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CHEBYSHEV–LEGENDRE SPECTRAL VISCOSITY METHOD FOR NONLINEAR CONSERVATION LAWS*

HEPING MA†

Abstract. In this paper, a Chebyshev–Legendre spectral viscosity (CLSV) method is developed for nonlinear conservation laws with initial and boundary conditions. The boundary conditions are dealt with by a penalty method. The viscosity is put only on the high modes, so accuracy may be recovered by postprocessing the CLSV approximation. It is proved that the bounded solution of the CLSV method converges to the exact scalar entropy solution by compensated compactness arguments. Also, a new spectral viscosity method using the Chebyshev differential operator $D = \sqrt{1-x^2}\partial_x$ is introduced, which is a little weaker than the usual one while guaranteeing the convergence of the bounded solution of the Chebyshev Galerkin, Chebyshev collocation, or Legendre Galerkin approximation to nonlinear conservation laws. This kind of viscosity is ready to be generalized to a super viscosity version.

Key words. conservation laws, Chebyshev–Legendre method, spectral viscosity, convergence

AMS subject classifications. 35L65, 65M10, 65M15

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1. Introduction.

It is well known that the standard spectral method does not work for nonlinear conservation laws. Basically, there are two problems:

1. *Approximation:* When a function has discontinuities, the accuracy of the spectral approximation itself is very poor. However, this can be saved successfully for piecewise smooth functions by filter techniques or reconstruction methods such as the Gegenbauer partial sum. Much work has been done in this field [GT], [AGT], [CGS], [GSSV], [GS1], [GS2], [GS3]. The Gegenbauer partial sum can especially recover pointwise exponential accuracy at all points including those at the discontinuities themselves from the knowledge of a spectral partial sum of a piecewise analytic function. So it would still be very meaningful to get a spectral approximation even for a solution of discontinuity such as a shock.

2. *Stability:* The usual spectral approximation solution to nonlinear conservation laws may not converge to the exact entropy solution [Ta2]. This can be avoided by spectral viscosity methods, which were first established by E. Tadmor [Ta1]. The leading work [Ta1], [MT], [Ta2], and [MOT1] has shown that by adding a spectral viscosity to the high modes, one can achieve stability and convergence without sacrificing the spectral accuracy.

In this paper, we develop a CLSV method for the following conservation law:

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \qquad (x,t) \in (-1,1) \times (0,T) \tag{1.1}$$

provided with an initial condition at $t = 0$ and boundary data on the inflow boundaries. The main purpose of this approach is to replace the Legendre collocation in [MOT1] with the Chebyshev collocation so that the scheme may be implemented more efficiently at Gauss–Lobatto–Chebyshev points. Another difference is that we treat

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†Division of Applied Mathematics, Brown University, Providence, RI 02912 and Department of Mathematics, Shanghai University, Shanghai 201800, People's Republic of China. This work was supported by NSF grant DMS 9500814, AFOSR grant 95-1-0074, and China SMKP for Basic Researches.

the boundary condition by a penalty method, the advantage of which has been shown in [FG1], [FG2].

We also consider a new spectral viscosity (NSV) method introduced by Gottlieb, which uses the Chebyshev differential operator $D = \sqrt{1-x^2}\partial_x$ rather than the derivative ∂_x as in the usual one [MOT1]. Obviously, the former is a little weaker than the latter. It is always a problem to balance the stability and accuracy. We need the viscosity to improve the stability, but we hope it does not affect the accuracy too much. So a weaker viscosity, while still guaranteeing the stability, should be preferable. A more important feature is that the NSV method is ready to be generalized to a super viscosity version suggested by Gottlieb, which will be analyzed in another paper. For the periodic problem, Tadmor has established the Fourier superspectral viscosity method [Ta3], which is much weaker than the second-order viscosity method while still guaranteeing the stability and convergence.

Although the Chebyshev Galerkin or collocation projection is used to compute the nonlinear term in the scheme, we would not expect to get an energy estimate in the Chebyshev weighted norm because it disagrees with the wave propagation property as pointed out in [GO, p. 56], so we will basically work in the usual L^2-norm [R]. This is also the reason we still use a Legendre-kind viscosity. In fact, the scheme is a kind of Chebyshev–Legendre method [DG].

The paper is organized as follows. In section 2 we present the CLSV scheme and the NSV scheme with brief descriptions of the implementation. In section 3 we give some estimates concerning the viscosity term and the Chebyshev and Legendre approximation operators. In section 4 we work on some a priori estimates. In section 5 we prove that the bounded solutions of the CLSV and the NSV methods converge to the exact scalar entropy solution by the compensated compactness arguments.

2. The CLSV schemes. Let $I = (-1, 1)$ and $\rho(x)$ be a positive weight on I. The inner product and norm of $L^2_\rho(I)$ are denoted by $(\cdot,\cdot)_\rho$ and $\|\cdot\|_\rho$, respectively. We will drop the subscript ρ whenever $\rho \equiv 1$. Let $\mathcal{P}_N$ denote the space of algebraic polynomials of degree $\leq N$ and $\omega(x) = (1-x^2)^{-1/2}$.

2.1. The Chebyshev–Legendre pseudospectral viscosity scheme. We define the Chebyshev interpolation operator $I_N^C : C(\bar{I}) \to \mathcal{P}_N$ by

$$I_N^C\varphi(x_j) = \varphi(x_j), \qquad x_j = \cos(\frac{j\pi}{N}), \qquad 0 \leq j \leq N. \tag{2.1}$$

The Chebyshev–Legendre pseudospectral viscosity method for (1.1) is to find $u_N(t) \in \mathcal{P}_N$ such that

$$(\partial_t u_N + \partial_x I_N^C f(u_N), \varphi) = -\varepsilon_N(\partial_x Q u_N, \partial_x Q\varphi) - (B(u_N), \varphi), \quad \forall \varphi \in \mathcal{P}_N. \tag{2.2}$$

Here the boundary term is put in a penalty way [FG1], [FG2] such that $B(u_N) \in \mathcal{P}_N$ is defined by

$$B(u_N) = \sum_{j=0,N} b_j(t)\tau_j[u_N(x_j,t) - g_j(t)]R_N^{(j)}(x), \quad R_N^{(j)}(x) = \frac{1}{2}[L'_N(x_j x) + L'_{N+1}(x_j x)], \tag{2.3}$$

where L_k are Legendre polynomials standardized as $L_k(1) = 1$; $b_j(t) = 1$ on the inflow boundary prescribed with the data $g_j(t)$, and $b_j(t) = 0$ on the outflow boundary $(j = 0, N)$. The value of τ_j should be chosen to help the stability, and usually it does not affect the accuracy [FG1], [FG2], [DG]. It is easy to verify that

$$(B(u_N),\varphi)=\sum_{j=0,N} b_j(t)\tau_j(u_N(x_j,t)-g_j(t))\varphi(x_j),\qquad \forall\varphi\in\mathcal{P}_N. \tag{2.4}$$

The operator Q is defined by

$$Q\varphi=\sum_{k=0}^{N}\hat{Q}_k\hat{\varphi}_k L_k,\qquad \forall\varphi=\sum_{k=0}^{\infty}\hat{\varphi}_k L_k$$

with the coefficients satisfying

$$\begin{cases}\hat{Q}_k\equiv 0 & k\le m_N,\\ 1-\left(\dfrac{m_N}{k}\right)^2\le\hat{Q}_k\le 1 & k>m_N,\end{cases} \tag{2.5}$$

where the parameters $\varepsilon_N\downarrow 0$ and $m_N\uparrow\infty$ will be chosen later to balance the stability and accuracy. Obviously, we have

$$(\partial_x Qu_N,\partial_x QL_k)=0,\qquad 0\le k\le m_N. \tag{2.6}$$

Thus the viscosity is only added to the high modes as in [MOT1] so that we may recover the accuracy by postprocessing the numerical solution [GT], [AGT], [GSSV], [GS1], [GS2], [GS3], [MOT1]. The difference is that the operator Q is put after the derivative in [MOT1], while here it is put before the derivative, which enables us to get an estimate we will need in the proof of convergence.

In order to implement (2.2) efficiently, we should calculate it at the Gauss–Lobatto–Chebyshev points $\{x_j\}$. It can be done by rewriting (2.2) as a pointwise equation. To this end, we first seek a polynomial $V_N\in\mathcal{P}_N$ such that

$$(V_N(u_N),\varphi)=(\partial_x Qu_N,\partial_x Q\varphi),\qquad\forall\varphi\in\mathcal{P}_N. \tag{2.7}$$

The solution to (2.7) is

$$V_N(u_N)=\sum_{k=0}^{N}\frac{(\partial_x Qu_N,\partial_x QL_k)}{\|L_k\|^2}L_k. \tag{2.8}$$

Now the (2.2) reads

$$\partial_t u_N+\partial_x I_N^C f(u_N)=-\varepsilon_N V_N(u_N)-B(u_N). \tag{2.9}$$

Since the both sides of (2.9) are in $\mathcal{P}_N$, (2.9) is equivalent to its collocation equation at points x_j $(0\le j\le N)$. Thanks to the fast Legendre algorithm [AR], which gives the transformation between the coefficients of Legendre expansion and its values at Chebyshev points at the cost of $\mathcal{O}(N\log N)$, the right-hand side of (2.9) can also be computed efficiently. For example, the computation of $V_N(u_N)$ can be described as follows:

$$\begin{aligned}&\{u_N(x_j)\}\xrightarrow{FLT}\{(\widehat{u_N})_k^L\}\xrightarrow{SM}\{(\widehat{Qu_N})_k^L\}\xrightarrow{RF}\{(\widehat{\partial_x Qu_N})_k^L\},\{(\widehat{\partial_x QL_k})_k^L\}\\ &\qquad\xrightarrow{SM}\{(\widehat{V_N(u_N)})_k^L\}\xrightarrow{FLT}\{V_N(u_N)(x_j)\},\end{aligned} \tag{2.10}$$

where we have used the notation $(\hat{\varphi})_k^L$ as the Legendre expansion coefficients of φ, and the abbreviations FLT as the fast Legendre transformation, SM as the simple multiplication, and RF as the recurrence formula [GO, p. 117]. The FLT given in [AR] is basically a way to compute the Legendre coefficients from the Chebyshev coefficients (and vice versa) at the cost of $\mathcal{O}(N)$. After obtaining the Chebyshev coefficients, we can use fast Fourier transformation (FFT) to get the values at the Gauss–Lobatto–Chebyshev points $\{x_j\}$.

We note that an alternate scheme to (2.2) may be defined by using discrete inner product [MOT1]. Since we want to implement the schemes on the Chebyshev points $\{x_j\}$, the costs of them are nearly the same.

2.2. A new spectral viscosity scheme. We denote by $\{T_k(x)\}$ the Chebyshev polynomials standardized as $T_k(1) = 1$. The spectral approximation operator P_N can be one of the following:

1. $P_N^C : L^2_\omega(I) \to \mathcal{P}_N$, the Chebyshev Galerkin projection operator ($L^2_\omega(I)$-orthogonal);
2. $I_N^C : C(\bar{I}) \to \mathcal{P}_N$, the Chebyshev interpolation operator defined in (2.1);
3. $P_N^L : L^2(I) \to \mathcal{P}_N$, the Legendre Galerkin projection operator ($L^2(I)$-orthogonal).

As mentioned in section 1, a weaker viscosity term can be applied by using $D = \sqrt{1-x^2}\partial_x$ instead of ∂_x. The new spectral viscosity method for (1.1) is to find $u_N(t) \in \mathcal{P}_N$ such that

$$(\partial_t u_N + \partial_x P_N f(u_N), \varphi) = -\varepsilon_N (DQu_N, D\varphi)_\omega - (B(u_N), \varphi), \quad \forall \varphi \in \mathcal{P}_N, \tag{2.11}$$

where the operator Q is defined by

$$Q\varphi = \sum_{k=0}^{N} \hat{Q}_k \hat{\varphi}_k T_k, \qquad \forall \varphi = \sum_{k=0}^{\infty} \hat{\varphi}_k T_k$$

with the coefficients satisfying

$$\begin{cases} \hat{Q}_k \equiv 0 & k \le m_N, \\ 1 - \left(\dfrac{m_N}{k}\right)^3 \le \hat{Q}_k \le 1 & k > m_N. \end{cases} \tag{2.12}$$

The parameters $\varepsilon_N \downarrow 0$ and $m_N \uparrow \infty$ will be chosen later. The boundary term is the same as in (2.3). We note that the viscosity term here is also added only to the high modes for

$$(DQu_N, D\varphi)_\omega = \left(\sum_{k>m_N}^{N} k^2 \hat{Q}_k \hat{u}_k T_k, \varphi \right)_\omega = 0, \qquad \forall \varphi \in \mathcal{P}_{m_N}, \tag{2.13}$$

where $u_N = \sum_{k=0}^{N} \hat{u}_k T_k$ and we have used the fact that

$$D^2 T_k(x) + k^2 T_k(x) = 0, \qquad \forall x \in I. \tag{2.14}$$

We can see from (2.11) and (2.13) that in the transform space the viscosity term is of the following form:

$$\begin{cases} 0 & 0 \le k \le m_N, \\ -\varepsilon_N k^2 \hat{Q}_k \hat{u}_k & m_N < k \le N, \end{cases} \tag{2.15}$$

which is exactly what has been used in the Fourier spectral viscosity method [Ta1]. In the physical space, the scheme (2.11) is of the form

$$\partial_t u_N + \partial_x P_N f(u_N) = \varepsilon_N P_N^L(\omega D^2 Q u_N) - B(u_N). \tag{2.16}$$

Obviously, the scheme (2.16) with $P_N = I_N^C$ is the most efficient for we can implement it at the Chebyshev points x_j $(0 \le j \le N)$ with the help of the fast transformation. For example, the viscosity term can be calculated in the following way:

$$\{u_N(x_j)\} \overset{FFT}{\longrightarrow} \{\hat{u}_k\} \overset{CTL}{\longrightarrow} \{(\widehat{\omega D^2 Q u_N})_k^L\} \overset{FLT}{\longrightarrow} \{[\omega D^2 Q u_N](x_j)\},$$

where we have used the same notation $(\hat{\varphi})_k^L$ and abbreviations FFT, FLT as in (2.10). The second step, Chebyshev coefficients to Legendre ones (CTL), is done as follows:

$$\begin{aligned}
(\widehat{\omega D^2 Q u_N})_k^L &= 0, \qquad k \le m_N, \\
(\widehat{\omega D^2 Q u_N})_k^L &= -\left(k+\frac{1}{2}\right)\int_I \omega \sum_{l>m_N}^{N} l^2 \hat{Q}_l \hat{u}_l T_l \sum_{m=0}^{k} h_{mk} T_m \, dx \\
&= -\left(k+\frac{1}{2}\right)\sum_{l>m_N}^{k} l^2 \hat{Q}_l \hat{u}_l h_{lk}, \qquad k > m_N,
\end{aligned}$$

where h_{mk} can be found in [R], [AR] such that $L_k = \sum_{m=0}^{k} h_{mk} T_m$. We note that the method given in [AR] can also be used here so that the second step can be implemented at the cost $\mathcal{O}(N)$.

3. Preliminaries. In this section, we work on some estimates needed in the proof of convergence. Most of them are concerning approximation results for the Chebyshev and Legendre polynomials in different weighted norms.

LEMMA 3.1. *Let Q be defined in* (2.5). *We have that for any $u \in \mathcal{P}_N$,*

$$\|\partial_x u\| \le \|\partial_x Q u\| + C m_N^2 \sqrt{\ln N}\|u\|, \tag{3.1}$$

$$\|\partial_x Q u\| \le \|\partial_x u\| + C m_N^2 \sqrt{\ln N}\|u\|. \tag{3.2}$$

Proof. Let $u = \sum_{k=0}^{N} \hat{u}_k L_k$ and we have

$$u = Qu + Ru, \qquad Ru = \sum_{k=0}^{N} \hat{R}_k \hat{u}_k L_k,$$

where, according to (2.5),

$$\hat{R}_k \equiv 1 - \hat{Q}_k \begin{cases} = 1 & k \le m_N, \\ \le \left(\dfrac{m_N}{k}\right)^2 & k > m_N. \end{cases}$$

It is sufficient to prove that

$$\|\partial_x R u\|^2 \le C m_N^4 \ln N \|u\|^2.$$

We write $\partial_x R u$ as

$$\partial_x R u = \partial_x\left(\sum_{k=0}^{m_N} \hat{R}_k \hat{u}_k L_k\right) + \partial_x\left(\sum_{k>m_N}^{N} \hat{R}_k \hat{u}_k L_k\right).$$

The first term can be estimated by the inverse property [CHQZ, p. 288]. For the second term, we note that if $v = \sum_{k=0}^{N} \hat{v}_k L_k$ and $\partial_x v = \sum_{k=0}^{N} \hat{v}_k^{(1)} L_k$, then [MOT1]

$$\hat{v}_k^{(1)} = (2k+1) \sum_{j \in J_{k,N}} \hat{v}_j, \qquad J_{k,N} \equiv \{j | k+1 \le j \le N, \quad j+k \text{ odd } \}.$$

We define

$$J_{m_N} = \{j | j \in J_{k,N}, \quad j > m_N\}$$

and have

$$\begin{aligned} \left\| \partial_x \left(\sum_{k>m_N}^{N} \hat{R}_k \hat{u}_k L_k \right) \right\|^2 &= \sum_{k=0}^{N-1} (2k+1)^2 \left(\sum_{j \in J_{m_N}} \hat{R}_j \hat{u}_j \right)^2 \|L_k\|^2 \\ &\le 2 \sum_{k=0}^{N-1} (2k+1) \left(\sum_{j \in J_{m_N}} \frac{|\hat{R}_j|^2}{\|L_j\|^2} \right) \left(\sum_{j \in J_{m_N}} |\hat{u}_j|^2 \|L_j\|^2 \right) \\ &\le C m_N^4 \|u\|^2 \left(\sum_{k=0}^{m_N} + \sum_{k>m_N}^{N} \right) \left[(2k+1) \sum_{j \in J_{m_N}} \frac{1}{j^3} \right] \\ &\le C m_N^4 \|u\|^2 \left[\sum_{k=0}^{m_N} (2k+1) \frac{1}{m_N^2} + \sum_{k>m_N}^{N} (2k+1) \frac{1}{k^2} \right] \\ &\le C m_N^4 \ln N \|u\|^2. \qquad \square \end{aligned} \tag{3.3}$$

In order to estimate the term $\|\partial_x I_N^C u\|$, we need the following lemma, which will be proved in Appendix A.

LEMMA 3.2. *If* $v \in H^1_{\omega^{1-2\theta}}(I)$ $(0 \le \theta \le 1)$, *then*

$$\|\partial_x I_N^C v\|_{\omega^{1-2\theta}} \le C \|\partial_x v\|_{\omega^{1-2\theta}}. \tag{3.4}$$

By the same argument as in the proof of (A.6), we can obtain

$$\|I_N^C u - u\|_\omega \le C N^{-1} \|\partial_x u\|_{\omega^{-1}}, \qquad \forall u \in H^1_{\omega^{-1}}(I). \tag{3.5}$$

Thus using (3.5) and (3.4) with $\theta = 1/2$ yields

$$\|\partial_x I_N^C u\| + N \|I_N^C u - u\| \le C \|\partial_x u\|, \qquad \forall u \in H^1(I). \tag{3.6}$$

We give the following approximation result.

LEMMA 3.3. *If* $u \in H^m(I)$ $(m \ge 1)$, *then*

$$\|I_N^C u - u\|_{H^l(I)} \le C N^{l-m} \|u\|_{H^m(I)}, \qquad 0 \le l \le 1. \tag{3.7}$$

Proof. Let I_N^L be the Legendre interpolant. Applying (3.6) to the function $u - I_N^L u$ and using the approximation result of I_N^L given in [BM], we get

$$\begin{aligned} &\|\partial_x (I_N^C u - I_N^L u)\| + N \|I_N^C u - u\| \\ &\qquad \le C \|\partial_x (u - I_N^L u)\| \le C N^{1-m} \|u\|_{H^m(I)}. \end{aligned} \tag{3.8}$$

Then the desired result follows from the triangle inequality and, again, the approximation result of I_N^L. ∎

We need the following inverse property of weight, which will be proved in Appendix A.

LEMMA 3.4. *If* $-1 \le \mu \le \sigma \le 1$, *then*

$$\|u\|_{\omega^\sigma} \le CN^{(\sigma-\mu)/2}\|u\|_{\omega^\mu}, \qquad \forall u \in \mathcal{P}_N. \tag{3.9}$$

Let $D = \omega^{-1}\partial_x$ and $D_L^2 = \partial_x\omega^{-2}\partial_x$.

LEMMA 3.5. *If* $\omega^{-2}\partial_x u \in H^1_{0,\omega}(I)$, *then we have*

$$\|D^2u\|_\omega^2 + \|\partial_x u\|_\omega^2 + \|x\partial_x u\|_\omega^2 = \|D_L^2 u\|_\omega^2, \qquad \|D_L^2 u\|_\omega \le 3\|D^2 u\|_\omega, \tag{3.10}$$

$$(D^2u, D_L^2 u)_\omega \ge \max\ \{\|D^2u\|_\omega^2,\ \frac{1}{3}\|D_L^2 u\|_\omega^2\}. \tag{3.11}$$

Proof. For any $v \in H^1_{0,\omega}(I)$, we have [MG]

$$\int_I (1+x^2)v^2\omega^5\,dx + \|\partial_x(v\omega)\|_{\omega^{-1}}^2 = \|\partial_x v\|_\omega^2. \tag{3.12}$$

Putting $v = \omega^{-2}\partial_x u$, we find that

$$\int_I (1+x^2)|\partial_x u|^2\omega\,dx + \|D^2u\|_\omega^2 = \|D_L^2 u\|_\omega^2, \tag{3.13}$$

which gives the first conclusion of (3.10).

On the other hand, according to the definition,

$$D_L^2 u = \partial_x(\omega^{-1}Du) = D^2u - x\omega Du = D^2u - x\partial_x u. \tag{3.14}$$

We have from (3.14) and (3.13) that

$$\begin{aligned} \|D_L^2 u\|_\omega^2 &\le 3\|D^2u\|_\omega^2 + \frac{3}{2}\int_I x^2|\partial_x u|^2\omega\,dx \\ &\le 3\|D^2u\|_\omega^2 + \frac{3}{4}\int_I (1+x^2)|\partial_x u|^2\omega\,dx \\ &\le 3\|D^2u\|_\omega^2 + \frac{3}{4}(\|D_L^2u\|_\omega^2 - \|D^2u\|_\omega^2), \end{aligned} \tag{3.15}$$

and the second conclusion of (3.10) follows.

Next, we have from (3.14) and (3.13) that

$$\begin{aligned} 2(D^2u, D_L^2u)_\omega &= \|D^2u\|_\omega^2 + \|D_L^2u\|_\omega^2 - \int_I x^2|\partial_x u|^2\omega\,dx \\ &\ge \|D^2u\|_\omega^2 + \|D_L^2u\|_\omega^2 - \frac{1}{2}\int_I (1+x^2)|\partial_x u|^2\omega\,dx \\ &\ge \frac{3}{2}\|D^2u\|_\omega^2 + \frac{1}{2}\|D_L^2u\|_\omega^2, \end{aligned} \tag{3.16}$$

which combined with (3.10) leads to (3.11). ∎

LEMMA 3.6. *If* $u \in L^2_\omega(I)$, *then*

$$\|P_N^L u\|_\omega \le C\ln N\|u\|_\omega, \tag{3.17}$$

$$\|P_N^C u\| \le C\ln N\|u\|. \tag{3.18}$$

Proof. We only prove the first conclusion. The second one can be obtained in the same way. We need the following result. For small positive ε,

$$\|P_N^L u\|_{\omega^{1-\varepsilon}} \le \frac{C}{\varepsilon}\|u\|_{\omega^{1-\varepsilon}}, \qquad \forall u \in L^2_{\omega^{1-\varepsilon}}(I), \tag{3.19}$$

which will be proved in Appendix B. By the inverse property of weight (3.9) and (3.19),

$$\|P_N^L u\|_\omega \le CN^{\varepsilon/2}\|P_N^L u\|_{\omega^{1-\varepsilon}} \le \frac{CN^{\varepsilon/2}}{\varepsilon}\|u\|_\omega. \tag{3.20}$$

Taking $\varepsilon = (\ln N)^{-1}$ to minimize the above bound yields the desired result. ∎

We next give some approximation results of spectral operators in the norm related to the high-order Chebyshev operator D^σ. We first introduce a Sobolev-type space. Let

$$u = \sum_{k=0}^{\infty} \hat{u}_k T_k, \qquad \hat{u}_k = \frac{(u, T_k)_\omega}{\|T_k\|^2_\omega}. \tag{3.21}$$

By the property (2.14) we have formally that

$$\|D^\sigma u\|_\omega = \left(\frac{\pi}{2}\sum_{k=1}^{\infty} k^{2\sigma}|\hat{u}_k|^2\right)^{1/2}, \qquad \sigma > 0. \tag{3.22}$$

We then define the Sobolev-type norms

$$\|u\|_{\sigma,D} = \left(\pi|\hat{u}_0|^2 + \frac{\pi}{2}\sum_{k=1}^{\infty} k^{2\sigma}|\hat{u}_k|^2\right)^{1/2}, \qquad \sigma \in R, \tag{3.23}$$

and we denote by $H^\sigma_D(I)$ the closure of the space of all polynomials with respect to this norm.

LEMMA 3.7. *If $u \in H^\sigma_D(I)$, then*

$$\|D^\mu(P_N^C u - u)\|_\omega \le CN^{\mu-\sigma}\|D^\sigma u\|_\omega, \qquad 0 \le \mu \le \sigma, \tag{3.24}$$

$$\|D^\mu(I_N^C u - u)\|_\omega \le CN^{\mu-\sigma}\|D^\sigma u\|_\omega, \qquad 0 \le \mu \le \sigma, \quad \sigma > \frac{1}{2}, \tag{3.25}$$

$$\|D^\mu(P_N^L u - u)\|_\omega \le CN^{\mu-\sigma}\ln N\|D^\sigma u\|_\omega, \qquad 0 \le \mu \le \sigma. \tag{3.26}$$

Proof. It can be seen from (2.14) that for $s \ge 0$,

$$\|D^s u\|^2_\omega = \sum_{k=0}^{\infty} k^{2s}|\hat{u}_k|^2\|T_k\|^2_\omega, \qquad \forall u = \sum_{k=0}^{\infty}\hat{u}_k T_k \in H^s_D(I). \tag{3.27}$$

Thus it is not difficult to get (3.24), and, by the alias relation, (3.25) can be obtained as in the proof of (A.6).

Next, from (3.27), we have the following inverse property:

$$\|D^s u\|_\omega \le CN^s\|u\|_\omega, \qquad \forall u \in \mathcal{P}_N. \tag{3.28}$$

Then, by (3.28), (3.17), and (3.24),

$$\begin{aligned}\text{(3.29)}\qquad \|D^\mu(P_N^L u - u)\|_\omega &\le \|D^\mu P_N^L(u - P_N^C u)\|_\omega + \|D^\mu(P_N^C u - u)\|_\omega \\ &\le CN^\mu \ln N\|u - P_N^C u\|_\omega + CN^{\mu-\sigma}\|D^\sigma u\|_\omega \\ &\le CN^{\mu-\sigma}\ln N\|D^\sigma u\|_\omega. \qquad \square\end{aligned}$$

LEMMA 3.8. *We have that*

$$\text{(3.30)}\qquad \|u\|_{L^\infty(I)} \le C\|u\|_\omega^{1/2}\|u\|_{1,D}^{1/2}, \qquad \forall u \in H_D^1(I),$$

$$\text{(3.31)}\qquad \|u\|_{L^\infty(I)} \le C\sqrt{N}\|u\|_\omega, \qquad \forall u \in \mathcal{P}_N.$$

Proof. The first result is the usual Sobolev inequality under the transformation $x = \cos\theta$. The second result is the inverse property [CHQZ, (9.5.3), p. 295]. $\square$

LEMMA 3.9. *Let Q be defined in* (2.12), $\|Du\|_{\omega,Q}^2 \equiv (DQu, Du)_\omega$, *and* $R \equiv I - Q$. *Then*

$$\text{(3.32)}\qquad \|Du\|_\omega^2 \le \|Du\|_{\omega,Q}^2 + Cm_N^3 \ln^3 N\|u\|^2, \qquad \forall u \in \mathcal{P}_N,$$

$$\text{(3.33)}\qquad \|D^2 Ru\|_\omega \le m_N\|Du\|_\omega, \qquad \forall u \in \mathcal{P}_N.$$

Proof. If $u = \sum_{k=0}^N \hat{u}_k T_k$, then

$$u = Qu + Ru, \qquad Ru = \sum_{k=0}^{N} \hat{R}_k \hat{u}_k T_k,$$

where, according to (2.12),

$$\hat{R}_k \equiv 1 - \hat{Q}_k \begin{cases} = 1 & k \le m_N, \\ \le \left(\dfrac{m_N}{k}\right)^3 & k > m_N. \end{cases}$$

Let $\|Du\|_{\omega,R}^2 \equiv (DRu, Du)_\omega$. We have $\|Du\|_\omega^2 = \|Du\|_{\omega,Q}^2 + \|Du\|_{\omega,R}^2$ and

$$\|Du\|_{\omega,R}^2 = \sum_{k=1}^{m_N} k^2|\hat{u}_k|^2\|T_k\|_\omega^2 + \sum_{k>m_N}^{N} k^2\hat{R}_k|\hat{u}_k|^2\|T_k\|_\omega^2 \equiv I_1 + I_2.$$

By (3.9) and (3.18), we get

$$I_1 \le m_N^2\|P_{m_N}^C u\|_\omega^2 \le m_N^3\|P_{m_N}^C u\|^2 \le m_N^3 \ln^2 m_N\|u\|^2.$$

Next, let $\Delta_- v(k) = v(k) - v(k-1)$ and $\Delta_+ v(k) = v(k+1) - v(k)$. We have from (3.9) and (3.18) that

$$\begin{aligned} I_2 &\le m_N^3 \sum_{k>m_N}^{N} \frac{1}{k}\Delta_-(\|P_k^C u\|_\omega^2) \\ &= m_N^3\left(\frac{1}{N}\|u\|_\omega^2 - \frac{1}{m_N+1}\|P_{m_N}^C u\|_\omega^2 - \sum_{k>m_N}^{N-1}\Delta_+\left(\frac{1}{k}\right)\|P_k^C u\|_\omega^2\right) \\ &\le Cm_N^3\left(\|u\|^2 + \sum_{k>m_N}^{N-1}\frac{\ln^2 k}{k+1}\|u\|^2\right) \le Cm_N^3\ln^3 N\|u\|^2. \end{aligned}$$

Therefore,

$$\|Du\|_{\omega,R}^2 \le C m_N^3 \ln^3 N \|u\|^2. \tag{3.34}$$

For the second result we have

$$\begin{aligned}\|D^2 Ru\|_\omega^2 &= \sum_{k=0}^{N} k^4 |\hat{R}_k|^2 |\hat{u}_k|^2 \|T_k\|_\omega^2 \\ &\le m_N^2 \sum_{k=0}^{N} k^2 |\hat{u}_k|^2 \|T_k\|_\omega^2 = m_N^2 \|Du\|_\omega^2. \qquad \square\end{aligned}$$

We conclude this section by a remark on the viscosity operator. We might define the operator Q in (2.16) with the coefficients satisfying

$$\begin{cases} \hat{Q}_k \equiv 0 & k \le m_N, \\ 1 - \left(\dfrac{m_N}{k}\right)^2 \le \hat{Q}_k \le 1 & k > m_N. \end{cases} \tag{3.35}$$

Then it is easy to show that

$$\|Du\|_\omega^2 \le \|Du\|_{\omega,Q}^2 + m_N^2 \|u\|_\omega^2, \qquad \forall u \in \mathcal{P}_N.$$

Although $\|u_N\|_\omega \le C\|u_N\|_{L^\infty}$ and we will assume that $\|u_N\|_{L^\infty}$ is bounded, we do not want to use the assumption in the first place. We think it is more reasonable to use $\hat{Q}_k$ as in (2.12).

4. A priori estimates. This section is devoted to some a priori estimates related to the approximation solution, which will be used in the proof of convergence. It is not difficult to show that the approximation scheme has a unique local solution by the theory of ordinary differential equations. As for the global solution, we refer to [MOT2], [O]. We will assume that the approximation solution is uniformly bounded, which can be confirmed by the numerical results [MOT1].

Assumption (L^∞*-boundedness*). There exists a finite constant M such that

$$\|u_N\|_{L^\infty(\bar{I}\times[0,T])} \le M. \tag{4.1}$$

We will denote by C_M the constant dependent only on the bound M and the flux function f. To simplify the presentation, we only consider the case where $x = -1$ is an inflow boundary that is prescribed with the data $g(t) \in H^1(0,T)$, while $x = 1$ is an outflow one. The boundary term $B(u_N)$ is now of the form

$$B(u_N) = \tau e(t) R_N^{(0)}(x), \qquad e(t) \equiv u_N(-1,t) - g(t).$$

4.1. A priori estimates for the CLSV approximation. In this subsection we give two basic energy estimates for the solution of CLSV scheme (2.2). Let $F(u) = \int^u \xi f'(\xi)\, d\xi$ and set $\varphi = u_N$ in (2.2). We have from (3.7) and (3.1) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + F(u_N)|_{-1}^{+1} + \varepsilon_N \|\partial_x Q u_N\|^2 + \tau e^2(t) + \tau e(t) g(t) \\ = (\partial_x (I - I_N^C) f(u_N), u_N) = -((I - I_N^C) f(u_N), \partial_x u_N) \\ \le \frac{C_M}{N}\|\partial_x u_N\|^2 \le \frac{C_M}{N}\|\partial_x Q u_N\|^2 + \frac{C_M m_N^4 \ln N}{N}\|u_N\|^2.\end{aligned} \tag{4.2}$$

To bound $\tau e(t)g(t)$, we set $\varphi = 1$ in (2.2) and obtain

$$\frac{d}{dt}(u_N, 1) + f(u_N)|_{-1}^{+1} = -\tau e(t),$$

which implies that

$$\tau|e(t)| \leq \sqrt{2}\|\partial_t u_N\| + |f|_\infty, \qquad |f|_\infty \equiv \max_{|\xi|\leq M} |f(\xi)| \tag{4.3}$$

and for $E(t) \equiv \tau \int_0^t e(s)\,ds$,

$$|E(t)| \leq \sqrt{2}(\|u_N(t)\|_\omega + \|u_N(0)\|_\omega) + t|f|_\infty. \tag{4.4}$$

Let $\Omega = I \times (0, T)$. We have the following estimates.

LEMMA 4.1. *Let ε_N and m_N satisfying*

$$\varepsilon_N = N^{-\theta}, \qquad m_N \leq CN^{q/4}, \qquad 0 < q < \theta < 1. \tag{4.5}$$

Then we have

$$\|u_N(T)\|^2 + \varepsilon_N\|\partial_x Q u_N\|_{L^2(\Omega)}^2 + \tau\|e\|_{L^2}^2 \leq C_M(1 + \|g\|_{H^1}^2), \tag{4.6}$$

$$\|\partial_t u_N\|_{L^2(\Omega)}^2 + \|\partial_x u_N\|_{L^2(\Omega)}^2 + \tau e^2(T) \leq \frac{C_M}{\varepsilon_N}(1 + \|g\|_{H^1}^2). \tag{4.7}$$

Proof. By (4.4), we get

$$\left|\int_0^T \tau e(t)g(t)\,dt\right| = \left|E(t)g(t)|_0^T - \int_0^T E(t)\frac{d}{dt}g(t)\,dt\right| \leq C_M\|g\|_{H^1}^2. \tag{4.8}$$

Hence we obtain from (4.2) that

$$\begin{aligned}\|u_N(T)\|^2 &+ 2\left(\varepsilon_N - \frac{C_M}{N}\right)\|\partial_x Q u_N\|_{L^2(\Omega)}^2 + 2\tau\|e\|_{L^2}^2 \\ &\leq \|u_N(0)\|^2 + \frac{Cm_N^4 \ln N}{N}\|u_N\|_{L^2(\Omega)}^2 \\ &\quad + C|F|_\infty + 2\left|\int_0^T \tau e(t)g(t)\,dt\right| \\ &\leq C_M(1 + \|g\|_{H^1}^2),\end{aligned} \tag{4.9}$$

which completes the proof of (4.6). This together with (3.1) also gives us

$$\begin{aligned}\varepsilon_N\|\partial_x u_N\|_{L^2(\Omega)}^2 &\leq 2\varepsilon_N\|\partial_x Q u_N\|_{L^2(\Omega)}^2 + C\varepsilon_N m_N^4 \ln N\|u_N\|_{L^2(\Omega)}^2 \\ &\leq C_M(1 + \|g\|_{H^1}^2).\end{aligned} \tag{4.10}$$

Next, we set $\varphi = \partial_t u_N$ in (2.2) and get

$$\begin{aligned}\|\partial_t u_N\|^2 &+ (\partial_x I_N^C f(u_N), \partial_t u_N) \\ &= -\varepsilon_N(\partial_x Q u_N, \partial_t\partial_x Q u_N) - \tau e(t)\partial_t u_N(-1, t) \\ &= -\frac{\varepsilon_N}{2}\frac{d}{dt}\|\partial_x Q u_N\|^2 - \frac{\tau}{2}\frac{d}{dt}e^2(t) - \tau e(t)\frac{d}{dt}g(t).\end{aligned} \tag{4.11}$$

By using (4.3), we have

$$\|\partial_t u_N\|^2 + \frac{\varepsilon_N}{2}\frac{d}{dt}\|\partial_x Q u_N\|^2 + \frac{\tau}{2}\frac{d}{dt}e^2(t) \tag{4.12}$$
$$\le \|\partial_x I_N^C f(u_N)\|\;\|\partial_t u_N\| + \tau|e(t)|\;\left|\frac{d}{dt}g(t)\right|$$
$$\le C_M\|\partial_x u_N\|^2 + \frac{1}{2}\|\partial_t u_N\|^2 + C\left(\left|\frac{d}{dt}g(t)\right|^2 + |f|_\infty^2\right).$$

The integration of (4.12) combined with (4.11) yields

$$\|\partial_t u_N\|^2_{L^2(\Omega)} + \varepsilon_N\|\partial_x Q u_N(T)\|^2 + \tau e^2(T) \tag{4.13}$$
$$\le C_M(\varepsilon_N\|\partial_x Q u_N(0)\|^2 + \tau e^2(0) + \|\partial_x u_N\|^2_{L^2(\Omega)} + \|g\|^2_{H^1} + |f|^2_\infty)$$
$$\le \frac{C_M}{\varepsilon_N}(1 + \|g\|^2_{H^1}).$$

The desired result (4.7) follows from (4.13) and (4.10). ◻

4.2. A priori estimates for the NSV approximation. In this subsection we consider the solution of NSV scheme (2.16). We begin with a $L^2(I)$-estimate. Let $\|\cdot\|_{\omega;0} \equiv \|\cdot\|_{L^2(0,T;L^2_\omega(I))}$.

LEMMA 4.2. *Let ε_N, m_N, and τ satisfying*

$$\varepsilon_N = N^{-\theta}, \quad m_N \le CN^{q/3}, \quad \tau = N^\delta, \quad 0 < q < \theta < 1,\ 0 < \delta < 1 - \theta. \tag{4.14}$$

Then we have

$$\|u_N(T)\|^2 + \varepsilon_N\|Du_N\|^2_{\omega;0} + \tau\|e\|^2_{L^2} \le C_M(1 + \|g\|^2_{H^1}). \tag{4.15}$$

Proof. Let $F(u) = \int^u \xi f'(\xi)\,d\xi$. We get from (2.16) that

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + F(u_N)|_{-1}^{+1} + \varepsilon_N\|Du_N\|^2_{\omega;0} + \tau e^2(t) + \tau e(t)g(t) \tag{4.16}$$
$$= (\partial_x(I - P_N)f(u_N), u_N) \equiv I(P_N).$$

We estimate $I(P_N)$ in the different cases as follows:

1. By (3.30) and (3.24),

$$|I(P_N^C)| = |\;[(I - P_N^C)f(u_N)u_N]|_{-1}^{+1} - ((I - P_N^C)f(u_N), \partial_x u_N)\;| \tag{4.17}$$
$$\le C_M\|(I - P_N^C)f(u_N)\|_{L^\infty(I)} + \|(I - P_N^C)f(u_N)\|_\omega\,\|Du_N\|_\omega$$
$$\le C_M\|(I - P_N^C)f(u_N)\|_\omega^{1/2}\,\|(I - P_N^C)f(u_N)\|_{1,D}^{1/2}$$
$$+ \frac{C}{N}\|Df(u_N)\|_\omega\,\|Du_N\|_\omega$$
$$\le \frac{C_M}{\sqrt{N}}\|Df(u_N)\|_\omega + \frac{C_M}{N}\|Du_N\|^2_\omega \le C_M\left(1 + \frac{1}{N}\|Du_N\|^2_\omega\right);$$

2. By (3.25),

$$|I(I_N^C)| = |((I - I_N^C)f(u_N), \partial_x u_N)| \tag{4.18}$$
$$= \|(I - I_N^C)f(u_N)\|_\omega\,\|Du_N\|_\omega \le \frac{C_M}{N}\|Du_N\|^2_\omega;$$

3. By (3.30) and (3.26),

$$
\begin{aligned}
|I(P_N^L)| &= |\,[(I-P_N^L)f(u_N)u_N]|_{-1}^{+1}\,| \\
&\le C_M\|(I-P_N^L)f(u_N)\|_\omega^{1/2}\,\|(I-P_N^L)f(u_N)\|_{1,D}^{1/2} \\
&\le C_M\frac{\ln N}{\sqrt{N}}\|Df(u_N)\|_\omega \le C_M\left(1+\frac{\ln^2 N}{N}\|Du_N\|_\omega^2\right).
\end{aligned}
\tag{4.19}
$$

Thus we obtain from (4.16) and (3.32) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_N\|^2+\left(\varepsilon_N-\frac{C_M\ln^2 N}{N}\right)\|Du_N\|_\omega^2+\tau e^2(t) \\
&\qquad\le C\varepsilon_N m_N^3\ln^3 N\|u_N\|^2+|F|_\infty-\tau e(t)g(t),
\end{aligned}
\tag{4.20}
$$

where $|F|_\infty \equiv \max_{|\xi|\le M}|F(\xi)|$. To bound $\tau e(t)g(t)$, we use (2.16) to get

$$
\frac{d}{dt}(u_N,1)+f(u_N)|_{-1}^{+1}+\tau e(t)=[(I-P_N)f(u_N)]|_{-1}^{+1}\equiv J(P_N),
\tag{4.21}
$$

where $J(I_N^C)=0$, and as we can see from (4.17) and (4.19) that

$$
|J(P_N^C)| \le \frac{C_M}{\sqrt{N}}\|Du_N\|_\omega,
\tag{4.22}
$$

$$
|J(P_N^L)| \le C_M\frac{\ln N}{\sqrt{N}}\|Du_N\|_\omega.
\tag{4.23}
$$

Hence, we have from (4.21) that

$$
\left|E(T)\equiv\tau\int_0^T e(t)\,dt\right| \le C_M+C_M\frac{\ln N}{\sqrt{N}}\|Du_N\|_{\omega;0}
\tag{4.24}
$$

and

$$
\begin{aligned}
\left|\int_0^T \tau e(t)g(t)\,dt\right| &= \left|E(t)g(t)|_0^T-\int_0^T E(t)\frac{d}{dt}g(t)\,dt\right| \\
&\le C_M\left(\frac{\ln^2 N}{N}\|Du_N\|_{\omega;0}^2+\|g\|_{H^1}^2\right).
\end{aligned}
\tag{4.25}
$$

The proof is completed by the temporal integration of (4.20) and use of (4.25). ∎

Next, we work on an $H^1(I)$-estimate.

LEMMA 4.3. *Assume that* (4.14) *holds and*

$$
\varepsilon_N\|Du_N(0)\|^2 \le C\tau N.
\tag{4.26}
$$

We have

$$
\varepsilon_N\|Du_N(T)\|^2+\varepsilon_N^2\|D^2u_N\|_{\omega;0}^2 \le C_M(1+\|g\|_{H^1}^2)\tau N.
\tag{4.27}
$$

Proof. Let $D_L^2=\partial_x\omega^{-2}\partial_x$. We have from (2.16) that

$$
(\partial_t u_N+\partial_x P_N f(u_N), D_L^2 u_N)=\varepsilon_N(D^2Qu_N, D_L^2u_N)_\omega-(B(u_N),D_L^2u_N).
\tag{4.28}
$$

This gives us

$$\frac{1}{2}\frac{d}{dt}\|Du_N\|^2 + \varepsilon_N(D^2u_N, D_L^2u_N)_\omega = \varepsilon_N(D^2Ru_N, D_L^2u_N)_\omega + (DP_Nf(u_N), D_L^2u_N) + \tau e(t)(D_L^2u_N)(-1,t). \tag{4.29}$$

By using Lemma 3.5, (3.33), and Lemma 3.7, we get

$$\begin{aligned}\frac{\varepsilon_N}{2}\frac{d}{dt}\|Du_N\|^2 + \varepsilon_N^2\|D^2u_N\|_\omega^2 &\le \varepsilon_N\|D_L^2u_N\|_\omega\left(\varepsilon_N\|D^2Ru_N\|_\omega + \|DP_Nf(u_N)\|_\omega + C\tau\sqrt{N}|e|\right)\\ &\le \frac{\varepsilon_N^2}{4}\|D^2u_N\|_\omega^2 + C(\varepsilon_N^2\|D^2Ru_N\|_\omega^2 + \|DP_Nf(u_N)\|_\omega^2 + N\tau^2e^2)\\ &\le \frac{\varepsilon_N^2}{4}\|D^2u_N\|_\omega^2 + C_M(\varepsilon_N^2m_N^2\|Du_N\|_\omega^2 + \ln^2 N\|Du_N\|_\omega^2 + N\tau^2e^2).\end{aligned} \tag{4.30}$$

The temporal integration of (4.30) yields the desired result,

$$\begin{aligned}\varepsilon_N\|Du_N(T)\|^2 + \varepsilon_N^2\|D^2u_N\|_{\omega;0}^2 &\le \varepsilon_N\|Du_N(0)\|^2 + C_M(1+\|g\|_{H^1}^2)\left(\varepsilon_N m_N^2 + \frac{\ln^2 N}{\varepsilon_N} + \tau N\right)\\ &\le C_M(1+\|g\|_{H^1}^2)\,\tau N. \qquad \square\end{aligned} \tag{4.31}$$

5. The convergence of the spectral viscosity approximation. In this section we prove the convergence of the CLSV approximation (2.2) and the NSV approximation (2.16), respectively, by compensated compactness arguments. Based on the framework of [Tr], we need only to prove that $\partial_tU(u_N) + \partial_xF(u_N)$ can be expressed as a sum of two terms such that one belongs to a compact subset of $H_{loc}^{-1}(\Omega)$ and the other is bounded in $L_{loc}^1(\Omega)$ for all convex entropy pairs $(U(u_N), F(u_N))$, where $\Omega = (-1,1)\times(0,T)$. We will simplify $C_M(1+\|g\|_{H^1}^2)$ as C_M and also use the following notations:

$$(\cdot,\cdot) \equiv (\cdot,\cdot)_{L^2(\Omega)}, \qquad \|\cdot\| \equiv \|\cdot\|_{L^2(\Omega)}, \qquad \|\cdot\|_\infty \equiv \|\cdot\|_{L^\infty(\Omega)},$$

$$(\cdot,\cdot)_\omega \equiv (\cdot,\cdot)_{L^2(0,T;L^2_\omega(I))}, \qquad \|\cdot\|_\omega \equiv \|\cdot\|_{L^2(0,T;L^2_\omega(I))}.$$

We follow the same line as in [MOT1]. So we will go through quickly but only pay attention on some differences.

5.1. The convergence of the CLSV approximation. Here we consider the CLSV approximation (2.2). For any $\varphi \in H_0^1(\Omega)$, we define

$$\varphi_N = \int_{-1}^{x} P_{N-1}^L(\partial_x\varphi)\,dx, \tag{5.1}$$

where $P_M^L : L^2(I) \to \mathcal{P}_M$ is the Legendre orthogonal projection operator. Thus $\varphi_N(\pm1,t) = 0$ and we have [MOT1] that

$$\|\partial_x\varphi_N\| + N\|\varphi - \varphi_N\| \le C\|\partial_x\varphi\|. \tag{5.2}$$

Then, we have

$$(5.3)\qquad (\partial_t u_N + \partial_x f(u_N), \varphi) \equiv (\partial_t u_N + \partial_x f(u_N), \varphi - \varphi_N) + (\partial_x (I - I_N^C) f(u_N), \varphi_N) + (\partial_t u_N + \partial_x I_N^C f(u_N), \varphi_N) \equiv \sum_{j=1}^{3} I_j(\varphi).$$

By Lemma 4.1, the first term can be bounded as

$$(5.4)\qquad |I_1(\varphi)| \leq (\|\partial_t u_N\| + C_M \|\partial_x u_N\|) \|\varphi - \varphi_N\| \leq \frac{C_M}{\sqrt{\varepsilon_N}} \|\varphi - \varphi_N\|.$$

We use Lemma 3.3 to estimate the second term,

$$(5.5)\qquad \begin{aligned} |I_2(\varphi)| &= |((I - I_N^C) f(u_N), \partial_x \varphi_N)| \\ &\leq \frac{C_M}{N} \|\partial_x u_N\| \, \|\partial_x \varphi_N\| \leq \frac{C_M}{\sqrt{\varepsilon_N} N} \|\partial_x \varphi_N\|. \end{aligned}$$

For the third term, we have from (4.6) and (3.2) that

$$(5.6)\qquad \begin{aligned} |I_3(\varphi)| &\leq \varepsilon_N |(\partial_x Q u_N, \partial_x Q \varphi_N)| \leq \varepsilon_N \|\partial_x Q u_N\| \, \|\partial_x Q \varphi_N\| \\ &\leq C_M \sqrt{\varepsilon_N} (\|\partial_x \varphi_N\| + m_N^2 \sqrt{\ln N} \|\varphi_N\|). \end{aligned}$$

Therefore, it follows from (5.2) that for any $\varphi \in H_0^1(\Omega)$,

$$(5.7)\qquad \begin{aligned} &|(\partial_t u_N + \partial_x f(u_N), \varphi)| \\ &\leq \frac{C_M}{\sqrt{\varepsilon_N} N} \|\partial_x \varphi\| + C_M \sqrt{\varepsilon_N} \left[\|\partial_x \varphi\| + m_N^2 \sqrt{\ln N} \left(\frac{1}{N} \|\partial_x \varphi\| + \|\varphi\|\right)\right] \\ &\leq C_M \sqrt{\varepsilon_N} \left(\|\partial_x \varphi\| + m_N^2 \sqrt{\ln N} \|\varphi\|_\infty\right) \to 0, \end{aligned}$$

and it implies that $\partial_t u_N + \partial_x f(u_N)$ belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

Next, for any entropy pair $(U(u_N), F(u_N))$, we have $\partial_t U(u_N) + \partial_x F(u_N) = (\partial_t u_N + \partial_x f(u_N)) U'(u_N)$. If we replace the function φ in the above procedure with the function $U'(u_N)\varphi$, then we obtain that for any $\varphi \in H_0^1(\Omega)$,

$$(5.8)\qquad \begin{aligned} |(\partial_t U(u_N) + \partial_x F(u_N), \varphi)| &= |(\partial_t u_N + \partial_x f(u_N), U'(u_N)\varphi)| \\ &\leq \sum_{j=1}^{3} |I_j(U'(u_N)\varphi)| \\ &\leq C_M \sqrt{\varepsilon_N} \left(\|\partial_x (U'(u_N)\varphi)\| \right.\\ &\quad \left. + m_N^2 \sqrt{\ln N} \|U'(u_N)\varphi\|_\infty\right) \\ &\leq C_M \sqrt{\varepsilon_N} \left(\|\partial_x u_N\| \, \|\varphi\|_\infty + \|U'(u_N)\|_\infty \|\partial_x \varphi\| \right.\\ &\quad \left. + m_N^2 \sqrt{\ln N} \|U'(u_N)\|_\infty \|\varphi\|_\infty\right). \end{aligned}$$

So $\partial_t U(u_N) + \partial_x F(u_N)$ also belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

Furthermore, we can show that $\partial_t U(u_N) + \partial_x F(u_N)$ tends weakly to a negative measure. To this end, we first note that the first two terms in (5.8) tend to 0,

$$(5.9)\qquad \begin{aligned} \sum_{1}^{2} |I_j(U'(u_N)\varphi)| &\leq \frac{C_M}{\sqrt{\varepsilon_N} N} \|\partial_x (U'(u_N)\varphi)\| \\ &\leq \frac{C_M}{\sqrt{\varepsilon_N} N} \left(\|\partial_x u_N\| \, \|\varphi\|_\infty + \|U'(u_N)\|_\infty \|\partial_x \varphi\|\right) \to 0. \end{aligned}$$

Next, we rewrite the third term in (5.8) as

$$
\begin{aligned}
(5.10)\qquad I_3(U'(u_N)\varphi) &= -\varepsilon_N(\partial_x Qu_N, \partial_x Q(U'(u_N)\varphi)_N)\\
&= -\varepsilon_N(\partial_x Qu_N, \partial_x (I-R)(U'(u_N)\varphi)_N)\\
&= -\varepsilon_N(\partial_x u_N, \partial_x (U'(u_N)\varphi)_N) + \varepsilon_N(\partial_x Ru_N, \partial_x (U'(u_N)\varphi)_N)\\
&\quad + \varepsilon_N(\partial_x Qu_N, \partial_x R(U'(u_N)\varphi)_N) \equiv \sum_{j=1}^{3} J_j(\varphi).
\end{aligned}
$$

Thus, for any $\varphi \geq 0$, we have from the definition (5.1) and the convexity of U that

$$
\begin{aligned}
(5.11)\qquad J_1(\varphi) &= -\varepsilon_N(\partial_x u_N, \partial_x (U'(u_N)\varphi)_N)\\
&= -\varepsilon_N(\partial_x u_N, U''(u_N)\varphi\partial_x u_N) - \varepsilon_N(\partial_x u_N, U'(u_N)\partial_x\varphi)\\
&\leq -\varepsilon_N(\partial_x u_N, U'(u_N)\partial_x\varphi) \leq C\sqrt{\varepsilon_N}\|\partial_x\varphi\| \to 0,
\end{aligned}
$$

and the other two terms tend to 0,

$$
\begin{aligned}
(5.12)\qquad |J_2(\varphi)| &= \varepsilon_N|(\partial_x Ru_N, \partial_x(U'(u_N)\varphi))|\\
&\leq \varepsilon_N\|\partial_x Ru_N\|\ \|\partial_x(U'(u_N)\varphi)\|\\
&\leq C\varepsilon_N m_N^2\sqrt{\ln N}(\|\partial_x u_N\|\ \|\varphi\|_\infty + \|U'(u_N)\|_\infty\|\partial_x\varphi\|_\infty) \to 0,
\end{aligned}
$$

$$
\begin{aligned}
(5.13)\qquad |J_3(\varphi)| &\leq \varepsilon_N\|\partial_x Qu_N\| \cdot Cm_N^2\sqrt{\ln N}\|(U'(u_N)\varphi)_N\|\\
&\leq C_M\sqrt{\varepsilon_N \ln N}m_N^2\left(\frac{1}{N}\|\partial_x(U'(u_N)\varphi)\| + \|U'(u_N)\varphi\|\right)\\
&\leq C_M\sqrt{\varepsilon_N \ln N}m_N^2\left[\frac{C}{N}(\|\partial_x u_N\|\ \|\varphi\|_\infty + \|U'(u_N)\|_\infty\|\partial_x\varphi\|)\right.\\
&\quad \left. + \|U'(u_N)\|_\infty\|\varphi\|_\infty\right] \to 0.
\end{aligned}
$$

Thus we arrive at the following convergence theorem.

THEOREM 5.1. *Let the spectral viscosity parameters ε_N and m_N satisfy* (4.5). *Then the bounded solution $u_N(x,t)$ of the CLSV approximation* (2.2) *converges strongly in $L^p_{loc}(\Omega)$ $(p < \infty)$ to the unique entropy solution of* (1.1).

5.2. The convergence of the NSV approximation. Now we consider the NSV approximation (2.16). For any $\varphi \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
(5.14)\qquad &(\partial_t u_N + \partial_x f(u_N), \varphi)\\
&\quad = \varepsilon_N(D^2Qu_N, P_N^L\varphi)_\omega + (\partial_x(I-P_N)f(u_N),\varphi) - (B(u_N),\varphi) \equiv \sum_{j=1}^{3} I_j(\varphi).
\end{aligned}
$$

By Lemma 4.2, (4.7), and (3.17), the first term can be bounded as

$$
\begin{aligned}
(5.15)\qquad |I_1(\varphi)| &= \varepsilon_N|(D^2Qu_N,\varphi)_\omega + (D^2Qu_N, (P_N^L - I)\varphi)_\omega|\\
&\leq \varepsilon_N(\|DQu_N\|_\omega\|D\varphi\|_\omega + \|D^2Qu_N\|_\omega\|(P_N^L - I)\varphi\|_\omega)\\
&\leq C_M\left(\sqrt{\varepsilon_N} + \frac{\sqrt{\tau}\ln N}{\sqrt{N}}\right)\|D\varphi\|_\omega.
\end{aligned}
$$

We use Lemma 3.7 to estimate the second term,

$$|I_2(\varphi)| = |((I-P_N)f(u_N), \partial_x\varphi)| \le \|(I-P_N)f(u_N)\|_\omega \|D\varphi\|_\omega \le \frac{C_M \ln N}{N}\|Du_N\|_\omega\|D\varphi\|_\omega \le \frac{C_M \ln N}{\sqrt{\varepsilon_N}N}\|D\varphi\|_\omega. \tag{5.16}$$

For the third term we have from (4.6), (3.31), and Lemma 3.7 that

$$|I_3(\varphi)| = \left|\int_0^T \tau e(t)\,[(P_N^L - I_N^C)\varphi](-1,t)\,dt\right| \le \tau\|e\|_{L^2}\cdot\sqrt{N}\|(P_N^L - I_N^C)\varphi\|_\omega \le C_M\frac{\sqrt{\tau}\ln N}{\sqrt{N}}\|D\varphi\|_\omega. \tag{5.17}$$

Thus, we have that for any $\varphi \in H_0^1(\Omega)$,

$$|(\partial_t u_N + \partial_x f(u_N), \varphi)| \le C_M\left(\sqrt{\varepsilon_N} + \frac{\sqrt{\tau}\ln N}{\sqrt{N}} + \frac{\ln N}{\sqrt{\varepsilon_N}N}\right)\|D\varphi\|_\omega \le C_M\left(\sqrt{\varepsilon_N} + \frac{\sqrt{\tau}\ln N}{\sqrt{N}}\right)\|\partial_x\varphi\| \to 0, \tag{5.18}$$

and it implies that $\partial_t u_N + \partial_x f(u_N)$ belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

Next, if we replace the function φ in the above procedure with the function $U'(u_N)\varphi$, then we obtain that for any $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned} |(\partial_t U(u_N) + \partial_x F(u_N), \varphi)| &= |(\partial_t u_N + \partial_x f(u_N), U'(u_N)\varphi)| \\ &\le \sum_{j=1}^{3}|I_j(U'(u_N)\varphi)| \le C_M\left(\sqrt{\varepsilon_N} + \frac{\sqrt{\tau}\ln N}{\sqrt{N}}\right)\|D(U'(u_N)\varphi)\|_\omega \\ &\le C_M\left(\sqrt{\varepsilon_N} + \frac{\sqrt{\tau}\ln N}{\sqrt{N}}\right)(\|Du_N\|_\omega\|\varphi\|_\infty + \|U'(u_N)\|_\infty\|D\varphi\|_\omega) \\ &\le C_M\left(1 + \frac{\sqrt{\tau}\ln N}{\sqrt{\varepsilon_N N}}\right)(\|\varphi\|_\infty + \sqrt{\varepsilon_N}\|\partial_x\varphi\|) \\ &\le C_M(1 + N^{-(1-\theta-\delta)/2}\ln N)\,(\|\varphi\|_\infty + \sqrt{\varepsilon_N}\|\partial_x\varphi\|). \end{aligned} \tag{5.19}$$

So $\partial_t U(u_N) + \partial_x F(u_N)$ also belongs to a compact subset of $H_{loc}^{-1}(\Omega)$.

Furthermore, we can prove that $\partial_t U(u_N) + \partial_x F(u_N)$ tends weakly to a negative measure. In fact, we have already shown in (5.16) and (5.17) that

$$\begin{aligned} \sum_{j=2}^{3}|I_j(U'(u_N)\varphi)| &\le C_M\left(\frac{\ln N}{\sqrt{\varepsilon_N}N} + \frac{\sqrt{\tau}\ln N}{\sqrt{N}}\right)\|D(U'(u_N)\varphi)\|_\omega \\ &\le C_M\frac{\sqrt{\tau}\ln N}{\sqrt{N}}(\|Du_N\|_\omega\|\varphi\|_\infty + \|U'(u_N)\|_\infty\|D\varphi\|_\omega) \\ &\le C_M N^{-(1-\theta-\delta)/2}\ln N\,(\|\varphi\|_\infty + \sqrt{\varepsilon_N}\|\partial_x\varphi\|) \to 0. \end{aligned} \tag{5.20}$$

For the first term in (5.19), we have

$$\begin{aligned} I_1(U'(u_N)\varphi) &= \varepsilon_N(D^2Qu_N, U'(u_N)\varphi)_\omega + \varepsilon_N(D^2Qu_N, (P_N^L - I)[U'(u_N)\varphi])_\omega \\ &= \varepsilon_N(D^2u_N, U'(u_N)\varphi)_\omega - \varepsilon_N(D^2Ru_N, U'(u_N)\varphi)_\omega \\ &\quad + \varepsilon_N(D^2Qu_N, (P_N^L - I)[U'(u_N)\varphi])_\omega \equiv \sum_{j=1}^{3}J_j(\varphi). \end{aligned}$$

Hence, for any $\varphi \geq 0$, we have from the convexity of U that

$$(5.21)\qquad \begin{aligned} J_1(\varphi) &= -\varepsilon_N(Du_N, D[U'(u_N)\varphi])_\omega \\ &= -\varepsilon_N(Du_N, U''(u_N)\varphi Du_N)_\omega - \varepsilon_N(Du_N, U'(u_N)D\varphi)_\omega \\ &\leq -\varepsilon_N(Du_N, U'(u_N)D\varphi)_\omega \leq C_M\sqrt{\varepsilon_N}\|\partial_x\varphi\| \to 0. \end{aligned}$$

On the other hand, by (3.34)

$$(5.22)\qquad \begin{aligned} |J_2(\varphi)| &= \varepsilon_N|(DRu_N, D[U'(u_N)\varphi])_\omega| \\ &\leq \varepsilon_N\|DRu_N\|_\omega\|D[U'(u_N)\varphi]\|_\omega \\ &\leq C_M\varepsilon_N(m_N\ln N)^{3/2}\left(\|Du_N\|_\omega\|\varphi\|_\infty + \|U'(u_N)\|_\infty\|D\varphi\|_\omega\right) \\ &\leq C_M N^{-(\theta-q)/2}\ln^{3/2}N\left(\|\varphi\|_\infty + \sqrt{\varepsilon_N}\|\partial_x\varphi\|\right) \to 0. \end{aligned}$$

Also, we have from (3.10), (4.7), and (3.26) that

$$(5.23)\qquad \begin{aligned} |J_3(\varphi)| &\leq \varepsilon_N\|D^2Qu_N\|_\omega\|(P_N^L - I)[U'(u_N)\varphi]\|_\omega \\ &\leq C_M\frac{\varepsilon_N\ln N}{N}\|D^2u_N\|_\omega\|D[U'(u_N)\varphi]\|_\omega \\ &\leq C_M\frac{\sqrt{\tau}\ln N}{\sqrt{N}}\left(\|Du_N\|_\omega\|\varphi\|_\infty + \|U'(u_N)\|_\infty\|D\varphi\|_\omega\right) \\ &\leq C_M N^{-(1-\theta-\delta)/2}\ln N\left(\|\varphi\|_\infty + \sqrt{\varepsilon_N}\|\partial_x\varphi\|\right) \to 0. \end{aligned}$$

We conclude by the following convergence theorem.

THEOREM 5.2. *Assume that* (4.14) *holds. Then the bounded solution* $u_N(x,t)$ *of the spectral viscosity scheme* (2.16) *converges strongly in* $L^p_{loc}(\Omega)$ $(p < \infty)$ *to the unique entropy solution of* (1.1).

6. Conclusion. It is shown that the CLSV method is an efficient way to solve conservation laws and enjoys the same convergence property as the Legendre spectral viscosity method. Basically, the schemes are formulated in Legendre methods except that the nonlinear term may be treated by Chebyshev methods. So it is more reasonable to expand u_N in Legendre polynomials and expect that the effect of the viscosity on the low modes is small. A suitable postprocessing procedure for the CLSV approximate solution will be interesting.

Appendix A. The proofs of Lemma 3.2 and Lemma 3.4. We will use operator interpolation methods to prove these lemmas.

Proof of Lemma 3.2. It seems not so easy to treat the term $\|\partial_x I_N^C u\|$ directly because we are unable to relate it to the exactness property of Gauss quadrature formula as in [BM]. We turn to the technique of interpolation of operators and start by quoting Theorem 3.6 of [BS, p. 213].

LEMMA A.1. *Let* (R,μ) *and* (S,ν) *be totally* σ*-finite measure spaces and let* T *be a linear operator defined on the* μ*-simple functions on* R *and taking values in the* ν*-measurable functions on* S*. Suppose that* u, v *are positive weights on* R *and* S*, respectively, and that* $1 \leq p_i, q_i \leq \infty$ $(i = 0, 1)$*. Suppose*

$$(A.1)\qquad \|(Tf)v_i\|_{L^{q_i}} \leq M_i\|fu_i\|_{L^{p_i}}, \qquad (i = 0, 1),$$

for all μ*-simple functions* f*. Let* $0 \leq \theta \leq 1$ *and define*

$$(A.2)\qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$u = u_0^{1-\theta} u_1^{\theta}, \qquad v = v_0^{1-\theta} v_1^{\theta}. \tag{A.3}$$

Then, if $p < \infty$, *the operator* T *has a unique extension to a bounded linear operator from* $L^p_{u^p}$ *to* $L^q_{v^q}$ *which satisfies*

$$\|(Tf)v\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \, \|fu\|_{L^p} \tag{A.4}$$

for all f *in* $L^p_{u^p}$.

To serve our purpose, we take $(R,\mu) = (S,\nu)$ as usual Lebesgue spaces; $p_i = q_i = 2$ $(i = 0,1)$ so that $p = q = 2$; and $u_0^2 = v_0^2 = \omega \equiv \ (1-x^2)^{-1/2}, u_1^2 = v_1^2 = \omega^{-1}$ so that $u^2 = v^2 = \omega^{1-2\theta}$. Next, we define the operator T: $L^2_{u_i^2}(I) \to L^2_{v_i^2}(I)$ $(i = 0,1)$ by

$$T = \partial_x I_N^C \partial_x^{-1}, \qquad \partial_x^{-1} u \equiv \int_{-1}^{x} u(\xi)\, d\xi.$$

The following lemma aims at meeting the condition of Lemma A.1.

LEMMA A.2. *We have*

$$\|Tu\|_{\omega} \le C\|u\|_{\omega}, \qquad \forall u \in L^2_{\omega}(I), \tag{A.5}$$

$$\|Tu\|_{\omega^{-1}} \le C\|u\|_{\omega^{-1}}, \qquad \forall u \in L^2_{\omega^{-1}}(I). \tag{A.6}$$

Proof. For any $u \in L^2_{\omega}(I)$, let $v \equiv \partial_x^{-1} u \in H^1_{\omega}(I)$. Corollary 4.6 of [BM] provides us with

$$\|\partial_x I_N^C v\|_{\omega} \le C \inf_{p_0 \in \mathcal{P}_0} \|v + p_0\|_{H^1_{\omega}} \le C\|\partial_x v\|_{\omega}. \tag{A.7}$$

So the first conclusion (A.5) follows.

Next, for the Chebyshev polynomial $T_k(x)$, we have

$$\partial_x(\omega^{-1}\partial_x T_k) + k^2 \omega T_k = 0, \tag{A.8}$$

which results in the orthogonal property in the weight ω^{-1},

$$(\partial_x T_k, \omega^{-1}\partial_x T_l) = k^2 \delta_{lk} \|T_k\|^2_{\omega}.$$

By this property, it is not difficult to show that for any $v = \sum_{k=0}^{\infty} \hat{v}_k T_k \in H^1_{\omega^{-1}}(I)$,

$$\|\partial_x v\|^2_{\omega^{-1}} = \|\sum_{k=0}^{\infty} \hat{v}_k \partial_x T_k\|^2_{\omega^{-1}} = \sum_{k=1}^{\infty} k^2 |\hat{v}_k|^2 \|T_k\|^2_{\omega}.$$

Let $I_N^C v = \sum_{k=0}^{N} \tilde{v}_k T_k$. We have the aliasing relation [CHQZ, p. 68]:

$$\tilde{v}_k = \sum_{j \in J_k} \hat{v}_k, \qquad J_k \equiv \{ j | j = 2lN \pm k \ge 0, \quad l \text{ is any integer } \}.$$

Then we get from (A.8),

$$\begin{aligned} \|\partial_x I_N^C v\|^2_{\omega^{-1}} &= \sum_{k=1}^{N} k^2 |\tilde{v}_k|^2 \|T_k\|^2_{\omega} = \sum_{k=1}^{N} k^2 \, | \sum_{j \in J_k} \hat{v}_j |^2 \, \|T_k\|^2_{\omega} \\ &\le \sum_{k=1}^{N} k^2 \left(\sum_{j \in J_k} \frac{1}{j^2} \right) \left(\sum_{j \in J_k} j^2 |\hat{v}_j|^2 \|T_j\|^2_{\omega} \right) \\ &\le C(k) \sum_{l=1}^{\infty} l^2 |\hat{v}_l|^2 \|T_l\|^2_{\omega} = C(k) \|\partial_x v\|^2_{\omega^{-1}}, \end{aligned} \tag{A.9}$$

where, since $2lN \pm k \ge lN \ge lk$ for $l \ge 1$ and $k \le N$,

$$C(k) \equiv \max_{1\le k\le N} \sum_{j\in J_k} \frac{k^2}{j^2} \le 1 + \sum_{l=1}^{\infty} \frac{1}{l^2} \le C,$$

which leads to the second conclusion (A.6). ∎

Now we are ready to derive the general result (3.4). Let $u = \partial_x v \in L^2_{\omega^{1-2\theta}}(I)$. By the notations used above, we know from Lemma A.2 that the conditions of Lemma A.1 hold in our case so that we have the conclusion

$$\|\partial_x I_N^C \partial_x^{-1} u\|_{\omega^{1-2\theta}} \le C\|u\|_{\omega^{1-2\theta}}, \qquad 0 \le \theta \le 1,$$

which complete the proof of Lemma 3.2 ∎

Proof of Lemma 3.4. The results with $(\sigma, \mu) = (1, 0)$ has been proved in [GO, p. 98] and [R] in different ways. Here we use the operator interpolation method to prove the generalized result. It seems that the following way is not so natural, but it keeps us from being involved in the operator interpolation theory too much. We define

$$T = \partial_x I_{N+1}^C \partial_x^{-1}, \qquad \partial_x^{-1} u \equiv \int_{-1}^{x} u(\xi)\, d\xi.$$

Then the result (3.9) with $(\sigma, \mu) = (1, -1)$, which can be proved easily by using the property of Gauss quadrature formula, and (A.6) lead to

$$\text{(A.10)} \qquad \|Tu\|_{\omega} \le CN\|Tu\|_{\omega^{-1}} \le CN\|u\|_{\omega^{-1}}, \qquad \forall u \in L^2_{\omega^{-1}}(I).$$

Thus, by Lemma A.1, we get from (A.5) and (A.10) that for $0 \le \theta \le 1$,

$$\text{(A.11)} \qquad \|Tu\|_{\omega} \le CN^{\theta}\|u\|_{\omega^{1-2\theta}}, \qquad \forall u \in L^2_{\omega^{1-2\theta}}(I).$$

Using Lemma A.1 again, we get from (A.11) and (A.6) that for $0 \le \delta \le 1$,

$$\text{(A.12)} \quad \|Tu\|_{\omega^{1-\delta}\omega^{-\delta}} \le CN^{\theta(1-\delta)}\|u\|_{\omega^{(1-2\theta)(1-\delta)}\omega^{-\delta}}, \quad \forall u \in L^2_{\omega^{(1-2\theta)(1-\delta)}\omega^{-\delta}}(I).$$

Now let $1-2\delta = \sigma$, $\theta(1-\delta) = (\sigma-\mu)/2$. Then $(1-2\theta)(1-\delta)-\delta = 1-2\delta-2\theta(1-\delta) = \sigma - (\sigma-\mu) = \mu$, and (A.12) reads

$$\text{(A.13)} \qquad \|Tu\|_{\omega^{\sigma}} \le CN^{(\sigma-\mu)/2}\|u\|_{\omega^{\mu}}, \qquad \forall u \in L^2_{\omega^{\mu}}(I).$$

The desired results follow from the fact that for any $u \in \mathcal{P}_N$, $Tu = u$. ∎

Appendix B. The proof of (3.19). We shall show that (3.19) is true, which is a special case of the following lemma with $(\alpha, \beta) = (0, -(1-\varepsilon)/2)$.

Let $\rho(x) = 1 - x^2$ and $P_N^{(\alpha)}$: $L^2_{\rho^\alpha}(I) \longrightarrow \mathcal{P}_N$ be $L^2_{\rho^\alpha}(I)$-orthogonal projection. To serve our purpose, we only consider the case of $\alpha \ge -1/2$. A similar result for $\alpha > -1$ is possible.

LEMMA B.1. *Assume that* $\alpha \ge -1/2$ *and* ε *is a small positive number. Then there exists a constant* C *independent of* N *and* f *such that*

$$\text{(B.1)} \qquad \|P_N^{(\alpha)} f\|_{\rho^\beta} \le \frac{C}{\varepsilon}\|f\|_{\rho^\beta}, \qquad |\beta - \alpha| = \frac{1}{2} - \varepsilon.$$

Proof. We will follow the line in [M1]. Let $J_k^{(\alpha)}(x)$ be the family of Jacobi polynomials orthogonal in the weight $\rho^\alpha(x)$ normalized as $J_k^{(\alpha)}(1) = \binom{k+\alpha}{k}$ and $K_N(x,y)$ be the Christoffel–Darboux kernel

$$K_N(x,y) = \sum_{k=0}^{N} \frac{J_k^{(\alpha)}(x) J_k^{(\alpha)}(y)}{\|J_k^{(\alpha)}\|_{\rho^\alpha}^2}. \tag{B.2}$$

As shown in [M1], we have

$$K_N(x,y) = a_N h_1(N,x,y) + b_N[h_2(N,x,y) + h_3(N,x,y)], \tag{B.3}$$

where $|a_N|, |b_N|$ are bounded above by a constant independent of N and

$$h_1(N,x,y) = (N+1) J_N^{(\alpha)}(x) J_N^{(\alpha)}(y), \tag{B.4}$$

$$h_2(N,x,y) = h_3(N,y,x) = \frac{N\rho(y) J_N^{(\alpha)}(x) J_{N-1}^{(\alpha+1)}(y)}{x-y}. \tag{B.5}$$

Also, we have [M1]

$$|J_N^{(\alpha)}(x)| \le C N^{-1/2} \rho^{-\alpha/2-1/4}(x), \quad |x| \le 1, \quad \alpha \ge -\frac{1}{2}. \tag{B.6}$$

It is easy to see that

$$\begin{aligned} \|P_N^{(\alpha)} f\|_{\rho^\beta}^2 &= \int_I \left| \int_I K_N(x,y) f(y) \rho^\alpha(y)\, dy \right|^2 \rho^\beta(x)\, dx \\ &\le C \sum_{j=1}^{3} \int_I \left| \int_I h_j(N,x,y) f(y) \rho^\alpha(y)\, dy \right|^2 \rho^\beta(x)\, dx. \end{aligned} \tag{B.7}$$

Due to the symmetry, we need only consider the following integrals:

$$I_j \equiv \int_I \left| \int_0^1 h_j(N,x,y) f(y) \rho^\alpha(y)\, dy \right|^2 \rho^\beta(x)\, dx, \qquad j = 1,2,3. \tag{B.8}$$

For I_1 we use (B.4), (B.6), and the Cauchy inequality to get

$$\begin{aligned} I_1 &\le C \int_I \left(\int_0^1 |f(y)| \rho^{\alpha/2-1/4}(y)\, dy \right)^2 \rho^{\beta-\alpha-1/2}(x)\, dx \\ &\le C \int_0^1 |f(y)|^2 \rho^\beta(y)\, dy \int_0^1 \rho^{\alpha-\beta-1/2}(y)\, dy \int_I \rho^{\beta-\alpha-1/2}(x)\, dx \\ &\le C \int_I \rho^{\varepsilon-1}(x)\, dx \|f\|_{\rho^\beta}^2 \le \frac{C}{\varepsilon} \|f\|_{\rho^\beta}^2. \end{aligned} \tag{B.9}$$

For I_2 we decompose it as

$$I_2 = \left(\int_{-1}^{-1/2} + \int_{-1/2}^{1} \right) \left| \int_0^1 h_2(N,x,y) f(y) \rho^\alpha(y)\, dy \right|^2 \rho^\beta(x)\, dx \equiv J_1 + J_2. \tag{B.10}$$

Since $|x-y| \geq 1/2$ in J_1, we have as in (B.9) that

$$J_1 \leq C \int_{-1}^{-1/2} \left(\int_0^1 |f(y)| \rho^{\alpha/2+1/4}(y)\, dy \right)^2 \rho^{\beta-\alpha-1/2}(x)\, dx$$
$$\leq C \int_0^1 |f(y)|^2 \rho^{\beta}(y)\, dy \int_0^1 \rho^{\alpha-\beta+1/2}(y)\, dy \int_{-1}^{-1/2} \rho^{\beta-\alpha-1/2}(x)\, dx \leq \frac{C}{\varepsilon} \|f\|^2_{\rho^\beta}.$$

For J_2 we let $f(y) = 0$ $(-\infty \leq y \leq 0)$ and make the variable transformations $x = 1 - X,\ y = 1 - Y$. Then we have

$$J_2 \leq C \int_0^{3/2} \left| \left(\int_0^{X/2} + \int_{X/2}^{3X/2} + \int_{3X/2}^{\infty} \right) \frac{f(y)}{X-Y} \sqrt{N} J_{N-1}^{(\alpha+1)}(y) \rho^{\alpha+1}(y)\, dY \right|^2 X^{\beta-\alpha-1/2}\, dX$$
$$\leq J_{21} + J_{22} + J_{23}.$$

For J_{21} use $|X - Y| \geq X/2$, (B.6), and the Hardy inequality [MPF, p. 145],

$$J_{21} \leq C \int_0^{3/2} \left(\int_0^{X/2} |f(1-Y)| Y^{\alpha/2+1/4}\, dY \right)^2 X^{\beta-\alpha-5/2}\, dX$$
$$\leq C \int_0^{\infty} \left(\int_0^{X} |f(1-\frac{Y}{2})| Y^{\alpha/2+1/4}\, dY \right)^2 X^{\beta-\alpha-5/2}\, dX$$
$$\leq C \int_0^{\infty} |f(1-\frac{X}{2})|^2 X^{\alpha+1/2} X^{\beta-\alpha-1/2}\, dX \leq C \|f\|^2_{\rho^\beta}.$$

For J_{23} use $|Y - X| \geq Y/3$, (B.6), and the Hardy inequality [MPF, p. 145],

$$J_{23} \leq C \int_0^{3/2} \left(\int_{3X/2}^{\infty} |f(1-Y)| Y^{\alpha/2-3/4}\, dY \right)^2 X^{\beta-\alpha-1/2}\, dX$$
$$\leq C \int_0^{\infty} \left(\int_X^{\infty} \left|f\left(1-\frac{3}{2}Y\right)\right| Y^{\alpha/2-3/4}\, dY \right)^2 X^{\beta-\alpha-1/2}\, dX$$
$$\leq \frac{C}{\varepsilon^2} \int_0^{\infty} \left|f\left(1-\frac{3}{2}X\right)\right|^2 X^{\alpha-3/2} X^{\beta-\alpha+3/2}\, dX \leq \frac{C}{\varepsilon^2} \|f\|^2_{\rho^\beta}.$$

For J_{22} let $X_{-2} = 9/4,\ X_{-1} = 3/2$, and $X_n = 1/2^n$ $(n \geq 0)$. Then

$$J_{22} \leq C \sum_{n=-1}^{\infty} \int_{X_{n+1}}^{X_n} \left| \int_{X/2}^{3X/2} \frac{f(1-Y)}{X-Y} \sqrt{N} J_{N-1}^{(\alpha+1)}(1-Y) \rho^{\alpha+1}(1-Y)\, dY \right|^2 X^{\beta-\alpha-1/2}\, dX.$$

According to the Hilbert inequality, we have that [M2] for $\sigma \leq 0$,

$$\int_{X_{n+1}}^{X_n} \left| \int_{X/2}^{3X/2} \frac{g(Y)}{X-Y}\, dY \right|^2 X^{\sigma}\, dX \leq X_{n+1}^{\sigma} \int_{X_{n+1}}^{X_n} \left| \int_{X/2}^{3X/2} \frac{g(Y)}{X-Y}\, dY \right|^2 dX$$
$$\leq C X_{n+1}^{\sigma} \int_{X_{n+2}}^{X_{n-1}} |g(X)|^2\, dX \leq C \frac{X_{n+1}^{\sigma}}{X_{n-1}^{\sigma}} \int_{X_{n+2}}^{X_{n-1}} |g(X)|^2 X^{\sigma}\, dX,$$

and (B.6) implies that

$$J_{22} \leq C \sum_{n=-1}^{\infty} \int_{X_{n+2}}^{X_{n-1}} |f(1-X)|^2 X^{\alpha+1/2} X^{\beta-\alpha-1/2}\, dX \leq C \|f\|^2_{\rho^\beta}.$$

The estimation for I_3 is the same. Thus the proof of Lemma B.1 is completed. ∎

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