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Rotation prevents finite-time breakdown

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Abstract

We consider a two-dimensional (2D) convection model augmented with the rotational Coriolis forcing, $U_t + U \cdot \nabla_x U = 2kU^{\perp}$, with a fixed $2k$ being the inverse Rossby number. We ask whether the action of dispersive rotational forcing alone, $U^{\perp}$, prevents the generic finite-time breakdown of the free nonlinear convection. The answer provided in this work is a conditional yes. Namely, we show that the rotating Euler equations admit global smooth solutions for a subset of generic initial configurations. With other configurations, however, finite-time breakdown of solutions may and actually does occur. Thus, global regularity depends on whether the initial configuration crosses an intrinsic, $\mathcal{O}(1)$ critical threshold (CT), which is quantified in terms of the initial vorticity, $\omega_0 = \nabla \times U_0$, and the initial spectral gap associated with the 2×2 initial velocity gradient, $\eta_0 := \lambda_2(0) - \lambda_1(0)$, $\lambda_j(0) = \lambda_j(\nabla U_0)$. Specifically, global regularity of the rotational Euler equation is ensured if and only if $4k\omega_0(\alpha) + \eta_0^2(\alpha) < 4k^2, \forall \alpha \in \mathbb{R}^2$. We also prove that the velocity field remains smooth if and only if it is periodic. An equivalent Lagrangian formulation reconfirms the CT and shows a global periodicity of velocity field as well as the associated particle orbits. Moreover, we observe yet another remarkable periodic behavior exhibited by the *gradient* of the velocity field. The spectral dynamics of the Eulerian formulation [SIAM J. Math. Anal. 33 (2001) 930] reveals that the vorticity and the divergence of the flow evolve with their own path-dependent period. We conclude with a kinetic formulation of the rotating Euler equation.

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1. Introduction and statement of main results

Finite-time breakdown is a familiar trademark of nonlinear convection mechanism. Consider the canonical example of an N-dimensional system of free transport equations, $U_t+U\cdot\nabla_x U = 0$. It follows—see Corollary 2.2—that the solution $U(t, \cdot)$ will lose its initial regularity at a finite-time if and only if an eigenvalue of the initial velocity gradient crosses the negative real axis, i.e., if and only if there exists at least one eigenvalue, $\lambda(0, x) := \lambda(\partial U_i(0, x)/\partial x_j)$, such

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that $\lambda(0, x) \in \mathbb{R}^-$. Consequently, finite-time breakdown is a generic phenomenon for the free nonlinear transport. Thus, for example, irrotational initial data $\nabla_x \times U(0, x) = 0$—where all eigenvalues $\lambda_j(t, x)$ remain real, will necessarily lead to finite-time breakdown, except for non-generic cases where $\lambda_j(0, x) \geq 0, \forall j, x$, requiring, in particular, that the initial divergence is *globally* positive, $\nabla_x \cdot U(0, x) > 0$. This general N-dimensional scenario is completely analogous to the one-dimensional (1D) inviscid Burgers' equation, $U_t + UU_x = 0$, where solutions of the latter will necessarily reach a finite-time breakdown except for the non-generic case of monotonically increasing initial data.

Physically relevant models are governed by the fundamental Eulerian convection equation augmented by proper forcing F,

$$U_t + U \cdot \nabla_x U = F. \tag{1.1}$$

Here, there is a competition between the finite-time breakdown dynamics driven by nonlinear convection and the balancing act of nonlinear forcing, F. Different models show up in different contexts dictated by the different modeling of such forcing. Three prototypes are dissipation, relaxation and dispersion. It is well known that if (1.1) is augmented with a sufficiently large amount of either dissipation or relaxation, then (1.1) admits a global smooth solution for a rich enough class of initial data. In both cases of dissipation and relaxation, global existence is secured by enforcing a sufficiently large amount of energy decay. Dispersive forcing, however, is different. The dispersive KdV equation, for example, $U_t + UU_x = U_{xxx}$, is a case in point. It admits global smooth solution while keeping the L^2-energy invariant in time.

1.1. *The rotational model*

In this paper, we study the regularity of the two-dimensional (2D) convection model augmented by rotational forcing,

$$U_t + U \cdot \nabla_x U = 2kJU, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.2}$$

subject to initial conditions, $U(0, x) = U_0(x)$. Here $2k = \epsilon^{-1}$, where ϵ is the Rossby number [24], $\epsilon = \bar{U}/2\Omega\bar{L}$, expressed in terms of the characteristic length $\bar{L}$, characteristic speed $\bar{U}$, and the amplitude of angular velocity Ω of the rotating body, see [16,23]. With these parameters the system evolves on a characteristic time scale $t \sim \bar{L}/\bar{U}$. The system admits a global energy invariant in time, which is independent of the amplitude of rotation encoded by the constant k on the RHS of (1.2). To see this, we note that (1.2) is formally equivalent to the extended 3×3 system,

$$\partial_t \rho + \nabla_x \cdot (\rho U) = 0, \quad x \in \mathbb{R}^2, \ \ t \in \mathbb{R}^+, \tag{1.3}$$

$$\partial_t (\rho U) + \nabla_x \cdot (\rho U \otimes U) = 2k\rho JU, \tag{1.4}$$

which are the usual statements of conservation of mass and Newton's second law, governing the local density $\rho = \rho(t, x)$ and the velocity field $U := (u, v)(t, x)$. The usual manipulation, $-1/2|U|^2 \times$ (1.3) $+ U^\top \times$ (1.4) and the skew-symmetry form induced by the rotational forcing imply

$$\partial_t (\tfrac{1}{2}\rho |U|^2) + \nabla_x \cdot (\tfrac{1}{2}\rho U |U|^2) = 2k\rho \langle U, JU \rangle = 0.$$

The global invariance of the energy follows:

$$E(t) := \frac{1}{2} \int_x \rho(t, x) |U(t, x)|^2 \, dx = E(0).$$

The system (1.3) and (1.4) coincides with a simplified version of the 2D shallow-water equations (SWEs), lacking the additional pressure terms. Although (1.3) and (1.4) should not be claimed as a faithful approximation to the general SWEs, it does arise as a meaningful simplification, for example, when centrifugal forces are counterbalanced by underlying gravity waves, see e.g. [11,15]. The only remaining forcing in (1.3) and (1.4) is the rotational Coriolis forcing, and our main quest in this paper is whether the action of dispersive rotational forcing alone prevents the generic finite-time breakdown of nonlinear convection. The answer outlined in Section 4 is a conditional yes. Namely, we show that (1.2) admits global smooth solutions for a subset of generic initial configurations, U_0. With other initial configurations, however, the finite-time breakdown of solutions may—and actually does occur. Thus, global regularity depends on whether the initial configuration crosses an intrinsic, $\mathcal{O}(1)$ critical threshold (CT), which is quantified in terms of the initial vorticity, $\omega_0 := \nabla_x \times U_0$ and the initial spectral gap, $\Gamma_0 := (\lambda_2(0) - \lambda_1(0))^2$.

Theorem 1.1 (CT for rotation forcing). *Consider the 2D rotational flow* (1.2) *with* $k > 0$. *Then the solution of* (1.2) *with initial data* $U(0,x)|_{x=\alpha} = U_0(\alpha)$ *remains smooth for all time,* $-\infty < t < \infty$, *if and only if the initial data* U_0 *satisfy*

$$i_0(\alpha) := 4k[k - \omega_0(\alpha)] - \Gamma_0(\alpha) > 0, \quad \forall \alpha \in \mathbb{R}^2. \tag{1.5}$$

Moreover, if $X(t) \equiv X(t,\alpha)$ *is the particle path governed by* $X_t = U(t,X)$, $X(0,\alpha) = \alpha$, *then the vorticity,* $\omega(t) \equiv \omega(t,\alpha)$ *and the divergence,* $d(t) \equiv d(t,\alpha) := \mathrm{div}_x U(t,\alpha)$ *form a periodic orbit in phase space* $(\omega, d) \in \mathbb{R}^2$, *with a period,* $\bar{T} = \bar{T}(\alpha)$, *given by*

$$\bar{T} = \frac{2}{k} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{(\theta_0^{-1} + \theta_0) + (\theta_0^{-1} - \theta_0)\sin\theta}. \tag{1.6}$$

Here $\theta_0 = \theta(\alpha) < 1$ *is determined by the initial data*

$$\theta_0 = \frac{\sqrt{1 + 8kp_0} - 1}{\sqrt{1 + 8kp_0} + 1}, \qquad p_0 := \frac{\sqrt{i_0}}{d_0^2 + (\sqrt{i_0} - 2k)^2}, \qquad d_0 = d(0). \tag{1.7}$$

Several remarks are in order.

1. We note that the CT (1.5) is independent of the initial divergence $d_0 := \mathrm{div}_x U_0$.
2. Let us point out that system (1.2) could be viewed as a crossroad between the 2D SWEs and the so-called 2D pressureless equations, e.g. [1–4,8,14], corresponding to Theorem 1.1 with $k = 0$,

$$\partial_t \rho + \nabla_x \cdot (\rho U) = 0, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}^+, \tag{1.8}$$

$$\partial_t (\rho U) + \nabla_x \cdot (\rho U \otimes U) = 0. \tag{1.9}$$

   According to Theorem 1.1, the pressureless system admits a global smooth solution forward in time, and respectively—reversible in time, if and only if $\lambda_j(0) \notin \mathbb{R}^-$, and respectively—if and only if $\lambda_j(0) \notin \mathbb{R}$. The latter is equivalently expressed in Theorem 1.1 by the requirement $\Gamma_0 := (\lambda_2(0) - \lambda_1(0))^2 < 0$.
3. In particular, (1.2) does admit global smooth solutions with negative initial divergence in contrast to the free transport ($k = 0$) equation discussed in Section 1. It follows that rotation prevents finite-time breakdown, either by a large Coriolis forcing ($k \gg 1$) or by a large initial rotation ($\Gamma_0 \ll 0$).

1.2. Global solutions are periodic

We continue with couple of remarks on the periodicity of the global solutions discussed in Theorem 1.1. If we set y-independent initial data, then (1.2) is reduced to the 1D system

$$u_t + uu_x = 2kv, \qquad v_t + uv_x = -2ku$$

with CT $(u_0'(\alpha))^2 - 4kv_0'(\alpha) < 4k^2$. To interpret Theorem 1.1 in this simplified setting, we observe that the gradient $(\omega, d) := (-v_x, u_x)$ solves a coupled system

$$(\partial_t + u\partial_x)\omega + d\omega = 2kd, \tag{1.10}$$

$$(\partial_t + u\partial_x)d + d^2 = -2k\omega \tag{1.11}$$

and a straightforward computation reveals the global invariant along the particle path, $\dot{X}(t) = U(t, X)$, $X(0) = \alpha$,

$$\frac{(2k-\omega)^2}{d^2+\omega^2} = B_0, \qquad B_0 = B_0(\alpha) := \frac{(2k-\omega_0(\alpha))^2}{d_0^2(\alpha)+\omega_0(\alpha)^2}.$$

The CT statement in this case reads $B_0 > 1$, stating that the gradient (ω, d) forms a closed elliptical orbit in the phase plane (whereas $B_0 \leq 1$ corresponds to unbounded parabolic/hyperbolic orbits). Following the analysis in Section 4, we also obtain a path-dependent period for the gradient

$$\bar{T} = \frac{2}{k}\int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{(\theta_0^{-1}+\theta_0)+(\theta_0^{-1}-\theta_0)\sin\theta}, \qquad \theta_0 = \frac{\sqrt{B_0}-1}{\sqrt{B_0}+1}.$$

Such path-dependent period of the gradient reflects the fact that its governing system (1.10) and (1.11), is a nonlinear perturbation of the harmonic oscillator. As the Rossby number approaches zero, however, $k \gg 1$, $\theta_0 \sim 1$, and the above path-dependent period $\bar{T}$ is approaching the global inertial period π/k (the harmonic oscillator period).

As we shall see in Theorem 1.2, sub-critical initial data yield smooth velocity fields, $U(t, X(t))$, with time period $\bar{T} = \pi/k$, or—expressed in terms of the original non-scaled time units, a period $T = \bar{T}\bar{U}/\bar{L} = \pi/\Omega$. This period of the particle orbits is related to the global revolution of the plane. Theorem 1.1 points out yet another remarkable property for a portion of the gradient of $U(t,\cdot)$, namely, the divergence $d(t,\alpha)$, and the vorticity $\omega(t,\alpha)$ which exhibit a local period dictated by the unique initial parameter, $8kp_0$. It is instructive to compute the period predicted in Theorem 1.1, using configurations similar to those encountered in various applications. Let us illustrate a couple of examples taken from [16]. For the Gulf Stream, with the Rossby number $\epsilon = 0.07$, $\bar{L} = 100\,\mathrm{km}$ and $\bar{U} = 1\,\mathrm{m/s}$, we find that the vorticity and divergence of the flow keep repeating themselves every $T = \bar{T}\bar{L}/\bar{U} \sim 11.7\,\mathrm{h}$; for the weather system we have $\epsilon = 0.14$ with $\bar{L} = 1000\,\mathrm{km}$, $\bar{U} = 20\,\mathrm{m/s}$, and the vorticity/divergence exhibit a period of $T = \bar{T}\bar{L}/\bar{U} \sim 12.2\,\mathrm{h}$. It is also interesting to see how this gradient period be influenced by the small Rossby numbers. After rescaling we may assume initial configuration such that $d_0 \sim \omega_0 \sim 1$, for which a small Rossby number yields $i_0 \sim 4k^2$, $p_0 \sim 2k$ and hence $\theta_0 \sim 1$. Restored in terms of the original time scale, $T = \bar{T}\bar{L}/\bar{U}$, the period is given by the θ_0-dependent elliptic integral,

$$T = \frac{2}{\Omega}\int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{(\theta_0^{-1}+\theta_0)+(\theta_0^{-1}-\theta_0)\sin\theta} \sim \frac{\pi}{\Omega},$$

which is close to the inertial period when the Rossby number is small. For the earth core, for example, we have $T = 11.95\,\mathrm{h}$ with $\epsilon = 2 \times 10^{-7}$ ($\bar{L} = 3000\,\mathrm{km}$ and $\bar{U} = 0.1\,\mathrm{cm/s}$), whereas for Jupiter's Red Spot we have a Rossby number $\epsilon = 0.015$ (with $\bar{L} = 10^4\,\mathrm{km}$ and $\bar{U} = 0.1\,\mathrm{cm/s}$) and the velocity gradient period $T \sim 5.13\,\mathrm{h}$. We should point out the difference between the period of the velocity field vs. the velocity gradient periods, which is due to our tracking of the flow dynamics along the particle path.

Of course, one should not expect the current cartoon model to provide a faithful description of the full model since other forces which are ignored at this stage, such as magnetic forces, pressure, etc., play a decisive role in the dynamics of the problem. Nevertheless, the 'pure' rotational model is interesting in its own sake, in particular, since the rotational flow is predicted to be periodic once smooth solutions are secured for subcritical initial data.

Surprisingly, the periods computed above fall within the physically relevant range. It will be challenging to refine the estimates for these periods by taking into account other forces which should complement the rotational model.

To put our study in a proper perspective we recall a few of the references from the considerable amount of available literature on the global behavior of nonlinear convection driven by rotational forcing. Let us mention the rotating shallow-water (SW) model studies in [13,17,25] and the rotating incompressible Euler and Navier–Stokes equations in [5–7,10]. The common feature of these studies is rotation dominated flows with sufficiently small Rossby number ϵ. The flow structure has been extensively studied in terms of $\epsilon \ll 1$. In particular, based on averaging the interaction of fast waves of the rotating Euler equation, 2D structures were shown to emerge in the limit $\epsilon \to 0$, see [5,13]. For bounds on the vertical gradients of the Lagrangian displacement that vanish linearly with the maximal local Rossby number we refer to [10]. It is well known that large-scale atmospheric (or oceanic) fields are in permanent process of Rossby (or geostrophic) adjustment [22]. A nonlinear theory of geostrophic adjustment for the rotating SW model for small Rossby number is developed in [25]. The analysis of an approximation for the rotating SWE can be found in [17].

When dealing with the questions of time regularity for Eulerian dynamics without damping, one encounters several limitations with the classical stability analysis. Among other issues, we mention that:

(i) the usual stability analysis does not tell us how large the perturbations can be before losing stability—indeed, the smallness of the initial perturbation is essential to make the energy method work, e.g. the 3D incompressible Navier–Stokes equation [18]; in particular, see [9, Theorem 1] for a precise statement on the size of initial perturbations which give rise to *global* regularity in time.

(ii) the steady solution may be only conditionally stable due to the weak dissipation present in the system, say in the 1D Euler–Poisson equations [12].

To address these difficulties we advocated in [12,19,20], a new notion of CT which describes conditional stability, where the answer to the question of global vs. local existence depends on whether the initial configuration crosses an intrinsic, $\mathcal{O}(1)$ CT. Little or no attention has been paid to this remarkable phenomenon, and our goal is to bridge the gap of previous studies on the behavior of rotational Euler equations, a gap between the regularity of Eulerian solutions in the small and their finite-time breakdown in the large. The CT was completely characterized for the 1D Euler–Poisson system in terms of the relative size of the initial velocity slope and the initial density; see [19,26] for the CT for the convolution model for conservation laws; moving to the multi-D setup, one has first to identify the proper quantities which govern the CT phenomena. In [21] we have shown that these quantities depend in an essential manner on the *eigenvalues* of the velocity gradient matrix, $\lambda(\nabla_x U)$.

1.3. On the Lagrangian and kinetic formulations

The CT for the current rotation model can be also obtained, in a straightforward manner, through a Lagrangian flow formulation. This is summarized in Theorem 1.2. We should point out that it was the spectral dynamics analysis of $\lambda(\nabla_x U)$ that led us to the CT formulation in the first place, which in turn was then sought within Lagrangian formulation. In Section 5 we prove the following theorem.

Theorem 1.2 (Flow map for rotation forcing). *The flow map associated with (1.2), $\dot{X}_\alpha := \mathrm{d}X_\alpha/\mathrm{d}t = U(t, X_\alpha)$, subject to initial condition $X_\alpha(0) = \alpha$, is given by*

$$X_\alpha(t) = \frac{1}{2k} J^{-1} \mathrm{e}^{2kJt} U_0(\alpha) + \alpha - \frac{1}{2k} J^{-1} U_0(\alpha).$$

For sub-critical initial data (1.5), this flow map is invertible and periodic with an inertial period $\bar{T} = \pi/k$. The velocity field, $U(t, X_\alpha(t)) = U_0(\alpha) + 2kJ(X_\alpha(t) - \alpha)$, shares the same inertial period.

At this point, one may wonder whether this inertial period is none other than the planet rotation. Actually the two are not the same; the rotating planet completes one revolution in a time equal to $2\pi/\Omega$, while the period of the particle path expressed in the original non-scaled variables is $T = \bar{T}\bar{L}/\bar{U} = \pi/\Omega$. Thus, the particle goes around its orbit twice as the planet accomplishes a single revolution, which is consistent with the observation in [11].

The periodicity of both the flow map and the "Lagrangian" velocity field stated in Theorem 1.2 enable us to conclude that the time-periodicity of the "Eulerian" velocity field as well as its gradient at any fixed location x.

Corollary 1.3. *For subcritical initial data (1.5), the velocity field $U(t, x)$ and its gradient $\nabla_x U(t, x)$ are periodic in time with period π/k, i.e. $(U, \nabla_x U)(t + \pi/k, x) = (U, \nabla_x U)(t, x)$ for any $t > 0$.*

Note that the different period for the velocity gradient stated in Theorem 1.1 is due to its expression in terms of Lagrangian coordinate.

Finally, we conclude in Section 6 with a kinetic formulation of the current rotation model.

Theorem 1.4. *The rotation model (1.2) admits for the following kinetic formulation*:

$$\partial_t f + \xi \cdot \nabla_x f + 2kJ\xi \cdot \nabla_\xi f = \frac{1}{\epsilon}(M - f),$$

where $M_{\{\rho,U\}}(\xi)$ is the Maxwellian given by

$$M = \frac{\rho}{\sqrt{\pi T}}\, \mathrm{e}^{-|\xi - U|^2/T}, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where ρ and U are macroscopic density and velocity, respectively, and T is an arbitrary fixed temperature.

In Sections 4 and 5, we quantify the same CT using the Eulerian and Lagrangian formulations, and it would be of interest to derive the same CT directly using the kinetic formulation in Theorem 1.4.

2. Spectral dynamics

We consider a general nonlinear transport equation (1.1), $U_t + U \cdot \nabla_x U = F$, and we trace the evolution of $\nabla_x U$ in terms of its eigenvalues, $\lambda := \lambda(\nabla_x U)(t, x)$. The following result is in the heart of matter.

Lemma 2.1 (Spectral dynamics [20, Lemma 3.1]). *Let $\lambda := \lambda(\nabla_x U)(t, x)$ denote an eigenvalue of $\nabla_x U$ with corresponding left and right normalized eigenpair, $\langle \ell, r\rangle = 1$. Then λ is governed by the forced Riccati equation*

$$\partial_t \lambda + U \cdot \nabla_x \lambda + \lambda^2 = \langle \ell, \nabla_x F r\rangle.$$

As an immediate corollary we obtain the precise description for finite-time breakdown of free nonlinear transport.

Corollary 2.2 (Finite-time breakdown of free transport [20, Lemma 4.1]). *The free nonlinear N-dimensional transport*

$$\partial_t U + U \cdot \nabla_x U = 0, \quad x \in \mathbb{R}^N, \tag{2.1}$$

admits global smooth solution forward in time, $t > 0$, if and only if the eigenvalues of its initial velocity gradient, $\lambda := \lambda(\nabla_x U)$, satisfy $\lambda(0, x) \notin \mathbb{R}^-$. Likewise, it admits a globally smooth, time-reversible solution for $-\infty < t < \infty$ if and only if $\lambda(0, x) \notin \mathbb{R}$.

For the proof, we note that the eigenvalues, governed by the homogeneous Riccati equations (2.1), propagate along the particle path $x = x(t, \alpha)$,

$$\lambda(t, x) = \frac{\lambda(0, \alpha)}{t\lambda(0, \alpha) + 1}.$$

We note in passing that the rotational system (1.2) is to the full SWEs as the free transport model (2.1) is to the full Euler equations. The existence of a CT phenomena associated with global linear forcing model was first identified by us [20], although the exact configuration cannot be obtained in such generality. The current paper provides a precise description of the CT for the 2D rotational system (1.2). In particular, we use the spectral dynamics lemma to obtain remarkable explicit formulae for the CT surface summarized in the main Theorem 1.1. Taking the gradient of the velocity equation (1.2), we find that the velocity gradient field, $\nabla_x U$, solves the following matrix equation:

$$\partial_t(\nabla_x U) + U \cdot \nabla_x(\nabla_x U) + (\nabla_x U)^2 = 2kJ\nabla_x U. \tag{2.2}$$

Using the spectral dynamics Lemma 2.1, we obtain the spectral dynamics equations:

$$\partial_t\lambda_1 + U \cdot \nabla_x\lambda_1 + \lambda_1^2 = 2k\lambda_1\langle l_1, Jr_1\rangle, \tag{2.3}$$

$$\partial_t\lambda_2 + U \cdot \nabla_x\lambda_2 + \lambda_2^2 = 2k\lambda_2\langle l_2, Jr_2\rangle, \tag{2.4}$$

where $\lambda_i, i = 1, 2$ are eigenvalues of the velocity gradient field $\nabla_x U$ associated with left (row) eigenvectors l_i and right (column) eigenvectors r_i. Since J is skew-symmetric we have $Jr_1 = \alpha_1 l_2^\top$ and $Jr_2 = \alpha_2 l_1^\top$. Noting that $l_2 r_2 = l_1 r_1 = 1$, one then has $\alpha_1 = \langle r_2, Jr_1\rangle$ and

$$\alpha_2 = \langle r_1, Jr_2\rangle = -\langle r_2, Jr_1\rangle = -\alpha_1.$$

Therefore, (2.3) and (2.4) now read

$$\partial_t\lambda_1 + U \cdot \nabla_x\lambda_1 + \lambda_1^2 = 2k\lambda_1\langle r_2, Jr_1\rangle\langle l_1, l_2\rangle, \tag{2.5}$$

$$\partial_t\lambda_2 + U \cdot \nabla_x\lambda_2 + \lambda_2^2 = -2k\lambda_2\langle r_2, Jr_1\rangle\langle l_1, l_2\rangle, \tag{2.6}$$

from which we deduce that the spectral gap $\eta := \lambda_2 - \lambda_1$ and divergence $d := \lambda_2 + \lambda_1$, satisfy

$$\partial_t\eta + U \cdot \nabla_x\eta + d\eta = -2kd\langle r_2, Jr_1\rangle\langle l_1, l_2\rangle$$

and

$$\partial_t d + U \cdot \nabla_x d + \tfrac{1}{2}(d^2 + \eta^2) = -2k\eta\langle r_2, Jr_1\rangle\langle l_1, l_2\rangle. \tag{2.7}$$

On the other hand, differentiation of (1.2) yields the $\nabla_x U$ (Eq. (2.2)), i.e.,

$$(\partial_t + U \cdot \nabla_x)\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} + \begin{pmatrix} u_x^2 + v_x u_y & \mathrm{d}u_y \\ \mathrm{d}v_x & v_x u_y + v_y^2 \end{pmatrix} = 2k\begin{pmatrix} v_x & v_y \\ -u_x & -u_y \end{pmatrix}, \tag{2.8}$$

which in turn—using the LHS of (2.5) and (2.6) to express $\lambda_1^2 + \lambda_2^2 \equiv (d^2 + \eta^2)/2$, leads to

$$\partial_t d + U \cdot \nabla_x d + \tfrac{1}{2}(d^2 + \eta^2) = -2k\omega, \tag{2.9}$$

$$\partial_t\omega + U \cdot \nabla_x\omega + d\omega = 2kd. \tag{2.10}$$

Equating the expressions on the right of (2.9) and (2.7) we find

$$-2k\omega = -2k\eta\langle r_2, Jr_1\rangle\langle l_1, l_2\rangle.$$

Thus, the scaled product of the eigenvectors measures the ratio of vorticity over the spectral gap in the following manner

$$\langle r_2, Jr_1\rangle\langle l_1, l_2\rangle = \frac{\omega}{\eta}. \tag{2.11}$$

When the spectral gap η shrinks to zero, the scaled product becomes unbounded due to the degeneracy of eigenvectors. When the vorticity ω shrinks to zero, (2.11) recovers the symmetry of $\nabla_x U$ which is reflected through the orthogonality of ℓ_1 and ℓ_2. Equipped with the above relations we come up with a closed system for (ω, d, η) along the particle path (here and below $' \equiv \partial_t + U\cdot\nabla_x$)

$$\omega' + d\omega = 2kd, \qquad d' + \frac{d^2+\eta^2}{2} = -2k\omega, \qquad \eta' + d\eta = -2k\frac{d\omega}{\eta}.$$

Note that the spectral gap may become purely imaginary when eigenvalues are complex. To avoid the discussion on the complex solution of the above system, we introduce the following real variable

$$\Gamma := \eta^2.$$

Using the above equations we have

$$\Gamma' = 2\eta\eta' = 2d[-2k\omega - \Gamma].$$

Note that the sign of $2k$ indicates the direction of the rotational forcing, and the vorticity measures the rotation in the flow. In order to combine these two effects we introduce $\varphi := 4k^2 - 2k\omega$, and thus obtain a closed system for $W := (\varphi, d, \Gamma)^\top$

$$\varphi' = -d\varphi, \tag{2.12}$$

$$d' = -\tfrac{1}{2}(d^2+\Gamma) + \varphi - 4k^2, \tag{2.13}$$

$$\Gamma' = 2d[\varphi - 4k^2 - \Gamma]. \tag{2.14}$$

We shall use this system to describe the dynamics of the velocity gradient field. Linearization of the above system around $W^* = (\varphi^*, d^*, \Gamma^*)^\top$ gives the linear system $W' = A(W^*)(W - W^*)$ with

$$A = \begin{pmatrix} -d^* & -\varphi^* & 0 \\ 1 & -d^* & -\frac{1}{2} \\ 2d^* & 2\varphi^* & -2d^* \end{pmatrix}.$$

The corresponding eigenvalues of A at critical points $(\varphi^*, 0, \Gamma^*)$ are $\lambda_1 = 0, \lambda_{2,3} = \pm\sqrt{-2\varphi^*}$. The classical stability analysis based on linearization is not sufficient to predict the global time dynamics.

3. Material invariants

It follows from Eqs. (2.12) and (2.14) that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\Gamma} = \frac{-\varphi}{2(\varphi - 4k^2 - \Gamma)},$$

which upon integration gives the first material invariant

$$\left.\frac{2\varphi - \Gamma - 4k^2}{\varphi^2}\right|_{(t,X_\alpha(t))} = \mathcal{C}_0, \quad \mathcal{C}_0 \equiv \mathcal{C}_0(\alpha) := \left.\frac{2\varphi_0 - \Gamma_0 - 4k^2}{\varphi_0^2}\right|_{x=\alpha}. \tag{3.1}$$

This material invariant enables us to reduce the full system (2.12)–(2.14) to the following system

$$\varphi' = -\varphi d, \tag{3.2}$$

$$d' = -\tfrac{1}{2}[d^2 + 4k^2 - \mathcal{C}_0\varphi^2]. \tag{3.3}$$

In order to have global bounded solution it is necessary to assume $\mathcal{C}_0(\alpha) > 0$, i.e.,

$$\Gamma_0 < 2\varphi_0 - 4k^2 \equiv 4k(k - \omega_0), \tag{3.4}$$

for otherwise, (3.3) will be majorized by the Riccati equation $d' \le -(d^2/2) - 2k^2$, which would lead to finite-time breakdown. As we shall see in Section 4, the positivity of $C_0(\alpha)$ is also sufficient for global bounded solutions. Another material invariant is obtained along the lines of [21, Lemma 2.2]: we set $q = d^2$ to find

$$\frac{\mathrm{d}q}{\mathrm{d}\varphi} = 2d\frac{d'}{\varphi'} = \frac{q + 4k^2 - \mathcal{C}_0\varphi^2}{\varphi}.$$

Integration yields

$$\left.\frac{d^2 + 4k^2 + \mathcal{C}_0\varphi^2}{\varphi}\right|_{(t,X_\alpha(t))} = \mathcal{D}_0, \qquad \mathcal{D}_0 \equiv \mathcal{D}_0(\alpha) := \left.\frac{d_0^2 + 4k^2 + \mathcal{C}_0\varphi_0^2}{\varphi_0}\right|_{x=\alpha} \tag{3.5}$$

and together with (3.1) we end up with a second independent material invariant

$$\left.\frac{d^2 - \Gamma + 2\varphi}{\varphi}\right|_{(t,X_\alpha(t))} = \mathcal{D}_0(\alpha).$$

In summary we have

Lemma 3.1. *Let* $\varphi := 4k^2 - 2k\omega$, $d := \nabla_x \cdot U = \mathrm{tr}(\nabla_x U)$ *and* $\Gamma := (\lambda_2 - \lambda_1)^2$ *be the solution of the dynamical system* (2.12)–(2.14), *associated with the rotational system* (1.2). *Then we have the following material invariants along particle path* $(t, X_\alpha(t))$,

$$\left.\frac{2\varphi - \Gamma - 4k^2}{\varphi^2}\right|_{(t,X_\alpha(t))} = \frac{2\varphi_0(\alpha) - \Gamma_0(\alpha) - 4k^2}{\varphi_0^2(\alpha)}, \tag{3.6}$$

$$\left.\frac{d^2 - \Gamma}{\varphi}\right|_{(t,X_\alpha(t))} = \frac{d_0^2(\alpha) - \Gamma_0(\alpha)}{\varphi_0(\alpha)}. \tag{3.7}$$

4. Critical thresholds

As we observed earlier, the positivity of condition, $\mathcal{C}_0(\alpha) > 0$, is necessary for global bounded solution, for otherwise

$$d' < -\tfrac{1}{2}[4k^2 + d^2], \tag{4.1}$$

which would imply that d, and hence φ, become unbounded in a finite time. We shall show that the same positivity condition, $\mathcal{C}_0(\alpha) > 0$, is in fact sufficient for the existence of global bounded solution. For $\mathcal{C}_0 > 0$, the reduced system (3.2) and (3.3) has two unique equilibrium points in the phase plane $(\rho, d) \in \mathbb{R}^2$, $(\varphi^*_\pm, d) = (\pm(2k/\sqrt{\mathcal{C}_0}), 0)$. The local behavior of the solution depends on the properties of these critical points. We note that since $\varphi = 0$ is an invariant set, then $\varphi_0\varphi(t) > 0$ for all time, and we therefore concentrate on the solution behavior for $\varphi_0 > 0$, with the other case of $\varphi_0 < 0$ being handled similarly. On the right plane $\varphi > 0$, the coefficient matrix of linearized system of (3.2) and (3.3) around the equilibrium point $(\varphi^*_+, d) = (2k/\sqrt{\mathcal{C}_0}, 0)$ is

$$\begin{pmatrix} 0 & -\varphi^*_+ \\ \mathcal{C}_0\varphi^*_+ & 0 \end{pmatrix}$$

with purely imaginary eigenvalues, $\pm(\sqrt{\mathcal{C}_0}\varphi^*_+)i$. This means that the bounded trajectory is possibly a periodic solution or limit circle. Observe that if $(\varphi(t), d(t))$ is a solution to (3.2) and (3.3), so is $(\varphi(-t), -d(-t))$. Such symmetry implies that $(\varphi^*_+, 0)$ is a center and the trajectory in a neighborhood of this equilibrium point is periodic.

In order to clarify the global behavior of the flow around the center, we appeal to the material invariant (3.5) which we rewrite as

$$V_+(\varphi, d) := \frac{d^2 + (\sqrt{\mathcal{C}_0}\varphi - 2k)^2}{\varphi}.$$

$V_+(\cdot)$ is a positive definite Liapunov function for $\varphi > 0$, and achieves its global minimum, $V_+ = 0$, at the equilibrium point $(\varphi^*_+ = 2k/\sqrt{\mathcal{C}_0}, 0)$. A family of closed orbits in the phase plane (φ, d) can be expressed as the level set curve

$$V_+(\varphi, d) = \text{Const} > 0,$$

since V_+ is material invariant in the sense that $\mathrm{d}V_+/\mathrm{d}t = 0$. Similarly, on the left plane $\varphi < 0$, one may use the Liapunov functional

$$V_-(\varphi, d) = \frac{d^2 + (\sqrt{\mathcal{C}_0}\varphi + 2k)^2}{-\varphi},$$

whose level set curve determines a family of closed orbit on the left plane centered around $(\varphi^*_- = -2k/\sqrt{\mathcal{C}_0}, 0)$. The global behavior of the solutions is summarized in the following lemma.

Lemma 4.1 (Bounded solutions are periodic). *The system (2.12)–(2.14) admits a global bounded solution if and only if its initial data* $(\varphi_0, d_0, \Gamma_0)$ *lie in the sub-critical region* (*independent of* d_0), *where*

$$\Gamma_0 < 2\varphi_0 - 4k^2. \tag{4.2}$$

Moreover, the bounded solutions of (2.12)–(2.14) are necessarily periodic. The periodic orbit on the right plane $\varphi > 0$ *lies on the ellipse* $d^2 + (\sqrt{\mathcal{C}_0}\varphi - 2k)^2 = (\mathcal{D}_0 - 4k\sqrt{\mathcal{C}_0})\varphi$, *where* $\mathcal{C}_0$ *and* $\mathcal{D}_0$ *are determined by the initial data in (3.1) and (3.5).*

Proof. Eq. (3.3) shows that if $\mathcal{C}_0 \le 0$ then (4.1) holds and the divergence blows up. Thus, the positivity of $\mathcal{C}_0(\alpha)\forall\alpha \in \mathbb{R}$—namely, the sub-critical condition (4.2) is necessary for global regularity. We turn to show that it is sufficient. Indeed, if $\mathcal{C}_0 > 0$ then the second material invariant in (3.5) takes the equivalent form

$$d^2 + (\sqrt{\mathcal{C}_0}\varphi - 2k)^2 = (\mathcal{D}_0 - 4k\sqrt{\mathcal{C}_0})\varphi.$$

The corresponding solution (φ, d) travels along this elliptical orbit while being kept bounded. In fact, φ and d are periodic and by the invariance of (3.6) or (3.7), Γ shares the same period along the elliptical orbit. Finally, it remains to show the boundedness of the whole velocity gradient, $\nabla_x U$. The latter follows from the boundedness of the divergence d along the lines of [21, Lemma 2.1]. To this end we note that the 'anti-trace' of (2.8), $r := v_x + u_y$, satisfies $r' + rd = -2ks$, where the 'anti-vorticity' $s := u_x - v_y$ is governed by $s' + sd = 2kr$. Solving the 2×2 couples system in terms of the divergence d we obtain

$$\frac{r}{s} = \tan\left(\tan^{-1}\left(\frac{r_0}{s_0}\right) - 2kt\right), \qquad r^2 + s^2 = (r_0^2 + s_0^2)\exp\left(-2\int_0^t d(\xi)\,d\xi\right).$$

Thus, $s^2 + r^2$ remain bounded. In fact, by the periodicity of d and its symmetry about the axis $d = 0$ we conclude that $s^2 + r^2$ shares the same period with d. A more precise statement follows from the identity $s^2 + r^2 = \Gamma + \omega^2$ which imply

$$r = \sin\left(\tan^{-1}\left(\frac{r_0}{s_0}\right) - 2kt\right)\sqrt{\Gamma + \omega^2}, \qquad s = \cos\left(\tan^{-1}\left(\frac{r_0}{s_0}\right) - 2kt\right)\sqrt{\Gamma + \omega^2}.$$

Being the product of two periodic functions with the corresponding periods π/k and $\bar{T}$, we conclude that r and s are periodic if the ratio of these periods,

$$\frac{\bar{T}}{\pi/k} = 2\int_0^1 \frac{d\xi}{(\theta_0 + \theta_0^{-1}) + (\theta_0 - \theta_0^{-1})\cos(\pi\xi)}$$

is a rational number. In this case, the overall gradient $\nabla_x U$ is periodic with a integer multiple of π/k as its period. This completes the proof. □

Once we identified bounded solutions as periodic, the next step is to seek the period for each periodic orbit.

Lemma 4.2. *The period of each bounded orbit associated with (2.12)–(2.14) is given by*

$$\bar{T} = \frac{2}{k}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\theta_0 + \theta_0^{-1} + (\theta_0^{-1} - \theta_0)\sin\theta}. \tag{4.3}$$

Here $\theta_0 = \theta_0(\alpha) < 1$ *is given by*

$$\theta_0 := \frac{\sqrt{1 + 8kp_0} - 1}{\sqrt{1 + 8kp_0} + 1}, \tag{4.4}$$

where p_0 *is determined by the initial data*

$$p_0(\alpha) = \frac{\sqrt{2\varphi_0 - \Gamma_0 - 4k^2}}{d_0^2 + (\sqrt{2\varphi_0 - 4k^2 - \Gamma_0} - 2k)^2}.$$

Proof. Due to the symmetry it suffices to compute the half period. The intersection points of the ellipse $V(\varphi, d) = V(\varphi_\pm, 0)$ with $d = 0$ can be written explicitly in terms of the initial data

$$\varphi_- = \frac{2k}{\sqrt{C_0}}\theta_0, \qquad \varphi_+ = \frac{2k}{\sqrt{C_0}}\theta_0^{-1}. \tag{4.5}$$

The trajectory from $(\varphi_-, 0)$ to $(\varphi_+, 0)$ in the lower-half (φ, d)-plane is given by (3.5)

$$d = -\sqrt{\mathcal{D}_0\varphi - C_0\varphi^2 - 4k^2} = -\sqrt{\mathcal{C}_0}\sqrt{(\varphi_+ - \varphi)(\varphi - \varphi_-)}.$$

By (3.2), $\dot{\varphi} = -\varphi d$ along this trajectory, whose period is therefore given by

$$\bar{T} = 2\int_{\varphi_-}^{\varphi_+} \frac{d\varphi}{-\varphi d} = \frac{2}{\sqrt{\mathcal{C}_0}}\int_{\varphi_-}^{\varphi_+} \frac{ds}{s\sqrt{(\varphi_+ - s)(s - \varphi_-)}}. \tag{4.6}$$

Let $s = (\varphi_- + \varphi_+)/2 + (\varphi_+ - \varphi_-)/2\tau$; using the expression of $\varphi_\pm$ in (4.5) we conclude

$$\bar{T} = \frac{4}{\sqrt{\mathcal{C}_0}} \int_{-1}^{1} \frac{\mathrm{d}\tau}{[\varphi_- + \varphi_+ + (\varphi_+ - \varphi_-)\tau]\sqrt{1-\tau^2}} = \frac{4}{\sqrt{\mathcal{C}_0}} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{\varphi_- + \varphi_+ + (\varphi_+ - \varphi_-)\sin\theta}$$
$$= \frac{2}{k} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}\theta}{\theta_0 + \theta_0^{-1} + (\theta_0^{-1} - \theta_0)\sin\theta},$$

which gives the desired result in (4.3). □

5. Flow map

For the smooth flow we may further study the structure of the flow map. Assume $x = X_\alpha(t)$ is the flow map started at the initial position α, then one has

$$\dot{X}_\alpha := \frac{\mathrm{d}X_\alpha}{\mathrm{d}t} = U(t, X_\alpha), \quad X_\alpha(0) = \alpha$$

and the momentum equation can be written as

$$\ddot{X}_\alpha = 2kJ\dot{X}_\alpha.$$

Integration once gives

$$\dot{X}_\alpha = U_0(\alpha) + 2kJ(X_\alpha - \alpha),$$

where $U_0(\alpha)$ is the initial velocity at location α. The above equation leads to the flow map expression

$$X_\alpha(t) = \frac{1}{2k} J^{-1}\,\mathrm{e}^{2kJt} U_0(\alpha) + \alpha - \frac{1}{2k} J^{-1} U_0(\alpha). \tag{5.1}$$

This flow map determines the unique smooth velocity field if and only if the indicator matrix,

$$\Gamma := \frac{\partial X_\alpha(t)}{\partial\alpha} = I - \frac{1}{2k} J^{-1}(I - \mathrm{e}^{2kJt})\nabla_\alpha U_0,$$

remains non-singular. Noting that $J^{-1} = -J$ and

$$\mathrm{e}^{2kJt} = \begin{pmatrix} \cos(2kt) & \sin(2kt) \\ -\sin(2kt) & \cos(2kt) \end{pmatrix},$$

we have

$$\Gamma = I + \frac{1}{2k} J \begin{pmatrix} 1 - \cos(2kt) & -\sin(2kt) \\ \sin(2kt) & 1 - \cos(2kt) \end{pmatrix} \nabla_\alpha U_0.$$

Hence, with $U = (u, v)$ we find

$$2k\Gamma = 2kI + \begin{pmatrix} \sin(2kt) & 1 - \cos(2kt) \\ \cos(2kt) - 1 & \sin(2kt) \end{pmatrix} \nabla_\alpha U_0$$
$$= \begin{pmatrix} 2k + v_{0x} + u_x \sin(2kt) - v_{0x}\cos(2kt) & v_{0y} + u_{0y}\sin(2kt) - v_{0y}\cos(2kt) \\ -u_{0x} + v_{0x}\sin(2kt) + u_{0x}\cos(2kt) & 2k - u_{0y} + v_{0y}\sin(2kt) + u_{0y}\cos(2kt) \end{pmatrix}.$$

A careful calculation gives its determinant as

$$\det(2k\Gamma) = 4k^2 - 2k\omega_0 + 2\det(\nabla_\alpha U_0) + (2k\omega_0 - 2\det(\nabla_\alpha U_0))\cos(2kt) + (2kd_0)\sin(2kt).$$

Thus $\Gamma(t)$ remains non-singular for all time if and only if $\det(2k\Gamma) \neq 0$, i.e.,

$$4k^2 - 2k\omega_0 + 2\det(\nabla_\alpha U_0) \notin \left(-\sqrt{(2k\omega_0 - 2\det(\nabla U_0))^2 + 4k^2 d_0^2}, \sqrt{(2k\omega_0 - 2\det(\nabla_\alpha U_0))^2 + 4k^2 d_0^2}\right),$$

which is equivalent to

$$(4k^2 - 2k\omega_0 + 2\det(\nabla_\alpha U_0))^2 > (2k\omega_0 - 2\det(\nabla_\alpha U_0))^2 + 4k^2 d_0^2. \tag{5.2}$$

We now invoke the relation between the spectral gap Γ_0, and the corresponding the determinant and divergence,

$$\Gamma_0 = d_0^2 - 4\det(\nabla_\alpha U_0).$$

Applied to the above inequality (5.2), this yields

$$\Gamma_0 < 4k^2 - 4k\omega_0,$$

which is exactly the CT (1.5) stated in Theorem 1.1.

Shifting our attention to the Eulerian framework, we conclude this section with a discussion on the time-periodicity of sub-critical velocity field for *fixed* location x. The solution determined by

$$U(t, x) = e^{2kJt} U_0(\alpha), \quad x = \alpha + \frac{1}{2k} J^{-1}(e^{2kJt} - 1)U_0(\alpha)$$

is implicit. For sub-critical initial data one could find α in terms of (t, x) and substitute back to obtain the velocity. In fact α is given by

$$\alpha = x + \frac{1}{2k} J^{-1}(e^{-2kJt} - 1)U(t, x),$$

which gives the velocity field implicitly determined by

$$U(t, x) = e^{2kJt} U_0 \left(x + \frac{1}{2k}(e^{-2kJt} - 1)U(t, x)\right).$$

Combined with the CT condition (1.5), this shows that for any fixed x, $U(t, x)$ and hence $\nabla_x U(t, x)$ are periodic in time with period π/k and Corollary 1.3 follows.

6. Kinetic formulation

This section describes a kinetic formulation for the rotational model (1.2) in terms of a density function, $f = f(t, x, \xi)$ governed by the BGK model

$$\partial_t f + \xi \cdot \nabla_x f + 2k\xi^\perp \cdot \nabla_\xi f = \frac{1}{\epsilon}(M - f), \quad \xi^\perp := J\xi. \tag{6.1}$$

Here $M = M_{\{\rho, U\}}(\xi)$ is the Maxwellian given by

$$M = \frac{\rho}{\sqrt{\pi T}} e^{-|\xi - U|^2/T}, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

The fixed temperature T, plays no role in this pressureless model. The connection between the distribution function f and macroscopic flow variable is realized in terms of the usual moments of density ρ, momentum $m = \rho U$ and total energy $E = \rho|U|^2/2$,

$$(\rho, \rho U, E)^\top = \int \psi^\top(\xi) f \, d\xi, \qquad \psi(\xi) := \left(1, \xi, \frac{|\xi|^2}{2}\right)^\top.$$

The conservation principle for mass, momentum and energy during the course of particle collisions requires the equilibrium to satisfy the compatibility condition

$$\int (M - f)\psi^\top(\xi) \, d\xi = 0,$$

while the rotational forcing is introduced through the potential

$$\int 2k\xi^\perp \cdot \nabla_\xi f \psi^\top(\xi) \, d\xi = (0, -2k\rho JU, 0).$$

Indeed, a straightforward computation yields $\int 2k\xi^\top \cdot \nabla_\xi f \, d\xi = 0$; for the momentum equation we compute

$$\int 2k\xi^\perp \cdot \nabla_\xi f \xi \, d\xi = 2k \int \xi(\xi_2 \partial_{\xi_1} f - \xi_1 \partial_{\xi_2} f) \, d\xi = 2k \int \begin{pmatrix} \xi_1\xi_2\partial_{\xi_1} f \\ -\xi_1\xi_2\partial_{\xi_2} f \end{pmatrix} d\xi = 2k \begin{pmatrix} -\rho v \\ \rho u \end{pmatrix} = -2k\rho JU,$$

while the presence of this forcing does not change the energy equation since

$$2k \int \xi^\perp \cdot \nabla_\xi f |\xi|^2 \, d\xi = 0.$$

The first three moments of (6.1) then yield the equivalent extended system of (1.3) and (1.4),

$$\partial_t \begin{pmatrix} \rho \\ \rho U \\ \rho \frac{1}{2}(|U|^2) \end{pmatrix} + \nabla_x \cdot \begin{pmatrix} F_\rho \\ F_m \\ F_E \end{pmatrix} = \begin{pmatrix} 0 \\ 2k\rho JU \\ 0 \end{pmatrix}.$$

The corresponding macroscopic fluxes are

$$(F_\rho, F_m, F_E)^\top := \int \psi^\top(\xi)\xi f \, d\xi$$

and under the closure $f = M_{\{\rho,U\}}$ we conclude

$$(F_\rho, F_m, F_E)^\top = (\rho U, \rho U \otimes U, \rho \tfrac{1}{2}|U|^2)^\top.$$

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