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Construction of Solutions and L^1-error Estimates of Viscous Methods for Scalar Conservation Laws with Boundary

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Abstract This paper is concerned with an initial boundary value problem for strictly convex conservation laws whose weak entropy solution is in the piecewise smooth solution class consisting of finitely many discontinuities. By the structure of the weak entropy solution of the corresponding initial value problem and the boundary entropy condition developed by Bardos–Leroux–Nedelec, we give a construction method to the weak entropy solution of the initial boundary value problem. Compared with the initial value problem, the weak entropy solution of the initial boundary value problem includes the following new interaction type: an expansion wave collides with the boundary and the boundary reflects a new shock wave which is tangent to the boundary. According to the structure and some global estimates of the weak entropy solution, we derive the global L^1-error estimate for viscous methods to this initial boundary value problem by using the matching travelling wave solutions method. If the inviscid solution includes the interaction that an expansion wave collides with the boundary and the boundary reflects a new shock wave which is tangent to the boundary, or the inviscid solution includes some shock wave which is tangent to the boundary, then the error of the viscosity solution to the inviscid solution is bounded by $O(\varepsilon^{1/2})$ in L^1-norm; otherwise, as in the initial value problem, the L^1-error bound is $O(\varepsilon|\ln\varepsilon|)$.

Keywords scalar conservation laws, initial boundary value problem, global weak entropy solution, error estimate of viscous methods
MR(2000) Subject Classification 35L65

1 Introduction

We consider the initial boundary problem for the scalar conservation laws:

$$\begin{cases} u_t + f(u)_x = 0, & x > 0, \quad t > 0, \\ u(x,0) = u_0(x), & x > 0, \\ u(0,t) = u_b(t), & t > 0, \end{cases} \tag{1.1}$$

where the flux $f \in C^2$ satisfies

(A_1) $$f'' \geq \alpha > 0, \quad f(0) = f'(0) = 0.$$

The viscosity method approach to (1.1) is to solve the parabolic equation with initial boundary conditions:

$$\begin{cases} (v_\varepsilon)_t + f(v_\varepsilon)_x = \varepsilon (v_\varepsilon)_{xx}, & x > 0, \quad t > 0, \\ v_\varepsilon(x,0) = v_0(x), & x \geq 0, \\ v_\varepsilon(0,t) = v_b(t), & t \geq 0, \end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is a small viscosity parameter.

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The initial boundary value problem of scalar conservation laws plays an important roles in the mathematical modelling and simulation of the practical problem of the one-dimensional sedimentation processes and traffic flow on highways [1–5]. Bardos et al. [6] established the existence and uniqueness of the weak entropy solution in the BV-setting for the initial boundary value problems of scalar conservation laws, respectively, by the vanishing viscosity method and by Kruzkov's method [7] (about proving the existence of the solution of the initial value problem for conservation laws by the vanishing viscosity method, see also [8–10] etc.). The main difficulty for scalar conservation laws with a boundary effect is to have a good formation of the boundary condition. Namely, for a fixed initial value as $(1.1)_2$, we really cannot impose such a condition on the boundary as $(1.1)_3$, and the boundary condition is necessarily linked to the entropy condition (The approach for dealing with the boundary in [6] has been followed by many authors [3, 11–15], who extended the previous results in different directions). In other words, the weak entropy solution $u(x,t)$ for (1.1) does not admit a trace at the boundary, namely, $u(0,t)$ does not always equal $u_b(t)$, whereas, as a viscosity approximation of the weak entropy solution for (1.1), the solution of the initial boundary value problems of parabolic Equation (1.2) does admit a fixed trace at the boundary. Therefore, it is very interesting to consider the error estimates for the viscosity approximation for the initial boundary problems of scalar conservation laws.

For the problem without a boundary, i.e., the initial value problem, the asymptotic convergence of the solution of the viscous problem to the corresponding discontinuous solution of the inviscid problem has been the main driving force for the mathematical theory of shock waves from both theoretical and numerical point of view. Substantial progress has been made in the past in this regard, pioneered by Hopf [16], Lax [17], Oleinik [18] and Kruzkov [7] etc. For the BV entropy solution, Kuznetsov [19] was the first to establish the half-order rate of L^1-convergence for the viscosity approximation and monotone scheme. Tang–Teng [20] and Sabac [21] proved that this half-order rate of convergence is optimal in the BV solution class. However, for convex conservation laws with a piecewise smooth solution, the L^1-convergence rate can be improved to the first-order, see e.g., Teng–Zhang [22] for the monotone scheme, Tang–Teng [23] for the viscosity approximation and Teng [24] for the relaxation approximation, where the matching travelling wave solutions method, developed by Goodman–Xin [25], was used to get the first-order error estimates. This method relies on the L^1-stability properties of the approximation equations and nonlinear large time asymptotic stability of viscous shock profiles. For the initial value problems of systems of viscous conservation laws, the stability theory of viscous shock profiles was extensively studied by many authors in the past decade, see e.g., [26–29]. For the initial boundary problems, see also [30–37] for recent progress. By using the matching method, Tang–Teng [23] proved that, for convex conservation laws whose entropy solution consists of finitely many discontinuities, the L^1-error between the viscosity solution and the inviscid solution is bounded by $O(\varepsilon|\ln\varepsilon|)$, and the error bound is improved to $O(\varepsilon)$ if there is neither central rarefaction wave nor spontaneous shock included in the inviscid solution (see also [38]). Later, Tadmor–Tang [39, 40] used the energy method with some bootstrap extrapolation technique to obtain some first-order *pointwise* convergence results for viscous approximations to convex scalar conservation laws with piecewise smooth solutions. Very recently, for nonconvex scalar conservation laws whose weak entropy solution is in the piecewise smooth solution class consisting of finitely many discontinuities, Tang et al. [41] proved that the optimal rate of L^1-convergence of viscous approximations is a fractional number α satisfying $\frac{1}{2} < \alpha \leq 1$.

The key point on the error estimates for the viscous approximations of the initial boundary value problems of scalar conservation laws is a detailed description of the geometric structure of the weak entropy solution. However, the geometric structure of the solution is much more difficult due to the presence of the boundary. The authors in paper [3] constructed the global weak entropy solution to the initial boundary problem on a bounded interval for some special initial boundary data with three pieces of constant data corresponding to the practical problem of continuous sedimentation of an ideal suspension, but they have not obtained the general result for all cases of three pieces of constant data. In our previous work [42], we considered the initial boundary value problem (1.1) with three pieces of constant data, i.e., $u_b(t)(t > 0)$ is a constant function and $u_0(x)(x > 0)$ is a function with two pieces of constant. By investigating the interaction of elementary waves and the boundary $x = 0$, we clarified the structure and boundary behavior of the weak entropy solution. It is of interest to note that there is a new phenomenon on the geometric structure of the weak entropy solution, that is, if the central rarefaction wave collides with the boundary $x = 0$, then the boundary will reflect a shock wave under some condition. Moreover, by using the matching method, we derived the uniform L^1-error estimates for the viscosity methods. The error bound is stated as follows: If the inviscid solution includes the interaction that the central rarefaction wave collides with the boundary $x = 0$ and the boundary reflects a new shock wave, then the error of the viscosity solution to the inviscid solution is bounded by $O(\varepsilon^{1/2})$ in L^1-norm; otherwise, the L^1-convergence rate is similar to that of the initial value problem in [23].

In this paper, we will construct the global weak entropy solution and establish an L^1-convergence rate for the viscosity methods to the initial boundary value problem (1.1) with piecewise smooth initial data and constant boundary data. In Section 2, we give a construction method to the weak entropy solution of this initial boundary value problem by using the weak entropy solution of the initial value problem and boundary entropy condition. Then from the geometric structure of the weak entropy solution, we establish some global estimates of the weak entropy solution, which are very important in discussing the rate of convergence for viscosity approximation methods to conservation laws. In Section 3, we extend the analysis used in [23, 42] to our problem and derive the global error estimates for viscous approximations. Specifically, if the inviscid solution includes the interaction that an expansion wave collides with the boundary $x = 0$ and the boundary reflects a new shock wave which is tangent to the boundary, or includes some shock wave which is tangent to the boundary and is not reflected by the boundary at this tangent time, then the error of the viscosity solution to the inviscid solution is bounded by $O(\varepsilon^{1/2})$ in L^1-norm; otherwise, the L^1-convergence rate is similar to that of the initial value problem in [23].

2 Construction of Piecewise Smooth Solution

In this section, we will construct the piecewise smooth weak entropy solution for the initial boundary problem (1.1) and establish some lemmas according to its structures, which are needed for the error estimates.

We give the definition of the weak entropy solution to the initial boundary problems (1.1) (also see [3, 6, 11, 15]).

Definition 2.1 *A bounded and local bounded variation function $u(x,t)$ on $[0,\infty) \times [0,\infty)$ is called a weak entropy solution of the initial boundary problem* (1.1)*, if for each $k \in (-\infty,\infty)$, and for any nonnegative test function $\phi \in C_0^\infty([0,\infty) \times [0,\infty))$, it satisfies the following inequality*

$$\int_0^\infty \int_0^\infty \{|u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\phi_x\}dxdt + \int_0^\infty \operatorname{sgn}(u_b(t)-k)(f(u(0,t))-f(k))\phi(0,t)dt + \int_0^\infty |u_0(x)-k|\phi(x,0)dx \geq 0. \quad (2.1)$$

For the initial boundary value problem (1.1) whose initial data and boundary data are general bounded variation functions, the existence and uniqueness of the global weak entropy solution in the sense of (2.1) have been obtained, and the global weak entropy solution satisfies the following boundary entropy condition (2.2) (see also [11, 3, 15]):

Lemma 2.2 *If $u(x,t)$ is a weak entropy solution of* (1.1)*, then,*

$$u(0,t)=u_b(t) \quad \text{or} \quad \frac{f(u(0,t))-f(k)}{u(0,t)-k}\le 0, \quad k\in I(u(0,t),u_b(t)),\ k\neq u(0,t),\ \text{a.e. } t\ge 0, \tag{2.2}$$

where $I(u(0,t),u_b(t))=[\min\{u(0,t),u_b(t)\},\max\{u(0,t),u_b(t)\}]$.

The following lemma is easy to prove by Definition 2.1 and Lemma 2.2 (see also [15, 43]), which will be used to construct the piecewise smooth solution of (1.1):

Lemma 2.3 *Under the assumption* (A_1)*, a piecewise smooth function $u(x,t)$ with piecewise smooth discontinuity curves is a weak entropy solution of* (1.1) *in the sense of* (2.1)*, if and only if the following conditions are satisfied*:

(1) *$u(x,t)$ satisfies equation $(1.1)_1$ on its smooth domains*;

(2) *If $x=x(t)$ is a weak discontinuity of $u(x,t)$, then $\frac{dx(t)}{dt}=f'(u(x(t),t))$; if $x=x(t)$ is a strong discontinuity of $u(x,t)$, then $\frac{dx(t)}{dt}=\frac{f(u^-)-f(u^+)}{u^--u^+}$ (Rankine–Hugoniot condition) and $u^->u^+$ (Lax′s shock condition), where $u^\pm=u(x(t)\pm 0,t)$*;

(3) *The boundary entropy condition* (2.2) *holds*;

(4) *$u(x,0)=u_0(x)$ a.e. $x\ge 0$.*

Before carrying out our construction work, we make the following assumptions to the initial boundary data:

(A_2) $u_0(x)(x>0)$ is a bounded and piecewise C^2-smooth function with a finite number of discontinuious points γ_i, $u_0(\gamma_i\pm 0)$ and $\dot u_0(\gamma_i\pm 0)$ exist and are finite;

(A_3) $\ddot u_0\in L^1([0,\infty))$ and $\lim_{x\to+\infty}\dot a(u_0(x))=0$;

(A_4) $a(u_0(x))$ has a finite number of inflection points and $a(u_0)$ is sufficiently smooth near its negative minimum points;

(A_5) $u_b(t)\equiv u_-$ is a constant function;

where $a(u)=f'(u)$, $\dot a(u_0(x)):=\frac{d}{dx}(a(u_0(x)))$, $\dot u_0(x):=\frac{du_0(x)}{dx}$, $\ddot u_0(x):=\frac{d^2u_0(x)}{dx^2}$.

The behavior and structure of the weak entropy solution for the initial value problem of scalar conservation laws have been studied for many years, see for example [17, 18, 44–47]. In particular, Tadmor–Tassa [44] proved that if the initial speed has a finite number of decreasing inflection points, then it bounds the number of future shock discontinuities.

Basing on the analysis method in [42, 44], we now use Lemma 2.3 to construct the weak entropy solution of the initial boundary value problem (1.1) under the assumptions (A_1)–(A_5).

Consider the following Cauchy problem:

$$\begin{cases} v_t^{(0)}+f(v^{(0)})_x=0, & -\infty<x<\infty, t>0,\\ v^{(0)}(x,0)=\begin{cases} u_-, & x<0,\\ u_0(x), & x>0.\end{cases} \end{cases} \tag{2.3}$$

As proved in [44], under the assumptions (A_1)–(A_4), the weak entropy solution $v^{(0)}(x,t)$ of (2.3) is bounded and consists of a finite number of C^2-smooth pieces, and the number of disjoint shock curves is less than or equal to the number of negative minima of $\dot a(v^{(0)}(x,0))$ and the negative jump of $v^{(0)}(x,0)$.

Remark 2.1 For the Cauchy problem (2.3), a negative minimum point of $\dot a(u_0)$ may form a new shock at a future time.

According to the solution structures of the Cauchy problem (2.3), we construct the weak entropy solution of the initial boundary problem (1.1) by dividing our problem into three cases.

Case (I) A shock wave $x=X_0(t)$ emanates at the point $(0,t_0)$ in the x-t plane (where $t_0=0$), with negative original speed for the problem (2.3).

The shock wave $x=X_0(t)$ perhaps interacts with those elementary waves (including the shock wave, expansion wave, compression wave and constant state) lying on its right. We denote the resulting shock still by $x=X_0(t)$, it is regarded as an extension of the original shock $x = X_0(t)$. Again, if it interacts with those waves on its right, then we also denote the resulting shock as $x = X_0(t)$, and so on. The left state of $x = X_0(t)$ is u_-. By the Rankine–Hugoniot condition, we have

$$\dot{X}_0(t) := \frac{dX_0(t)}{dt} = \frac{f(u^+) - f(u_-)}{u^+ - u_-},$$

$$\ddot{X}_0(t) := \frac{d^2X_0(t)}{dt^2} = \frac{(f'(u^+)(u^+ - u_-) - (f(u^+) - f(u_-)))\frac{du^+}{dt}}{(u^+ - u_-)^2} = \frac{f''(\eta)}{2}\frac{du^+}{dt},$$

where $u^+ := u(X_0(t) + 0, t)$, η is a number between u^+ and u_-. Thus from (A$_1$), $\ddot{X}_0(t)$ and $\frac{du^+}{dt}$ take the same sign. In other words, if $\ddot{X}_0(t) > 0$ for $t \in (\tau_1, \tau_2)(\tau_2 > \tau_1 \geq 0)$, then the right state of $x = X_0(t)(t \in (\tau_1, \tau_2))$ is an expansion wave; if $\ddot{X}_0(t) < 0(t \in (\tau_1, \tau_2))$, then the right state of $x = X_0(t)(t \in (\tau_1, \tau_2))$ is a compression wave; if $\ddot{X}_0(t) \equiv 0(t \in (\tau_1, \tau_2))$, then the right state of $x = X_0(t)(t \in (\tau_1, \tau_2))$ is constant state. Moreover, noticing the property of the initial value function, we know that $x = X_0(t)(t > t_0)$ is a continuous curve consisting of a finite number of C^2-smooth pieces and has a finite number of inflection points.

If for the time interval $[t_0, \infty)$, $\dot{X}_0(t + 0) \leq 0$, then $x = X_0(t)$ lies in the second quadrant of the x-t plane. Let $u(x, t) = v^{(0)}(x, t)|_{R^+ \times R^+}$, where R^+ denotes the interval $(0, \infty)$ and $v^{(0)}(x, t)$ is the weak entropy solution of (2.3); then by Lemma 2.3, we can verify that $u(x, t)$ is the weak entropy solution of (1.1). This completes the construction work.

If there exists some $t_1^* > t_0$ such that the speed of the shock $x = X_0(t)$ is non-positive for $t < t_1^*$, and zero at $t = t_1^*$, and positive for t in some right deleted neighborhood of t_1^*, then we can not take $u(x, t) = v^{(0)}(x, t)|_{R^+ \times R^+}$ as the weak entropy solution of (1.1), since this $u(x, t)$ does not satisfy the boundary entropy condition (2.2) for $t > t_1$, where $t_1(> t_0)$ is the time at which the characteristic line from the point $(X_0(t_1^*), t_1^*)$ backward to $t = 0$ intersects the t-axis (see Fig. 1(a)).

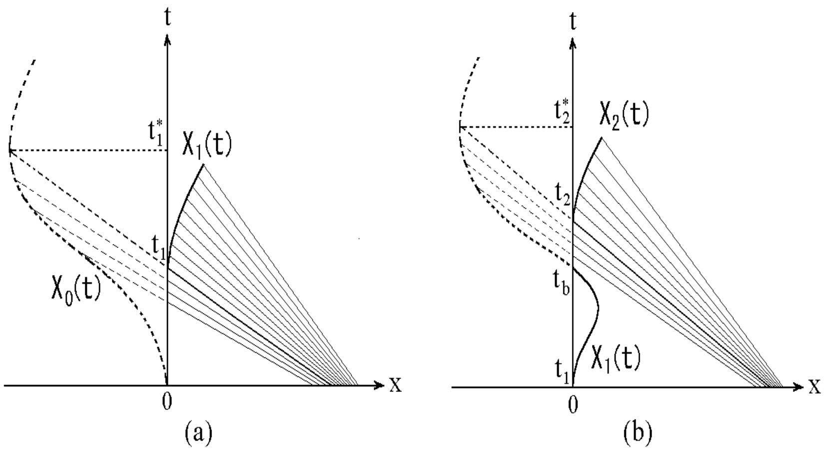

Figure 1: An expansion wave collides with the boundary $x = 0$ and the boundary reflects a new shock wave which is tangent to the boundary.

Now we reconstruct the solution of (1.1). First let $u(x, t) = v^{(0)}(x, t)|_{R^+ \times [t_0, t_1)}$. Then in view of Lemma 2.3, this $u(x, t)$ is the local weak entropy solution of (1.1) on $R^+ \times [t_0, t_1)$. Next we will extend this solution $u(x, t)$ to $R^+ \times [t_0, \infty)$. Consider the following Cauchy problem:

$$\begin{cases} (v^{(1)})_t + f(v^{(1)})_x = 0, & -\infty < x < \infty, t > t_1, \\ v^{(1)}(x, t_1) = \begin{cases} u_-, & x < 0, \\ v^{(0)}(x, t_1 - 0), & x > 0. \end{cases} \end{cases} \tag{2.4}$$

Then according to [44], the weak entropy solution $v^{(1)}(x, t)$ of (2.4) is bounded and consists of a finite number of C^2-smooth pieces, and the number of disjoint shock curves is less than or equal to the number of negative minima of $\dot{a}(v^{(1)}(x,t_1))$ and the negative jump of $v^{(1)}(x,t_1)$. For the Cauchy problem (2.4), a negative minimum point of $\dot{a}(v^{(1)}(x,t_1))$ may form a new shock at a future time. By the Rankine–Hugoniot condition and Lax's shock condition, a shock wave $x = X_1(t)$ with zero original speed and the left state u_-, starting at the point $(0,t_1)$, appears in the weak solution of (2.4) and $\ddot{X}_1(t) > 0$ in some right neighborhood of t_1. Perhaps $x = X_1(t)$ interacts with other elementary waves, the resulting shock is still denoted as $x = X_1(t)$. Then by the analysis on the interaction of elementary waves and the Rankine–Hugoniot condition and Lax's shock condition, $x = X_1(t)$ is continuous and consists of a finite number of C^2-smooth pieces and has a finite number of inflection points. We also notice that the right state of $x = X_1(t)$ in some right neighborhood of t_1 is an expansion wave.

According to the position of the shock $x = X_1(t)$, we construct the weak entropy solution of (1.1) on $[t_1,\infty)$ by dividing this case into the following two sub-cases:

(i) If the shock $x = X_1(t)(t > t_1)$ does not enter the second quadrant of the x-t plane, or enters from the first quadrant including the t-axis and keeps staying in the second quadrant after some time $t = t_b > t_1$ and the shock speed of the part in the second quadrant is non-positive, then by Lemma 2.3, $u(x,t) := v^{(1)}(x,t)|_{R^+\times[t_1,\infty)}$ is the weak entropy solution of (1.1) on $R^+ \times [t_1,\infty)$. Thus the construction of the solution to (1.1) is complete. In this sub-case, the weak entropy solution of (1.1) has the following geometric structure near the point $(0,t_1)$: an expansion wave collides with the boundary $x = 0$, then the boundary reflects a new shock wave which is tangent to the boundary $x = 0$ (at the time $t = t_1$), which is similar to the new geometric structure of the weak entropy solution of the initial boundary problem with three pieces of constant data in [42] (see also Fig. 1(a)). Besides, if in sub-case (i), there holds the possibility that there is one time interval $[t_b,t_b^*](t_b^* > t_b > t_1)$ such that the shock wave $x = X_1(t)$ stays on the first quadrant including the t-axis for $t_1 < t < t_b$ and on the t-axis for $t \in [t_b,t_b^*]$ and the speed of $x = X_1(t)$ is positive for t in some right deleted neighborhood of t_b^*, then the weak entropy solution of (1.1) has also the new geometric structure near the point $(0,t_b^*)$, that is, an expansion wave hits the boundary $x = 0$ at $t = t_b^*$ and in the meantime the boundary reflects a new shock wave which is tangent to the boundary at $t = t_b^*$ (see Fig. 2(a)). If in sub-case (i), there holds the possibility that there exists $t_b > t_1$ such that $x = X_1(t)$ lies in the first quadrant for t in some deleted neighborhood of t_b and intersects the t-axis at $t = t_b$, then for t in this neighborhood of t_b, the weak entropy solution of (1.1) includes a shock wave which is just the shock $x = X_1(t)$ and is tangent to the boundary $x = 0$ at $t = t_b$ (see Fig. 2(b)).

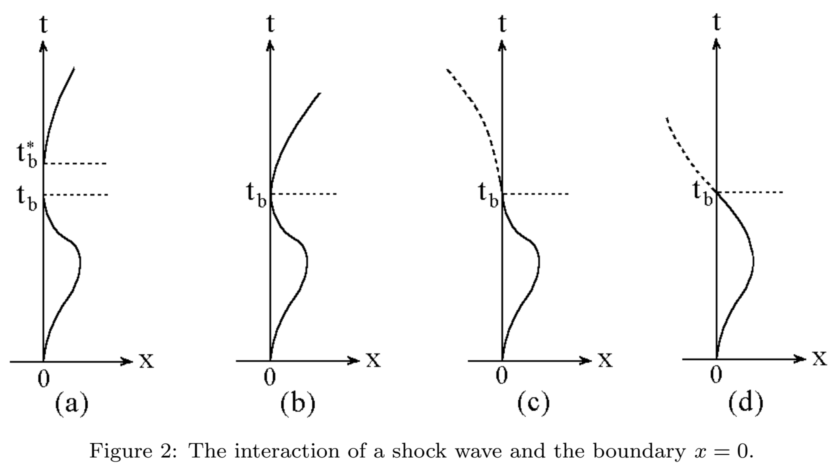

Figure 2: The interaction of a shock wave and the boundary $x = 0$.

(ii) If the shock $x = X_1(t)$ enters the second quadrant from the first quadrant including the t-axis after some time $t = t_b$, and there exists $t_2^* > t_b$, such that the speed of $x = X_1(t)$ is non-positive for $t_b < t < t_2^*$ and zero for $t = t_2^*$ and positive for t in some right deleted neighborhood of t_2^*, then we can extend the local weak entropy solution $u(x,t)$ of (1.1) to $R^+ \times [t_0, t_2)$ by taking $u(x,t) = v^{(1)}(x,t)|_{R^+\times[t_0,t_2)}$, where t_2 is the time at which the characteristic line from the point $(X_1(t_2^*), t_2^*)$ backward to $t = 0$ (see Fig. 1(b)) intersects the t-axis. In order to extend our solution to $R^+ \times R^+$, we need to consider the Cauchy problem again

$$\begin{cases} (v^{(2)})_t + f(v^{(2)})_x = 0, & -\infty < x < \infty, t > t_2, \\ v^{(2)}(x,t_2) = \begin{cases} u_-, & x < 0, \\ v^{(1)}(x, t_2 - 0), & x > 0. \end{cases} \end{cases}$$

Repeating the above procedure, by a finite number of steps (because, from (A_4), it follows that $a(v^{(2)}(x,t_2))$ has a finite number of inflection points), we can construct the global weak entropy solution of (1.1).

Case (II) A shock wave $x = X_0(t)$ emanates at the point $(0, t_0)$ in the x-t plane, with non-negative original speed for the problem (2.3).

As in Case (I), if $x = X_0(t)$ interacts with other waves, then we denote the resulting shock still as $x = X_0(t)$, and so on.

According to the position of the shock $x = X_0(t)$, we construct the weak entropy solution of (1.1) by dividing this case into the following three sub-cases: (i) The shock $x = X_0(t)(t > t_0)$ does not enter the second quadrant of the x-t plane, or enters from the first quadrant including the t-axis and keeps staying in the second quadrant after some time $t = t_b > t_0$ and the shock speed of the part in the second quadrant is non-positive. (ii) The shock $x = X_0(t)$ enters the second quadrant from the first quadrant including the t-axis after some time $t = t_b$, and there exists $t_1^* > t_b$ such that the speed of $x = X_0(t)$ is non-positive for $t_b < t < t_1^*$ and zero for $t = t_1^*$ and positive for t in some right deleted neighborhood of t_1^*. The construction method is similar to that of Case (I).

Case (III) No shock wave emanates at the point $(0, t_0)$ for the problem (2.3).

In this case, we let x_1 be the minimal point of the negative jump points of $v^{(0)}(x, 0)$, then $x_1 > 0$.

If every negative minimum point of $\dot{a}(v^{(0)}(x,0))$ lying on the interval $[0, x_1)$ does not lead to a new shock, then we trace the shock, denoted by $x = X_1(t)$, starting at the point $(x_1, 0)$, appearing in the weak entropy solution of the initial problem (2.3). As previously, we still denote by $x = X_1(t)$ the resulting shock wave in such a way that $x = X_1(t)$ interacts with other waves. For the position of the shock $x = X_1(t)$ we have one of the following cases: (i) $x = X_1(t)(t > t_0)$ does not enter the second quadrant of the x-t plane forever, or enters from the first qudrant including the t-axis and keeps staying in the second quadrant after some time and the shock speed of the part in the second quadrant is non-positive. (ii) $x = X_1(t)(t > t_0)$ enters the second quadrant from the first quadrant including the t-axis after some time $t = t_b$, and there exists $t_1^* > t_b$ such that the speed of $x = X_1(t)$ is non-positive for $t_b < t < t_1^*$ and zero for $t = t_1^*$ and positive for t in some right deleted neighborhood of t_1^*. The construction method of the weak entropy solution of (1.1) is similar to that of Case (I).

If some negative minimum points $\zeta_i (i = 1, 2, \ldots, k_0)$ $(\zeta_1 < \zeta_2 < \cdots < \zeta_{k_0})$ of $\dot{a}(v^{(0)}(x,0))$ lying on the interval $[0, x_1)$ lead to new shocks after some time, then as previously, by tracing the resulting shock $x = X_1(t)$ of the new shock to which was led by the negative minimum point ζ_1 of $\dot{a}(v^{(0)}(x,0))$, we can construct the solution of (1.1). By the position and the speed sign of the shock $x = X_1(t)$, we construct the solution of (1.1) to the following cases, respectively: (i) $x = X_1(t)$ lies in the second quadrant all the time and the shock speed is non-positive, or $x = X_1(t)$ never enters the second quardrant, or $x = X_1(t)$ enters the second quadrant from the first quadrant including the t-axis and keeps staying in the second quadrant after some time and the shock speed of the part in the second quadrant is non-positive. (ii) Before some time, $x = X_1(t)$ always stays in the second quadrant and the sign of the shock speed is changed from negative to positive, or $x = X_1(t)$ enters the second quadrant from the first quadrant including the t-axis after some time $t = t_b$ and keeps staying in the second quadrant and the sign of the corresponding shock speed is changed from negative to positive between the times $t = t_b$ and $t = t_c$ $(t_b < t_c)$. The construction method of the weak entropy solution of (1.1) is also similar to that of Case (I).

Therefore, we accomplish the construction of the weak entropy solution to the initial boundary problem (1.1).

From the above construction procedure, we know that under the assumptions (A_1)–(A_5), the weak entropy solution of (1.1) is bounded and piecewise smooth with finitely many discontinuities. The key point of the above given construction method for the weak entropy solution of this initial boundary value problem is to use the structure of the weak entropy solution of the corresponding initial value problem (2.3). It mainly refers to the behavior of the leftmost shock wave appearing in the weak entropy solution of the initial value problem (2.3). In particular, if the leftmost shock wave stays in the second quadrant and the sign of the shock speed is changed from negative to positive before some time, or enters the second quadrant from the first quadrant including the t-axis after one time and keeps staying in the second quadrant, and the sign of the corresponding shock speed is changed from negative to positive before another time, then we need to find some time, such as t_1 or t_2 in Case (I), and take the time as the new initial time to reconstruct the solution, layer by layer. Compared with the initial value problem, the weak entropy solution of the initial boundary value problem (1.1) includes the following new interaction type: an expansion wave collides with the boundary $x = 0$ and the boundary reflects a new shock wave which is tangent to the boundary $x = 0$. The interaction of a shock wave and the boundary $x = 0$ includes the following two types: the shock wave is absorbed by the boundary $x = 0$ at some time, which is called the absorbed time of the shock wave (see Fig. 2(a),(c),(d)); the shock wave hits the boundary $x = 0$ at some time and comes back to the first quadrant after the same time (see Fig. 2(b)), at this time the shock wave is tangent to the boundary $x = 0$.

From now on, let $t_0 = 0$ and $t_p (p = 1, 2, \dots, P)$ (where $t_0 < t_1 < t_2 < \cdots < t_P$) be the absorbed time of the shock wave, or the intersection time of two or more shock waves, or the time at which the boundary $x = 0$ reflects a new shock or there is a spontaneous formation of a shock by a compression wave, or the time such that the shock wave is tangent to the boundary $x = 0$ at this time and lies in the first quadrant of the x-t plane in some deleted neighborhood of this time. For a fixed $T > t_P$, we let $t_{P+1} = T$.

Although the weak entropy solution of the initial boundary value problem (1.1) includes a new type in the sense of the global solution structure near the boundary, the solution is the restiction of that solution of some initial value problem on the corresponding domain $R^+ \times [t_p, t_{p+1}) (p = 0, 1, \dots, P)$. From this character on the solution construction, we can still establish the following global estimates on derivatives of the weak entropy solution of (1.1), which are similar to those of the initial value problem (see [23]), and are necessary for the error analysis.

Lemma 2.4 *Assume that $x = X(t)$ is a shock curve satisfying*

$$\dot{X}(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}, \qquad u^{\pm} := u(X(t) \pm 0, t),$$

and that $a(u_0)$ is sufficiently smooth in the neighborhood of its negative minimum points. If $x = X(t)$ is formed at $t = 0$, then

$$\int_0^T |u_x(X(t) \pm 0, t)| dt \le C.$$

If $x = X(t)$ is formed at $t = t_p > 0$, then

$$\int_{t_p+\delta}^T |u_x(X(t) \pm 0, t)| dt \le C|\ln \delta| + C,$$

where δ is a sufficiently small constant, $T > t_P$ is a fixed time before which the shock $x = X(t)$ is not absorbed by the boundary $x = 0$, and $C > 0$ is a constant independent of δ.

Lemma 2.5 *Assume that a central rarefaction wave is formed at $x = z$. Let $x = X_L(t)$ and $x = X_R(t)$ be left and right boundaries of the rarefaction wave, respectively. If $\dot{a}(u_0(z + 0))$ is not a negative minimum, then*

$$|u_x(X_R(t) + 0, t)| \leq C, \quad |u_x(X_R(t) - 0, t)| \leq Ct^{-1}.$$

The above results hold before the rarefaction wave is interacted by a shock and $x = X_R(t)$ hits the boundary. If $\dot{a}(u_0(z + 0))$ is a negative minimum whose minimal point will lead to a new shock at a future time, then

$$u_x(X_R(t) + 0, t)| \leq C(t - t_p)^{-1}, \quad |u_x(X_R(t) - 0, t)| \leq Ct^{-1},$$

for $t < t_p$, where $t_p = -1/\dot{a}(u_0(z + 0))$. The curve $x = X_R(t)$ will become a shock after $t = t_p$. Similar results, based on $\dot{a}(u_0(z - 0))$, hold for $X_L(t)$.

Lemma 2.6 *Under the assumptions* (A$_1$)–(A$_5$)*, the weak entropy solution $u(x, t)$ of* (1.1) *satisfies*

$$\int_{t_p+\delta}^{t_{p+1}-\delta} |u_x(0+, t)| \leq C|\ln \delta| + C$$

and

$$\int_{t_p+\delta}^{t_{p+1}-\delta} \|u_{xx}(\cdot, \tau)\|_{L^1([0,\infty))} d\tau \leq C|\ln \delta| + C, \quad p = 0, 1, 2, \ldots, P,$$

provided that δ is sufficiently small, where C is a constant independent of δ.

3 Error Estimates

Throughout this section, the norm $\|\cdot\|$ denotes the standard L^1-norm $\|\cdot\|_{L^1([0,\infty))}$, C or $C(t)$ denotes a positive constant independent of ε, and c denotes a positive constant independent of t and ε, but with different values at different places.

We now give the main result in this paper.

Theorem 3.1 *Suppose that $v_0(x) - u_0(x) \to 0$ (as $x \to \infty$) and $v_0(\cdot) - u_0(\cdot) \in L^1$. Under the assumptions* (A$_1$)–(A$_5$)*, if v_ε is the smooth solution of* (1.2) *and u is the weak entropy solution of* (1.1)*, then the following error estimate holds for any $T \geq 0$:*

$$\sup_{0 \leq t \leq T} \|v_\varepsilon(\cdot, t) - u(\cdot, t)\| \leq \|v_0(\cdot) - u_0(\cdot)\| + C(T)\varepsilon^{1/2}. \tag{3.1}$$

We use the matching method to prove this theorem. As mentioned before, this method relies on the behavior of the travelling solutions of the approximation equation and L^1-stability properties of the nonhomogeneous viscous equations. We first introduce the L^1-stability lemma and the travelling wave solution lemma.

Lemma 3.2 *Let $v^{(i)}(x, t)$ $(i = 1, 2)$ be continuous and piecewise smooth solutions of the following equations*:

$$(v^{(i)})_t + f(v^{(i)})_x - \varepsilon(v^{(i)})_{xx} = g_i(x, t), \quad x > 0, \ t > d \geq 0, \ i = 1, 2. \tag{3.2}$$

The above equations hold for all values of $x > 0$ except on some curves $X_m(t)$, $1 \leq m \leq M$, where $v_x^{(i)}$ may not exist. If $\omega := v^{(1)} - v^{(2)} \to 0$ as $x \to \infty$, then

$$\begin{aligned} \|\omega(\cdot, t)\| \leq{}& \|\omega(\cdot, d)\| + \int_d^t \|g_1(\cdot, \tau) - g_2(\cdot, \tau)\| d\tau + \varepsilon \int_d^t |\omega_x(0+, \tau)| d\tau \\ &+ \int_d^t \operatorname{sgn} \omega(0, \tau)(f(v^{(1)}(0, \tau)) - f(v^{(2)}(0, \tau))) d\tau \\ &+ \varepsilon \sum_{m=1}^{M} \int_d^t |\omega_x(X_m(\tau) + 0, \tau) - \omega_x(X_m(\tau) - 0, \tau)| d\tau. \end{aligned} \tag{3.3}$$

Proof We now prove this lemma by a similar technique to that in [23] (see also [42]). It follows from (3.2) that

$$\omega_t + (f(v^{(1)}) - f(v^{(2)}))_x = \varepsilon\omega_{xx} + g_1(x,t) - g_2(x,t). \tag{3.4}$$

If $\omega \geq 0$ or $\omega \leq 0$ for all x, then straightforward integration on the above equation gives (3.3). Let $(0 <)p_1(t) < p_2(t) < \cdots$ be the points such that at those points ω changes signs. Let α_j be the sign of ω in (p_j, p_{j+1}) $(j = 0, 1, 2, 3, \ldots, p_0 = 0)$. Multiplying (3.4) by α_j $(j = 1, 2, \ldots)$ and integrating the resulting equation over (p_j, p_{j+1}) gives

$$\begin{aligned}\alpha_j \int_{p_j}^{p_{j+1}} \omega_t dx = {} & \varepsilon(\alpha_j\omega_x(p_{j+1} - 0, t) - \alpha_j\omega_x(p_j + 0, t)) \\ & + \alpha_j \int_{p_j}^{p_{j+1}} (g_1(x,t) - g_2(x,t))dx \\ & + \varepsilon \sum_{p_j < X_m < p_{j+1}} \alpha_j(\omega_x(X_m(t) + 0, t) - \omega_x(X_m(t) - 0, t)).\end{aligned} \tag{3.5}$$

Since $\omega(p_j, t) = \omega(p_{j+1}, t) = 0$ and $\alpha_j\omega \geq 0$ for $x \in (p_j, p_{j+1})$, we have

$$\frac{d}{dt}\int_{p_j}^{p_{j+1}} |\omega| dx = \alpha_j \int_{p_j}^{p_{j+1}} \omega_t dx.$$

Moreover, observing that $\alpha_j\omega_x(p_{j+1} - 0, t) \leq 0$ and $\alpha_j\omega_x(p_j + 0, t) \geq 0$ $(j = 1, 2, \ldots)$ (because $\alpha_j\omega_x(p_{j+1} - 0, t) = \alpha_j \lim_{x \to p_{j+1}^-} \frac{\omega(x,t) - \omega(p_{j+1}, t)}{x - p_{j+1}} = \lim_{x \to p_{j+1}^-} \frac{\alpha_j\omega(x,t)}{x - p_{j+1}} \leq 0$), we obtain from (3.5) that

$$\begin{aligned}\frac{d}{dt}\int_{p_j}^{p_{j+1}} |\omega| dx \leq {} & \sum_{p_j < X_m < p_{j+1}} \varepsilon\alpha_j |\omega_x(X_m(t) + 0, t) - \omega_x(X_m(t) - 0, t)| \\ & + \int_{p_j}^{p_{j+1}} \big|g_1(x,t) - g_2(x,t)\big| dx.\end{aligned}$$

Since the above inequality is true for all $j \geq 1$, we have

$$\frac{d}{dt}\int_{p_1}^{p^*} |\omega| dx \leq \sum_{p_1 < X_m < p^*} \varepsilon|\omega_x(X_m(t) + 0, t) - \omega_x(X_m(t) - 0, t)| + \int_{p_1}^{p^*} |g_1(x,t) - g_2(x,t)| dx, \tag{3.6}$$

where $p^* = \sup_j p_j$. If $p^* < \infty$, using a similar method to that above gives

$$\frac{d}{dt}\int_{p^*}^{\infty} |\omega| dx \leq \varepsilon \sum_{X_m > p^*} |\omega_x(X_m(t) + 0, t) - \omega_x(X_m(t) - 0, t)| + \int_{p^*}^{\infty} |g_1(x,t) - g_2(x,t)| dx. \tag{3.7}$$

In obtaining the last inequality, we have used the fact that $\omega \to 0$ as $x \to \infty$. In order to get (3.3), we need to estimate $\frac{d}{dt}\int_0^{p_1} |\omega| dx$. Multiplying (3.4) by α_0 and integrating the resulting equation over $(0, p_1)$ gives

$$\begin{aligned}\frac{d}{dt}\int_0^{p_1} |\omega| dx = {} & \alpha_0(f(v^{(1)}(0,t)) - f(v^{(2)}(0,t))) + \alpha_0 \int_0^{p_1} (g_1(x,t) - g_2(x,t))dx \\ & + \varepsilon(\alpha_0\omega_x(p_1 - 0, t) - \alpha_0\omega_x(0+, t)) \\ & + \sum_{0 < X_m < p_1} \alpha_0\varepsilon(\omega_x(X_m(t) + 0, t) - \omega_x(X_m(t) - 0, t)),\end{aligned}$$

then

$$\begin{aligned}\frac{d}{dt}\int_0^{p_1} |\omega| dx \leq {} & \varepsilon|\omega_x(0+, t)| + \int_0^{p_1} |g_1(x,t) - g_2(x,t)| dx \\ & + \operatorname{sgn}\omega(0,t)(f(v^{(1)}(0,t)) - f(v^{(2)}(0,t))) \\ & + \sum_{0 < X_m < p_1} \varepsilon|\omega_x(X_m(t) + 0, t) - \omega_x(X_m(t) - 0, t)|.\end{aligned} \tag{3.8}$$

Using (3.6)–(3.8), we obtain the desired result.

From [23], we have the following travelling wave lemma.

Lemma 3.3 *Let $u^+(t) < u^-(t)$ be two given functions, and $X(t)$ be defined by the equation*

$$\frac{dX(t)}{dt} = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

Then there is a unique travelling wave $V_\varepsilon(x - X(t); u^-, u^+)$ of $(1.2)_1$, where $V_\varepsilon(\xi; u^-, u^+)$ be defined implicitly by

$$\xi = \varepsilon \int_{\frac{1}{2}(u^- + u^+)}^{V_\varepsilon} (\Phi(u; u^-, u^+))^{-1} du, \quad \phi(u; u^-, u^+) = f(u) - f(u^-) - \frac{f(u^+) - f(u^-)}{u^+ - u^-}(u - u^-).$$

With respect to ξ, the function $V_\varepsilon(\xi; u^-, u^+)$ is a decreasing function satisfying

$$V_\varepsilon(-\infty; u^-, u^+) = u^-,\ V_\varepsilon(0; u^-, u^+) = \frac{1}{2}(u^- + u^+),\ V_\varepsilon(+\infty; u^-, u^+) = u^+.$$

It also satisties the following properties:

(1) $|V_\varepsilon(\xi; u^-, u^+) - H(\xi; u^-, u^+)| \le (u^- - u^+) \exp\{-\alpha(u^- - u^+)|\xi|/2\varepsilon\}$;

(2) $\|V_\varepsilon(\bullet; u^-, u^+) - H(\bullet; u^-, u^+)\|_{L^1(-\infty,\infty)} \le c\varepsilon$;

(3) $\|(V_\varepsilon)_{u^-}(\bullet; u^-, u^+)\dot{u}^- + (V_\varepsilon)_{u^+}(\bullet; u^-, u^+)\dot{u}^+ - H(\bullet; \dot{u}^-, \dot{u}^+)\| \le C\varepsilon(|u_x(X(t) + 0, t)| + |u_x(X(t) - 0, t)|)$;

where α is determined by (A_1), *$(V_\varepsilon)_{u^\pm}$ indicate the partial derivatives for V_ε with respect to the parameters $u^\pm$, respectively, $\dot{u}^\pm = d(u^\pm)/dt$, H is the so-called Heaviside function defined by*

$$H(x; u^-, u^+) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases}$$

We state the main clue of the proof of Theorem 3.1. Following [23], if the weak entropy solution $u(x, t)$ of (1.1) contains no shock discontinuity for $t \in (\tau_1, \tau_2)$ $(0 < \tau_1 < \tau_2 < T < \infty)$, then in view of the continuity of u in the time interval (τ_1, τ_2), we can directly apply Lemma 3.2 to $v_\varepsilon(x, t)$ and $u(x, t)$. Otherwise, we construct an auxiliary continuous approximation $\overline{v}_\varepsilon$ by replacing jumps of all shocks in the weak entropy solution u at each fixed time t in (τ_1, τ_2) with their corresponding travelling wave solutions of the approximation equation. Using the travelling wave solution Lemma 3.3 can estimate the L^1-error of $\overline{v}_\varepsilon - u$. As for the L^1-error bound of $v_\varepsilon - \overline{v}_\varepsilon$, we apply the L^1-stability Lemma 3.2 to $\overline{v}_\varepsilon(x, t)$ and $v_\varepsilon(x, t)$.

Proof of Theorem 3.1 We need to prove only that the following estimate is valid:

$$\|v_\varepsilon(\cdot, t) - u(\cdot, t)\| \le \|v_\varepsilon(\cdot, t_p) - u(\cdot, t_p)\| + C(\varepsilon^{1/2} + \varepsilon|\ln\varepsilon| + \varepsilon), \quad t \in [t_p, t_{p+1}], \tag{3.9}$$

where $t_p (p = 0, 1, 2, \ldots, P)$ is defined in Section 2. In fact, if (3.9) is true for all $0 \le p \le P$, then we can obtain that, for any $t \in [0, T]$,

$$\|v_\varepsilon(\cdot, t) - u(\cdot, t)\| \le \|v_\varepsilon(\cdot, t_0) - u(\cdot, t_0)\| + C(T)(\varepsilon^{1/2} + \varepsilon|\ln\varepsilon| + \varepsilon) \le \|v_0(\cdot) - u_0(\cdot)\| + C(T)\varepsilon^{1/2}.$$

Next we prove the estimate (3.9) by dividing our problem into four cases.

3.1 Zero Shock

In this case, the weak entropy solution u of (1.1) contains no shock discontinuity in the domain $R^+ \times (t_p, t_{p+1})$. We directly apply Lemma 3.2 to v_ε and u, and get, for $t \in (t_p + \varepsilon, t_{p+1} - \varepsilon)$,

$$\begin{aligned} \|v_\varepsilon(\cdot, t) - u(\cdot, t)\| \le{}& \|v_\varepsilon(\cdot, t_p + \varepsilon) - u(\cdot, t_p + \varepsilon)\| + \varepsilon \int_{t_p+\varepsilon}^{t} \|u_{xx}(\cdot, \tau)\| d\tau \\ &+ \varepsilon \int_{t_p+\varepsilon}^{t} (|v_{\varepsilon x}(0, \tau)| + |u_x(0, \tau)|) d\tau \\ &+ \int_{t_p+\varepsilon}^{t} \mathrm{sgn}(u_- - u(0, \tau))(f(u_-) - f(u(0, \tau))) d\tau \\ &+ \varepsilon \sum_k \int_{t_p+\varepsilon}^{t} |u_x(Y_k(\tau)\pm, \tau)| d\tau \end{aligned}$$

$$\leq \|v_\varepsilon(\cdot, t_p + \varepsilon) - u(\cdot, t_p + \varepsilon)\| + C(T)(\varepsilon|\ln\varepsilon| + \varepsilon), \tag{3.10}$$

where $Y_k(t)(1 \leq k \leq K)$ denotes the left or right boundary of the rarefaction waves for $t \in (t_p, t_{p+1})$. If some boundary $Y_{k_0}(t)$ of the rarefaction wave hits the boundary $x = 0$ at $t = t_p^*$, then from now on, we stipulate that $u_x(Y_{k_0}\pm, t) = 0$ for $t_p^* < t < t_{p+1}$. In obtaining (3.10), we have used (2.2), Lemma 2.5 and Lemma 2.6. Since v_ε and u satisfy the following stability results:

$$\|v_\varepsilon(\cdot,\tau) - v_\varepsilon(\cdot,\tau_0)\| \leq C|\tau - \tau_0|, \quad \|u(\cdot,\tau) - u(\cdot,\tau_0)\| \leq C|\tau - \tau_0|, \tag{3.11}$$

by (3.10), we get $\|v_\varepsilon(\cdot,t) - u(\cdot,t)\| \leq \|v_\varepsilon(\cdot,t_p) - u(\cdot,t_p)\| + C(\varepsilon|\ln\varepsilon| + \varepsilon)$, $t \in [t_p, t_{p+1}]$.

3.2 One Shock

In this case, we suppose that the weak entropy solution u of (1.1) is continuous in $R^+ \times (t_p, t_{p+1})$ except on the shock $x = X(t)$, i.e., there is only one shock curve $x = X(t)$ included in u in $R^+ \times (t_p, t_{p+1})$. According to the structure of the weak entropy solution u of (1.1), $x = X(t)$ is C^2-smooth and satisfies one of the following possible cases:

(P$_1$) $X(t_p) = 0$ and $X(t) > 0$ for $t \in (t_p, t_{p+1}]$;
(P$_2$) $X(t_p) = X(t_{p+1}) = 0$ and $x = X(t) > 0$ for $t \in (t_p, t_{p+1})$;
(P$_3$) $X(t) > 0$ for $t \in [t_p, t_{p+1}]$;
(P$_4$) $X(t) > 0$ for $t \in [t_p, t_{p+1})$ and $X(t_{p+1}) = 0$.

Lemma 3.2 cannot be applied directly since $u \notin C(R^+ \times (t_p, t_{p+1}))$. In order to overcome the difficulty, we introduce the following approximation solution of u:

$$\overline{v}_\varepsilon(x,t) = u(x,t) + V_\varepsilon(x - X(t); u^-, u^+) - H(x - X(t); u^-, u^+), \tag{3.12}$$

where $u^\pm := u(X(t) \pm 0, t)$. It is easy to verify that, in $R^+ \times (t_p, t_{p+1})$, $\overline{v}_\varepsilon$ is continuous and piecewise smooth except on the curve $x = X(t)$, and $\overline{v}_\varepsilon(x,t) - v_\varepsilon(x,t) \to 0$ as $x \to +\infty$.

Using the same technique as in [23], by Lemmas 2.4–2.6 and Lemma 3.3, we can derive that in its smooth domains, $\overline{v}_\varepsilon$ satisfies the equation

$$(\overline{v}_\varepsilon)_t + f(\overline{v}_\varepsilon)_x - \varepsilon(\overline{v}_\varepsilon)_{xx} = \overline{g}(x,t),$$

with

$$\|\overline{g}(\cdot,t)\| \leq \frac{C\varepsilon}{t}. \tag{3.13}$$

Therefore, Lemma 3.2 can be applied to $\overline{v}_\varepsilon(x,t)$ and v_ε, and from (3.12), (3.13), Lemmas 2.4–2.6 and Lemma 3.3, it follows that, for $t \in (t_p + \varepsilon, t_{p+1} - \varepsilon)$,

$$\begin{aligned}
\|v_\varepsilon(\cdot,t) - \overline{v}_\varepsilon(\cdot,t)\| \leq{}& \|v_\varepsilon(\cdot,t_p+\varepsilon) - \overline{v}_\varepsilon(\cdot,t_p+\varepsilon)\| + \int_{t_p+\varepsilon}^{t} \|\overline{g}(\cdot,\tau)\| d\tau \\
&+ \varepsilon \int_{t_p+\varepsilon}^{t} (|v_{\varepsilon x}(0,\tau)| + |\overline{v}_{\varepsilon x}(0,\tau)|) d\tau \\
&+ \int_{t_p+\varepsilon}^{t} \operatorname{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-)(f(\overline{v}_\varepsilon(0,\tau)) - f(u_-)) d\tau \\
&+ \varepsilon \int_{t_p+\varepsilon}^{t} |u_x(X(\tau) \pm 0, \tau)| d\tau \\
&+ \varepsilon \sum_k \int_{t_p+\varepsilon}^{t} |u_x(Y_k(\tau) \pm 0, \tau)| d\tau \\
\leq{}& \|v_\varepsilon(\cdot,t_p+\varepsilon) - \overline{v}_\varepsilon(\cdot,t_p+\varepsilon)\| + C(\varepsilon + \varepsilon|\ln\varepsilon|) + I_0,
\end{aligned} \tag{3.14}$$

where $Y_k(t)$ is defined as in Section 3.1, and

$$I_0 := \int_{t_p+\varepsilon}^{t} \operatorname{sgn}(\overline{v}_\varepsilon(0,\tau) - u_-)(f(\overline{v}_\varepsilon(0,\tau)) - f(u_-)) d\tau.$$

Using Lemma 3.3 yields

$$\|\overline{v}_\varepsilon(\cdot,t)-u(\cdot,t)\|\le c\varepsilon,\quad t\in(t_p,t_{p+1}).\tag{3.15}$$

If we are able to prove

$$I_0\le C(\varepsilon+\varepsilon^{1/2}),\quad t\in(t_p+\varepsilon,t_{p+1}-\varepsilon),\tag{3.16}$$

then combining (3.14), (3.15) and the stability results (3.11) gives (3.9).

Next, we verify (3.16) for possible cases (P_1)–(P_4), respectively.

3.2.1 The Case of (P_1)

By the structure of the weak entropy solution of (1.1), $\dot{X}(t_p)\ge 0$ and for $t\in(t_p,t_{p+1})$, $u(0,t)\equiv u_-$, $u(X(t)-0,t)=u_->u^+$. Then from (3.12), the smoothness of f, the boundedness of u and Lemma 3.3, we have

$$I_0\le C\int_{t_p+\varepsilon}^t|(H-V_\varepsilon)(-X(\tau);u^-,u^+)|d\tau\le C\int_{t_p+\varepsilon}^t\exp\{-c\alpha X(\tau)/2\varepsilon\}d\tau.\tag{3.17}$$

Therefore, in order that (3.16) holds, the key point is to find a suitable *lower bound function* of $X(t)$ for $t\in[t_p,t_{p+1}]$.

When $\dot{X}(t_p)>0$, there exists some $t_*\in(t_p,t_{p+1})$ such that $\ddot{X}(t)>0$ or $\ddot{X}(t)\le 0$ for $t\in[t_p,t_*]$. If $\ddot{X}(t)\le 0$ for $t\in[t_p,t_*]$, then

$$X(t)\ge\frac{x_*}{t_{p+1}-t_p}(t-t_p),\quad t\in[t_p,t_{p+1}].\tag{3.18$_1$}$$

If $\ddot{X}(t)>0$ for $t\in[t_p,t_*]$, then

$$X(t)\ge\frac{x_{**}}{t_{p+1}-t_p}(t-t_p),\quad t\in[t_p,t_{p+1}],\tag{3.18$_2$}$$

where $x_*=\min_{t\in[t_*,t_{p+1}]}X(t)>0$ and $x_{**}>0$ is the minimum of x_* and $\dot{X}(t_p)(t_{p+1}-t_p)$.

When $\dot{X}(t_p)=0$, by the structure of the weak entropy solution of (1.1) (see Fig. 1(a),(b) and Fig. 2(a),(b) for $t_p\ne 0$), there is $t_{**}\in(t_p,t_{p+1})$ such that $\ddot{X}(t)>0$ for $t\in(t_p,t_{**})$. Since $X(t)(t_p<t<t_{**})$ is smooth and increasing, for any given constant $\varepsilon_0, 0<\varepsilon_0<X(t_{**})$, there exists $t_{\varepsilon_0}\in(t_p,t_{**})$ such that $X(t_{\varepsilon_0})=\varepsilon_0$. From the Rankine–Hugoniot condition and the conditions $(A_1),(A_2)$, we can easily verify that, for $t\in[t_p,t_{\varepsilon_0})$,

$$X(t)=X(t_p)+\dot{X}(t_p)(t-t_p)+\ddot{X}(l)(t-t_p)^2/2\ge c(t-t_p)^2,\quad l\in(t_p,t).$$

Thus we can get the following inequality:

$$X(t)\ge\begin{cases}c(t-t_p)^2, & t\in[t_p,t_{\varepsilon_0}),\\ \dfrac{x_{\varepsilon_0}}{t_{p+1}-t_{\varepsilon_0}}(t-t_{\varepsilon_0}), & t\in[t_{\varepsilon_0},t_{p+1}],\end{cases}\tag{3.18$_3$}$$

where $x_{\varepsilon_0}=\min_{t\in[t_{\varepsilon_0},t_{p+1}]}X(t)>0$. Applying $(3.18)_1$–$(3.18)_3$ to (3.17), we obtain (3.16).

3.2.2 The Case of (P_2)

By the structure of the weak entropy solution of (1.1), $\dot{X}(t_p)\ge 0$ and for $t\in(t_p,t_{p+1})$, $u(0,t)\equiv u_-$, $u(X(t)-0,t)=u_->u^+$. Then from (3.12), the smoothness of f, the boundedness of u and Lemma 3.3, it follows that (3.17) is also valid for this case.

When $\dot{X}(t_{p+1})<0$, there exists some $t_*\in[t_p,t_{p+1})$ such that $\ddot{X}(t)>0$ or $\ddot{X}(t)\le 0$ for $t\in[t_*,t_{p+1}]$. If $\ddot{X}(t)\le 0$ for $t\in[t_*,t_{p+1}]$, then

$$X(t)\ge\frac{X(t_*)}{t_{p+1}-t_*}(t_{p+1}-t),\quad t\in[t_*,t_{p+1}].\tag{3.19$_1$}$$

If $\ddot{X}(t)>0$ for $t\in[t_*,t_{p+1}]$, then

$$X(t)\ge\dot{X}(t_{p+1})(t-t_{p+1}),\quad t\in[t_*,t_{p+1}].\tag{3.19$_2$}$$

Thus, using $(3.19)_1$ and $(3.19)_2$ we obtain, for $t\in[t_*,t_{p+1}]$,

$$\int_{t_*}^t\exp\{-c\alpha X(\tau)/2\varepsilon\}d\tau\le C\varepsilon.\tag{3.20$_1$}$$

Since $X(t_p)=0$, by the same technique as in case (P_1), we can conclude, for $t\in(t_p,t_*]$, that

$$\int_{t_p+\varepsilon}^{t}\exp\{-c\alpha X(\tau)/2\varepsilon\}d\tau\le C(\varepsilon+\varepsilon^{1/2}). \tag{3.20$_2$}$$

Consequently, $(3.20)_1$ and $(3.20)_2$ result in (3.16).

When $\dot X(t_{p+1})=0$, from the structure of the weak entropy solution of (1.1) (see Fig. 2(a), (b), (c)), it follows that there is a $t_*\in(t_p,t_{p+1})$ such that $\ddot X(t)>0$ for $t\in(t_*,t_{p+1})$. Since $X(t)(t_*<t<t_{p+1})$ is smooth and decreasing, for any given constant $\varepsilon_0, 0<\varepsilon_0<X(t_*)$, there exists $t_{\varepsilon_0}\in(t_*,t_{p+1})$ such that $X(t_{\varepsilon_0})=\varepsilon_0$. Thus it is easy to verify that, for $t\in(t_{\varepsilon_0},t_{p+1}]$,

$$X(t)=X(t_{p+1})+\dot X(t_{p+1})(t-t_{p+1})+\ddot X(\theta)(t-t_{p+1})^2/2\ge c(t-t_{p+1})^2,\quad \theta\in(t,t_{p+1}),$$

from which, it follows that, for $t\in[t_{\varepsilon_0},t_{p+1}]$,

$$\int_{t_{\varepsilon_0}}^{t}\exp\{-c\alpha X(\tau)/2\varepsilon\}d\tau\le\varepsilon^{1/2},$$

whereas for $t\in(t_p,t_{\varepsilon_0}]$, by the same technique as in case (P_1), we can obtain

$$\int_{t_p+\varepsilon}^{t}\exp\{-c\alpha X(\tau)/2\varepsilon\}d\tau\le C(\varepsilon+\varepsilon^{1/2}).$$

Combining the above two inequalities gives (3.16).

3.2.3 The Case of (P_3)

Let $a_0:=\min_{t\in[t_p,t_{p+1}]}X(t)>0$, and $(H-V_\varepsilon)(t):=(H-V_\varepsilon)(-X(t);u^-,u^+)$. Then

$$X(t)\ge\frac{a_0}{t_{p+1}-t_p}(t-t_p),\quad t\in[t_p,t_{p+1}],$$

and, from which and Lemma 3.3, we have, for $t\in[t_p,t_{p+1}]$,

$$0<(H-V_\varepsilon)(t)\le C\exp\left\{-\frac{c\alpha a_0}{t_{p+1}-t_p}\cdot\frac{t-t_p}{2\varepsilon}\right\}:=e(t). \tag{3.21}$$

By the structure of the weak entropy solution $u(x,t)$ of (1.1), we know that $u(0,t)(t\in(t_p,t_{p+1}))$ is continuous and $u(x,t)$ only contains a finite number of compression waves or expansion waves or constant states near the segment of the boundary: $x=0,t\in(t_p,t_{p+1})$. Hence, the time interval $[t_p,t_{p+1}]$ is composed of a finite number of the closed sub-intervals on which $u(0,t)$ is strictly increasing or strictly decreasing or constant, and for $t\in(t_p,t_{p+1})$, the function $u(0,t)-u_-$ is continuous and changes its signs at most finitely many times. Let $[t_p,t_{p+1}]=\cup_{i=1}^{i_0}[t_p^{(i)},t_p^{(i+1)}]\cup[t_p,t_p^{(1)}]$, where i_0 is a nonnegative integer, $t_p^{(i_0+1)}=t_{p+1}$, $t_p^{(1)}<t_p^{(2)}<\cdots<t_p^{(i_0)}$ are the points such that at those points $u(0,t)-u_-$ changes signs, and in each of these open sub-intervals $(t_p,t_p^{(1)})$ and $(t_p^{(i)},t_p^{(i+1)})\,(i=1,\dots,i_0)$ $u(0,t)-u_->0$ or $u(0,t)-u_-<0$ or $u(0,t)-u_-\equiv0$. For small ε, $t_p+\varepsilon$ must lie on some closed sub-interval $[t_p^{(i)},t_p^{(i+1)}]$ or open sub-interval $(t_p,t_p^{(1)})$. There is no harm in assuming $t_p+\varepsilon\in(t_p,t_p^{(1)})$ (this can be done by letting ε be sufficiently small). Denote $t_p+\varepsilon$ by $t_p^{(0)}$, then $[t_p+\varepsilon,t_{p+1}]=\cup_{i=0}^{i_0}[t_p^{(i)},t_p^{(i+1)}]$.

If for each $i\,(i=0,1,2,\dots,i_0)$, the following inequality holds:

$$I_0^{(i)}:=\int_{t_p^{(i)}}^{t}\operatorname{sgn}(\overline v_\varepsilon(0,\tau)-u_-)(f(\overline v_\varepsilon(0,\tau))-f(u_-))d\tau\le C\varepsilon,\quad t\in(t_p^{(i)},t_p^{(i+1)}], \tag{3.22}$$

then (3.16) is valid. Now we verify (3.22).

If in $(t_p^{(i)},t_p^{(i+1)})$, $u(0,t)-u_-\equiv0$, then for $t\in(t_p^{(i)},t_p^{(i+1)}]$,

$$I_0^{(i)}\le C\int_{t_p^{(i)}}^{t}|\overline v_\varepsilon(0,t)-u_-|d\tau=C\int_{t_p^{(i)}}^{t}(H-V_\varepsilon)(\tau)d\tau.$$

By the above inequality and (3.21), (3.22) is true.

If in $(t_p^{(i)}, t_p^{(i+1)})$, $u(0,t)-u_- < 0$, then by the boundary entropy condition (2.2), $f(u(0,t))-f(u_-) \geq 0$ for $t \in [t_p^{(i)}, t_p^{(i+1)}]$, thus from (3.21) and (3.12), it follows that $\mathrm{sgn}(\overline{v}_\varepsilon(0,t)-u_-)(f(u(0,t))-f(u_-)) \leq 0$. Therefore, for $t \in (t_p^{(i)}, t_p^{(i+1)}]$, we have

$$\begin{aligned} I_0^{(i)} &\leq \int_{t_p^{(i)}}^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau)-u_-)(f(\overline{v}_\varepsilon(0,\tau))-f(u(0,t)))d\tau \\ &\leq C\int_{t_p^{(i)}}^t |\overline{v}_\varepsilon(0,t)-u(0,t)|d\tau \leq C\int_{t_p^{(i)}}^t (H-V_\varepsilon)(\tau)d\tau. \end{aligned}$$

The above inequality and (3.21) gives (3.22).

If in $(t_p^{(i)}, t_p^{(i+1)})$, $u(0,t)-u_- > 0$, then by the boundary entropy condition (2.2), one gets

$$f(u(0,t))-f(u_-) \leq 0, \quad t \in (t_p^{(i)}, t_p^{(i+1)}]. \tag{3.23}$$

Since the function $e(t)$ defined by (3.21) is strictly decreasing on $[t_p, t_{p+1}]$, by the properties of $u(0,t)$, we have that the function $u(0,t)-u_--e(t)$ changes its signs at most finitely many times on $[t_p^{(i)}, t_p^{(i+1)}]$, in other words, $[t_p^{(i)}, t_p^{(i+1)}]$ can be decomposed as follows: $[t_p^{(i)}, t_p^{(i+1)}]=\cup_{j=0}^{j_0^{(i)}}[t_p^{(i,j)}, t_p^{(i,j+1)}]$, where $j_0^{(i)}$ is a non-negative integer, $t_p^{(i,0)}=t_p^{(i)}$, $t_p^{(i,j_0^{(i)}+1)}=t_p^{(i+1)}$, $t_p^{(i,1)} < t_p^{(i,2)} < \cdots < t_p^{(i,j_0^{(i)})}$ are the points such that at those points $u(0,t)-u_--e(t)$ changes signs and in each open sub-interval $(t_p^{(i,j)}, t_p^{(i,j+1)})(j=0,1,\ldots,j_0^{(i)})$, $u(0,t)-u_--e(t) > 0$ or $u(0,t)-u_--e(t) < 0$ or $u(0,t)-u_--e(t) \equiv 0$. When $u(0,t)-u_--e(t) > 0$ or $u(0,t)-u_--e(t) \equiv 0$ in $(t_p^{(i,j)}, t_p^{(i,j+1)})$, by (3.12) and (3.21), we can get that for $t \in [t_p^{(i,j)}, t_p^{(i,j+1)}]$,

$$\overline{v}_\varepsilon(0,t)-u_- \geq u(0,t)-u_--e(t) \geq 0. \tag{3.24}$$

From (3.23), (3.24) and (3.21), it follows that for $t \in (t_p^{(i,j)}, t_p^{(i,j+1)}]$,

$$\begin{aligned} &\int_{t_p^{(i,j)}}^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau)-u_-)(f(\overline{v}_\varepsilon(0,\tau))-f(u_-))d\tau \leq \int_{t_p^{(i,j)}}^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau)-u_-)(f(\overline{v}_\varepsilon(0,\tau))-f(u(0,t)))d\tau \\ &\quad \leq C\int_{t_p^{(i,j)}}^t |\overline{v}_\varepsilon(0,t)-u(0,t)|d\tau \leq C\int_{t_p^{(i,j)}}^t (H-V_\varepsilon)(\tau)d\tau \leq C\int_{t_p}^{t_{p+1}} e(\tau)d\tau \leq C\varepsilon. \end{aligned} \tag{3.25}$$

This implies that (3.22) is valid. When $u(0,t)-u_--e(t) < 0$ in $(t_p^{(i,j)}, t_p^{(i,j+1)})$, by (3.12) and (3.21), we have that for $t \in (t_p^{(i,j)}, t_p^{(i,j+1)}]$,

$$\begin{aligned} &\int_{t_p^{(i,j)}}^t \mathrm{sgn}(\overline{v}_\varepsilon(0,\tau)-u_-)(f(\overline{v}_\varepsilon(0,\tau))-f(u_-))d\tau \\ &\quad \leq C\int_{t_p^{(i,j)}}^t |\overline{v}_\varepsilon(0,t)-u_-|d\tau \leq C\int_{t_p^{(i,j)}}^t ((u(0,\tau)-u_-)+(H-V_\varepsilon)(\tau))d\tau \\ &\quad \leq C\int_{t_p}^{t_{p+1}} e(\tau)d\tau \leq C\varepsilon. \end{aligned} \tag{3.26}$$

This shows that (3.22) holds.

3.2.4 The Case of (P_4)

In this case, $\dot{X}(t_{p+1}) \leq 0$. As in Case (P_3), $[t_p+\varepsilon, t_{p+1}]$ can be decomposed as follows: $[t_p+\varepsilon, t_{p+1}] = \cup_{i=0}^{i_0}[t_p^{(i)}, t_p^{(i+1)}]$, where i_0 is a nonnegative integer, $t_p^{(0)}=t_p+\varepsilon$, $t_p^{(i_0+1)}=t_{p+1}$, $t_p^{(1)} < t_p^{(2)} < \cdots < t_p^{(i_0)}$ are the points such that at those points $u(0,t)-u_-$ changes signs and in each open sub-interval $(t_p^{(i)}, t_p^{(i+1)})$ $(i=0,1,\ldots,i_0)$, $u(0,t)-u_- > 0$ or $u(0,t)-u_- < 0$ or $u(0,t)-u_- \equiv 0$.

As previously, we first estimate $X(t)(t \in (t_p^{(i_0)}, t_{p+1}])$. By the structure of the weak entropy solution of (1.1), there exists $t_* \in (t_p^{(i_0)}, t_{p+1})$ such that $\ddot{X}(t) > 0$ or $\ddot{X}(t) \leq 0$ for $t \in [t_*, t_{p+1}]$.

When $\ddot{X}(t) \le 0$ for $t \in [t_*, t_{p+1}]$, we have

$$X(t) \ge \frac{b_*}{t_{p+1} - t_p}(t_{p+1} - t), \quad t \in [t_p, t_{p+1}], \tag{3.27$_1$}$$

where $b_* = \min_{t\in[t_p,t_*]} X(t) > 0$.

When $\ddot{X}(t) > 0$ for $t \in [t_*, t_{p+1}]$ and $\dot{X}(t_{p+1}) < 0$, we have

$$X(t) \ge \frac{b_{**}}{t_{p+1} - t_p}(t_{p+1} - t), \quad t \in [t_p, t_{p+1}], \tag{3.27$_2$}$$

where $b_{**} > 0$ is the minimum of b_* and $\dot{X}(t_{p+1})(t_p - t_{p+1})$.

When $\ddot{X}(t) > 0$ for $t \in [t_*, t_{p+1}]$ and $\dot{X}(t_{p+1}) = 0$, as in (P$_2$), for any given constant $\varepsilon_0, 0 < \varepsilon_0 < X(t_*)$, there exists $t_{\varepsilon_0} \in (t_*, t_{p+1})$ such that $X(t_{\varepsilon_0}) = \varepsilon_0$; furthermore, for $t \in (t_{\varepsilon_0}, t_{p+1}]$, we also have

$$X(t) = X(t_{p+1}) + \dot{X}(t_{p+1})(t - t_{p+1}) + \ddot{X}(\theta)(t - t_{p+1})^2/2 \ge c(t - t_{p+1})^2, \quad \theta \in (t, t_{p+1}).$$

Accordingly,

$$X(t) \ge \begin{cases} \dfrac{a_{\varepsilon_0}}{t_{\varepsilon_0} - t_p}(t_{\varepsilon_0} - t), & t \in [t_p, t_{\varepsilon_0}], \\ c(t - t_{p+1})^2, & t \in (t_{\varepsilon_0}, t_{p+1}]), \end{cases} \tag{3.27$_3$}$$

where $a_{\varepsilon_0} = \min_{[t_p,t_{\varepsilon_0}]} X(t) > 0$.

From (3.27)$_1$–(3.27)$_3$ and Lemma 3.3, it follows that for $t \in [t_p, t_{p+1}]$,

$$0 < (H - V_\varepsilon)(t) \le C \exp\left\{ -\frac{c\alpha b_{**}}{t_{p+1} - t_p} \frac{t_{p+1} - t}{2\varepsilon} \right\} := e_1(t) \tag{3.28$_1$}$$

provided $\dot{X}(t_{p+1}) < 0$, and

$$\begin{aligned} 0 < (H - V_\varepsilon)(t) \quad &\le \begin{cases} C \exp\left\{ -\dfrac{c\alpha a_{\varepsilon_0}}{t_{\varepsilon_0} - t_p} \dfrac{t_{\varepsilon_0} - t}{2\varepsilon} \right\}, & t \in [t_p, t_{\varepsilon_0}], \\ C \exp\left\{ \dfrac{-c\alpha(t - t_{p+1})^2}{2\varepsilon} \right\}, & t \in (t_{\varepsilon_0}, t_{p+1}], \end{cases} \\ &:= e_2(t) \end{aligned} \tag{3.28$_2$}$$

provided $\dot{X}(t_{p+1}) = 0$.

Notice that $e_1(t)$ (in $[t_p, t_{p+1}]$), $e_2(t)$ (in $[t_p, t_{\varepsilon_0}]$ or $(t_{\varepsilon_0}, t_{p+1}]$) are strictly monotone functions, respectively. Therefore, by using the same technique as in case (P$_3$), from (3.28)$_1$ or (3.28)$_2$, we can prove that the following is valid for $t \in (t_p^{(i)}, t_p^{(i+1)}]$:

$$I_0^{(i)} \le \begin{cases} C\varepsilon, & \text{for } i = 0, 1, 2, \ldots, i_0 - 1, i_0 \ne 0, \\ C(\varepsilon + \varepsilon^{1/2}), & \text{for } i = i_0, \end{cases} \tag{3.29}$$

where $I_0^{(i)}$ is defined by (3.22). Thus (3.16) follows from (3.29).

3.3 More Shocks

Suppose that we have M smooth shock curves $x = X_m(t)$ included in u in $R_+ \times (t_p, t_{p+1})$, $1 \le m \le M$. According to the structure of the weak entropy solution of (1.1) and the definition of t_p and t_{p+1}, we know that

$$0 < X_1(t) < X_2(t) < \cdots < X_M(t), \quad t \in (t_p, t_{p+1}); \tag{3.30$_1$}$$

$$0 \le X_1(t_p) < X_2(t_p) < \cdots < X_M(t_p); \tag{3.30$_2$}$$

$$0 \le X_1(t_{p+1}) \le X_2(t_{p+1}) \le \cdots \le X_M(t_{p+1}). \tag{3.30$_3$}$$

The approximation $\overline{v}_\varepsilon$ to u is constructed as follows:

$$\overline{v}_\varepsilon(x, t) = u(x, t) + \sum_{m=1}^{M} (V_\varepsilon(x - X_m(t); u_m^-, u_m^+) - H(x - X_m(t); u_m^-, u_m^+)),$$

where $u_m^\pm := u(X_m(t) \pm 0, t)\,(m = 1, 2, \ldots, M)$.

From $(3.30)_1$–$(3.30)_3$, we may choose the lower bound function of $X_1(t)$ as that of $X_m(t)$ $(m = 2, 3, \ldots, M)$ for $t \in [t_p, t_{p+1}]$. Then by an analogous argument to that used in Section 3.2, and making use of Lemma 3.2 and Lemma 3.3, we can verify (3.9). The details are omitted.

This completes the proof of Theorem 3.1.

Remark 3.1 From the proof process of Theorem 3.1, we know the following fact: If the inviscid solution includes the interaction that an expansion wave (including central rarefaction waves) collides with the boundary $x = 0$ and the boundary reflects a new shock wave which is tangent to the boundary (see Fig. 1(a),(b) and Fig. 2(a)), or includes some shock wave which is tangent to the boundary and is not reflected by the boundary at this tangent time (see Fig. 2(a), (b), (c)), then the error of the viscosity solution to the inviscid solution is bounded by $O(\varepsilon^{1/2})$ in L^1-norm; otherwise, as in the initial value problem in [23], the L^1-error bound is $O(\varepsilon|\ln\varepsilon|)$.

Remark 3.2 The conclusion in this paper can be extended to more general initial boundary problems, in which $u_b(t)(t > 0)$ is a piecewise constant function with a finite number of discontinuous points.

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