\tau}} \tag{16}$$

and

$$\|R^-(\cdot,t)\|_\infty + \|R^+(\cdot,t)\|_\infty \le M \tag{17}$$

for all $t \ge 0$ as long as the C^1 solution exists.

Proof. Integrating the second equation in (12) along the second characteristics $x_2(t,\alpha)$

$$\frac{dx_2}{dt} = \lambda_2 = u + c_0,\ x_2(0,\alpha) = \alpha,$$

we have

$$R^+(x_2(t,\alpha),t) = R_0^+(\alpha)e^{-\frac{t}{\tau}},$$

which leads to the asserted bound (16).

Now integrating the first equation in (12) along the first characteristics $x_1(t,\beta)$

$$\frac{dx_1}{dt} = \lambda_1 = u - c_0,\ x_1(0,\beta) = \beta,$$

we have

$$R^-(x_1(t,\beta),t) = R_0^-(\beta) - \frac{1}{\tau}\int_0^t R^+(x_1(s,\beta),s)ds$$

Using the above decay result for $\|R^+(\cdot,t)\|_\infty$, we have

$$\|R^-(\cdot,t)\|_\infty \le \|R_0^-(\cdot)\|_\infty + \|R_0^+(\cdot)\|_\infty(1-e^{-\frac{t}{\tau}}).$$

This added upon (16) gives the desired bound (17). □

Lemma 3.2. *Assume that the initial data (2) are uniformly bounded with*

$$-\|u_0\|_\infty \le u_0(x) \le \|u_0\|_\infty, \quad 0 < \nu \le \rho_0(x) \le 1$$

for all $x \in R$.

Let (ρ, u) be a C^1 solution of (1) with (6) (7) determined from $R^\pm$, then the density satisfies

$$m^{-1} \le \rho(x,t) \le m \tag{18}$$

for all $t \ge 0$ as long as the C^1 solution exists, where

$$m = \frac{1}{\nu}e^{\frac{\|u_0\|_\infty}{c_0}}. \tag{19}$$

The velocity is also bounded

$$\|u(\cdot,t)\|_\infty \le \|u_0\|_\infty + c_0|ln\nu|$$

for all $t \geq 0$ as long as the C^1 solution exists.

Proof. From the relation $w = ln\rho$, it follows that

$$e^{-\|w(\cdot,t)\|_\infty} \leq \rho(x,t) \leq e^{\|w(\cdot,t)\|_\infty}, \quad x \in R. \tag{20}$$

Applying Lemma 3.1 to estimate

$$w = \frac{R^+ - R^-}{2c_0},$$

we have

$$\|w(\cdot,t)\|_\infty \leq \frac{1}{2c_0}(\|R^-(\cdot,t)\|_\infty + \|R^+(\cdot,t)\|_\infty) \leq \frac{1}{2c_0}(\|R_0^-\|_\infty + \|R_0^+\|_\infty).$$

Using expressions of the Riemann invariants (14) we have

$$\|w(\cdot,t)\|_\infty \leq \frac{1}{2c_0}(\|u_0 - c_0 ln\rho_0\|_\infty + \|u_0 + c_0 ln\rho_0\|_\infty) \leq \frac{\|u_0\|_\infty}{c_0} + \|ln\rho_0\|_\infty.$$

Under the conditions that $0 < \nu \leq \rho_0(x) \leq 1$ for all $x \in R$,

$$\|ln\rho_0\|_\infty \leq |ln\nu|.$$

Inserting this into (20) gives the asserted bounds (18) for the density. The bound for velocity follows immediately when recalling that (14) implies $u = (R^+ + R^-)/2$. $\square$

4. **Critical thresholds.**

In order to identify the upper threshold for global existence of smooth solutions, as well as the lower threshold for the finite time singularity formation as claimed in Theorems 1.1 and 1.2, we derive the *a priori* estimates of the derivatives of the Riemann invariants $R^\pm(x,t)$ of (1) with (6) (7).

Denote $r^- = R_x^-$ and $r^+ = R_x^+$, we shall show that $R_x^\pm$ are bounded when initial values of them, i.e., $r_0^\pm := R_{0,x}^\pm$, are bounded by some critical thresholds.

More precisely, we have the following.

Lemma 4.1. *Assume that $R_0^\pm(x) \in C^1(R)$ and $\|R_0^\pm\|_\infty$ are bounded. Let m be defined in (19). If further*

$$0 \geq \frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}} \geq -\frac{2}{\tau\sqrt{m}}, \quad x \in R$$

and

$$R_{0,x}^-(x) \geq 0, \quad x \in R,$$

then any C^1 solution of the Cauchy problem (12) (15) has the a priori estimates

$$0 \geq \frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R_{0,x}^+(x)}{\sqrt{\rho_0(x)}}$$

and

$$\max_{x\in R} \frac{R_{0,x}^-(x)}{\sqrt{\rho_0(x)}} \geq \frac{R_x^-(x,t)}{\sqrt{\rho(x,t)}} \geq \min_{x\in R} \frac{R_{0,x}^-(x)}{\sqrt{\rho_0(x)} + \frac{\sqrt{m}}{2}R_{0,x}^-(x)t}$$

for all $x \in R$ and $t \geq 0$ as long as the C^1 solution exists.

Proof. From (13) we derive that

$$\lambda_{1,x} = \lambda_{2,x} = \frac{r^+ + r^-}{2}.$$

We differentiate (12) with respect to x to obtain

$$\begin{cases} r_t^- + \lambda_1 r_x^- + \frac{r^- + r^+}{2} r^- = -\frac{1}{\tau} r^+ \\ r_t^+ + \lambda_2 r_x^+ + \frac{r^- + r^+}{2} r^+ = -\frac{1}{\tau} r^+. \end{cases} \tag{21}$$

From the first equation in (1) and (14), we derive

$$\rho_t + \lambda_2 \rho_x = -\rho u_x + c_0 \rho_x = -\rho r^-. \tag{22}$$

Multiplying the second equation in (21) by $\frac{1}{\sqrt{\rho}}$ and using equation (22), we get

$$\left(\frac{r^+}{\sqrt{\rho}}\right)_t + \lambda_2 \left(\frac{r^+}{\sqrt{\rho}}\right)_x = -\frac{r^+}{\sqrt{\rho}} \left(\frac{1}{\tau} + \frac{r^+}{2}\right).$$

Denote $a = \frac{r^+}{\sqrt{\rho}}$. Then a satisfies

$$a_t + \lambda_2 a_x = -a \left(\frac{1}{\tau} + \frac{\sqrt{\rho}}{2} a\right).$$

Along the second characteristics $x_2(t, \alpha)$: $\frac{dx_2}{dt} = \lambda_2,\ x_2(0, \alpha) = \alpha$, we have

$$\frac{d}{dt} a = -a \left(\frac{1}{\tau} + \frac{\sqrt{\rho}}{2} a\right),$$

which when using bounds for ρ from Lemma 3.2 leads to the following

$$-a \left(\frac{1}{\tau} + \frac{\sqrt{m}}{2} a\right) \le \frac{d}{dt} a \le -a \left(\frac{1}{\tau} + \frac{1}{2\sqrt{m}} a\right). \tag{23}$$

Solving these two differential inequalities, we conclude that a remains bounded

$$-\frac{2\sqrt{m}}{\tau} \le a(x_2(t, \alpha), t) \le \max\{0, a_0(\alpha)\} \tag{24}$$

provided

$$a_0(\alpha) \ge -\frac{2}{\tau\sqrt{m}}, \quad \forall \alpha \in R.$$

On the other hand, a will blow up in a finite time if there exists an $\alpha^* \in R$ such that

$$a_0(\alpha^*) < -\frac{2\sqrt{m}}{\tau}. \tag{25}$$

More precisely, the right differential inequality in (23) enables us to obtain

$$a(x_2(t, \alpha), t) \le \frac{2\sqrt{m} a_0(\alpha)}{(2\sqrt{m} + \tau a_0(\alpha)) e^{t/\tau} - \tau a_0(\alpha)}. \tag{26}$$

For initial data satisfying (25), the right hand side of (26) will become $-\infty$ at a finite time

$$T = \tau \min_{\alpha \in R} ln \left(\frac{\tau a_0(\alpha)}{2\sqrt{m} + \tau a_0(\alpha)}\right) < +\infty.$$

Therefore, there exists a $T^* < T$ such that

$$\lim_{t \to T^*} \min_x a(x, t) = -\infty. \tag{27}$$

Now we examine $r^- = R_x^-$. From the first equation in (1) we derive

$$\rho_t + \lambda_1 \rho_x = -\rho u_x - c_0 \rho_x = -\rho r^+.$$

Multiplying the first equation in (21) by $\frac{1}{\sqrt{\rho}}$ and using the above equation, we get

$$\left(\frac{r^-}{\sqrt{\rho}}\right)_t + \lambda_1 \left(\frac{r^-}{\sqrt{\rho}}\right)_x = -\frac{1}{\tau}a - \frac{(r^-)^2}{2\sqrt{\rho}}.$$

Let $b = \frac{r^-}{\sqrt{\rho}}$. Then b satisfies

$$b_t + \lambda_1 b_x = -\frac{1}{\tau}a - \frac{\sqrt{\rho}}{2}b^2.$$

It follows from (24) that if

$$0 \geq a_0(\alpha) \geq -\frac{2}{\tau\sqrt{m}}, \quad \forall \alpha \in R, \tag{28}$$

then

$$0 \geq a(x_2(\alpha, t), t) \geq -\frac{2\sqrt{m}}{\tau}, \quad (\alpha, t) \in R \times R^+.$$

Assuming (28) and letting $x_1(t, \beta)$ be the first characteristics, we have

$$\frac{2\sqrt{m}}{\tau^2} - \frac{1}{2\sqrt{m}}b^2 \geq \frac{d}{dt}b \geq -\frac{\sqrt{m}}{2}b^2, \quad \frac{d}{dt} := \partial_t + \lambda_1\partial_x.$$

If

$$b_0(\beta) \geq 0, \quad \forall \beta \in R,$$

then b stays bounded. Indeed

$$\frac{2\sqrt{m}}{\tau}h(t,\beta) \geq b(x_1(t,\beta),t) \geq \frac{b_0(\beta)}{1 + \frac{\sqrt{m}}{2}b_0(\beta)t}$$

where

$$h(t,\beta) = \frac{(b_0(\beta) + \frac{2\sqrt{m}}{\tau})e^{\frac{t}{\tau}} + (b_0(\beta) - \frac{2\sqrt{m}}{\tau})e^{-\frac{t}{\tau}}}{(b_0(\beta) + \frac{2\sqrt{m}}{\tau})e^{\frac{t}{\tau}} - (b_0(\beta) - \frac{2\sqrt{m}}{\tau})e^{-\frac{t}{\tau}}}, \quad \beta \in R.$$

Note that when $b_0 \geq 0$, $h(t, \beta)$ is a decreasing function in time and satisfies

$$1 \leq h(t,\beta) \leq \frac{\tau b_0(\beta)}{2\sqrt{m}}.$$

Therefore, for $b_0(\beta) \geq 0$ for all $\beta \in R$,

$$\frac{b_0(\beta)}{1 + \frac{\sqrt{m}}{2}b_0(\beta)t} \leq b(x_1(t,\beta),t) \leq b_0(\beta),$$

which when optimizing the bounds in terms of the parameter β leads to the desired estimates. □

In our analysis above we also found a threshold condition (25) for the finite time singularity in $a(x,t) = \frac{R_x^+(x,t)}{\sqrt{\rho(x,t)}}$, (27), which proves Theorem 2.2.

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