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CHEBYSHEV–LEGENDRE SUPER SPECTRAL VISCOSITY METHOD FOR NONLINEAR CONSERVATION LAWS*

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Abstract. In this paper, a super spectral viscosity method using the Chebyshev differential operator of high order $D^s = (\sqrt{1-x^2}\partial_x)^s$ is developed for nonlinear conservation laws. The boundary conditions are treated by a penalty method. Compared with the second-order spectral viscosity method, the super one is much weaker while still guaranteeing the convergence of the bounded solution of the Chebyshev–Galerkin, Chebyshev collocation, or Legendre–Galerkin approximations to nonlinear conservation laws, which is proved by compensated compactness arguments.

Key words. conservation laws, Chebyshev–Legendre method, super spectral viscosity, convergence

AMS subject classifications. 35L65, 65M10, 65M15

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1. Introduction.

We have discussed in [M] the second-order spectral viscosity (SV) method using the Chebyshev differential operator $D = \sqrt{1-x^2}\partial_x$ for the following nonlinear conservation law:

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \qquad (x,t) \in (-1,1) \times (0,T), \tag{1.1}$$

provided with an initial condition at $t = 0$ and boundary data on the inflow boundaries. The aim of this paper is to generalize the second-order SV method [MT], [Ta1], [Ta2] to a super spectral viscosity (SSV) version introduced by David Gottlieb, which uses the high-order Chebyshev differential operator D^s. We refer to [Ta3] for the Fourier SSV method of the periodic problems, where a more general version of SSV has been established. There is a switch in the SSV of [Ta3] controlled by the parameter m_N so that the viscosity is put only on the modes higher than m_N. The SSV method considered here corresponds to the case of $m_N = 1$.

We will see that the SSV method can be viewed as a special case of the SV method, but the former is much weaker than the latter for large s. Although the SSV method does not meet the stability requirement in [M], it is shown in this paper that the SSV method still guarantees the convergence of the bounded solution of the Chebyshev–Galerkin, Chebyshev collocation, or Legendre–Galerkin approximations to nonlinear conservation laws, which is proved by compensated compactness arguments.

Since the viscosity in the SSV method is much weaker, we may expect that the computed coefficients of the SSV solution are less affected by the viscosity, and are more accurate to the exact coefficients of the solution of (1.1). Therefore, spectral accuracy may be recovered from these coefficients by a postprocessing procedure such as filter or reconstruction methods [GT], [AGT], [GSSV], [GS1], [GS2], [GS3], [MOT].

The SSV method has a close relation with the exponential filter spectral method. The latter has been successfully applied to nonlinear conservation laws with the

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Fourier spectral method [SW] and to shock wave calculations with the Chebyshev spectral method [D]. We will give some details on this relation at the end of the paper.

The paper is organized as follows. In section 2 we describe the SSV scheme for (1.1) and its connection with the SV method. In section 3 we discuss some properties of the Chebyshev and Legendre differential operators of high order. Sections 4 and 5 work on some a priori estimates related to the solution of the SSV method and then prove that the bounded solution of the SSV method converges to the exact scalar entropy solution of (1.1) by compensated compactness arguments based on the standard framework.

2. The SSV method. Let $I = (-1, 1)$ and $\rho(x)$ be a positive weight on I. The inner product and norm of $L^2_\rho(I)$ are denoted by $(\cdot,\cdot)_\rho$ and $\|\cdot\|_\rho$. We will drop the subscript ρ whenever $\rho \equiv 1$. Let $\mathcal{P}_N$ denote the space of algebraic polynomials of degree $\le N$ and $\omega(x) = (1 - x^2)^{-1/2}$. We denote by $\{T_k(x)\}$ the Chebyshev polynomials standardized as $T_k(1) = 1$ and $\{L_k(x)\}$ the Legendre ones as $L_k(1) = 1$. The spectral approximation operator P_N can be one of the following:

1. $P^C_N : L^2_\omega(I) \to \mathcal{P}_N$, the Chebyshev–Galerkin projection operator ($L^2_\omega(I)$-orthogonal);
2. $I^C_N : C(\bar{I}) \to \mathcal{P}_N$, the Chebyshev interpolation operator at Gauss–Lobatto–Chebyshev points $x_j = \cos\frac{j\pi}{N}$, $0 \le j \le N$;
3. $P^L_N : L^2(I) \to \mathcal{P}_N$, the Legendre–Galerkin projection operator ($L^2(I)$-orthogonal).

The SSV method for (1.1) is to find $u_N(t) \in \mathcal{P}_N$ such that

$$\partial_t u_N + \partial_x P_N f(u_N) = (-1)^{s+1}\varepsilon_N P^L_N(\omega D^{2s} u_N) - B(u_N). \tag{2.1}$$

Here the boundary term is defined by

$$B(u_N) = \sum_{j=0,N} b_j(t)\tau[u_N(x_j,t) - g_j(t)]R^{(j)}_N(x), \quad R^{(j)}_N(x) = \frac{1}{2}[L'_N(x_j x) + L'_{N+1}(x_j x)],$$

where $b_j(t) = 1$ on the inflow boundary prescribed with the data $g_j(t)$, and $b_j(t) = 0$ on the outflow boundary ($j = 0, N$). We have that

$$(B(u_N), \varphi) = \sum_{j=0,N} b_j(t)\tau[u_N(x_j,t) - g_j(t)]\varphi(x_j) \qquad \forall \varphi \in \mathcal{P}_N.$$

This is a penalty-type treatment of boundary conditions [FG1], [FG2]. The parameters ε_N and τ are chosen such that

$$\varepsilon_N = N^{-(2s-1-\theta)}, \qquad \tau = N^\delta, \qquad 0 < \delta < \frac{4\theta}{2s-1}, \quad \theta < \frac{1}{2}. \tag{2.2}$$

The value of s may go to infinity, which will be described later.

To make a comparison with the second-order SV method, we rewrite the SSV term as

$$(-1)^{s+1}\varepsilon_N D^{2s} u_N = -\varepsilon_N \sum_{k=0}^{N} k^{2s}\hat{u}_k T_k = -\frac{1}{N^{1-\theta}}\sum_{k=0}^{N}\left(\frac{k}{N}\right)^{2s-2} k^2 \hat{u}_k T_k, \tag{2.3}$$

where we have used the fact that

$$D^2 T_k(x) + k^2 T_k(x) = 0 \qquad \forall x \in I. \tag{2.4}$$

So the SSV method can be viewed as a special case of the second-order SV method with $\hat{Q}_k = \hat{Q}_k^{(s)} = (\frac{k}{N})^{2s-2}$, and the implementation for the SSV method is almost the same as for the SV method, which can be done on the Chebyshev points $\{x_j\}$ efficiently [DG], [M]. However, in the usual SV method [M], we required that $\hat{Q}_k = 1$ for $k \geq N^{1/3}$ (we have ignored the small improvement in the second-order SV method since this kind of modification is also possible for the SSV [Ta3]). Obviously, the former is much weaker than the latter when the value of s is large.

3. Preliminaries. In this section, we discuss some properties of the Chebyshev and Legendre differential operators of high order, which are needed in the stability analysis. We first introduce a Sobolev-type space related to the Chebyshev differential operator D. Let

$$u = \sum_{k=0}^{\infty} \hat{u}_k T_k, \qquad \hat{u}_k = \frac{(u, T_k)_\omega}{\|T_k\|_\omega^2}. \tag{3.1}$$

By the property (2.4) we have formally that

$$\|D^\sigma u\|_\omega = \left(\frac{\pi}{2} \sum_{k=1}^{\infty} k^{2\sigma} |\hat{u}_k|^2 \right)^{1/2}, \qquad \sigma > 0. \tag{3.2}$$

We then define the Sobolev-type norms

$$\|u\|_{\sigma,D} = \left(\pi|\hat{u}_0|^2 + \frac{\pi}{2} \sum_{k=1}^{\infty} k^{2\sigma} |\hat{u}_k|^2 \right)^{1/2}, \qquad \sigma \in R, \tag{3.3}$$

and we denote by $H_D^\sigma(I)$ the closure of the space of all polynomials with respect to this norm. Accordingly, we should generalize the operator D to a distributional one in the usual way such that

$$(D^\sigma u, \varphi)_\omega = (-1)^\sigma (u, D^\sigma \varphi)_\omega \qquad \forall \varphi \in \mathcal{D}(I),$$

where $\mathcal{D}(I)$ is the space of infinitely differentiable functions with compact support in I.

It is easy to see that (3.2) is true for $u \in H_D^\sigma(I)$. For positive σ, we also have

$$D^{2\sigma} u = (-1)^\sigma \sum_{k=1}^{\infty} k^{2\sigma} \hat{u}_k T_k \qquad \forall u \in H_D^{2\sigma}(I). \tag{3.4}$$

If σ is negative, we define $D^{2\sigma} u$ by (3.4) and $D^{2\sigma+1} u \equiv D D^{2\sigma} u$. Thus (3.2) is also true for $u \in H_D^\sigma(I), \sigma < 0$. We have that, for $\sigma > 0$,

$$D^\sigma D^{-\sigma} u = u - \mu_\omega(u), \qquad \mu_\omega(u) \equiv \frac{1}{\pi} \int_I u(x)\omega(x)\, dx. \tag{3.5}$$

We will need the adjoint operator to D^σ for negative σ. If σ is even, we define $(D^\sigma)^* \equiv D^\sigma$, and if σ is odd, we define $(D^\sigma)^* \equiv D^{\sigma+1}(D^{-1})^*$ with

$$(D^{-1})^* u \equiv -\int_{-1}^{x} u(\xi)\omega(\xi)\, d\xi.$$

It is easy to check that

$$(D^\sigma u, v)_\omega = (u, (D^\sigma)^* v)_\omega. \tag{3.6}$$

We note that the behavior of u is quite different from Du. If $u \in H^{\sigma}_{D}(I)$, then $D^2u \in H^{\sigma-2}_{D}(I)$, but usually $Du \notin H^{\sigma-1}_{D}(I)$. If we connect them with the Fourier series under the transformation $x = \cos\theta$, we may say that u behaves *evenly* and Du behaves *oddly*. For example, let $u = x = \cos\theta$; then $Du = \sqrt{1-x^2} = \sin\theta$. We know $\sin\theta$ cannot be approximated well by Fourier cosine series.

Let $D^2_L = \partial_x\omega^{-2}\partial_x$. The following lemma is essential for getting a priori estimates and will be proved in the Appendix.

LEMMA 3.1. *If $u \in H^{s+2}_{D}(I)$ $(s \geq 0)$, then we have*

$$\|D^{s+2}u\|^2_\omega + \|D^s(x\partial_x u)\|^2_\omega + (2s+1)\|D^s\partial_x u\|^2_\omega \leq \|D^sD^2_L u\|^2_\omega, \tag{3.7}$$

$$\|D^sD^2_L u\|^2_\omega + 3\|D^{s+1}u\|^2_\omega \leq 9\|D^{s+2}u\|^2_\omega, \tag{3.8}$$

$$(D^{s+2}u, D^sD^2_L u)_\omega \geq \|D^{s+2}u\|^2_\omega + \frac{2s+1}{2}\|D^s\partial_x u\|^2_\omega, \tag{3.9}$$

$$(D^{s+2}u, D^sD^2_L u)_\omega \geq \frac{1}{3}\|D^sD^2_L u\|^2_\omega + \frac{1}{2}\|D^{s+1}u\|^2_\omega. \tag{3.10}$$

We will also use the following lemma, whose proof can be found in [M].

LEMMA 3.2. *If $u \in H^{\sigma}_{D}(I)$ $(\sigma \geq 0)$, then*

$$\|D^\mu(P^C_N u - u)\|_\omega \leq CN^{\mu-\sigma}\|D^\sigma u\|_\omega, \qquad 0 \leq \mu \leq \sigma, \tag{3.11}$$

$$\|D^\mu(I^C_N u - u)\|_\omega \leq CN^{\mu-\sigma}\|D^\sigma u\|_\omega, \qquad 0 \leq \mu \leq \sigma, \quad \sigma > \frac{1}{2}, \tag{3.12}$$

$$\|D^\mu(P^L_N u - u)\|_\omega \leq CN^{\mu-\sigma}\ln N\|D^\sigma u\|_\omega, \qquad 0 \leq \mu \leq \sigma. \tag{3.13}$$

LEMMA 3.3. *Assume that $f \in C^s(R)$ and $u \in H^s_D(I)$ $(s \geq 1)$. Let*

$$M = \max_I |u(x)|, \qquad |f|_{r,\infty} = \max_{|\xi|\leq M} |f^{(r)}(\xi)|.$$

We have that

$$\|D^s f(u)\|_\omega \leq C_s \left(\sum_{r=1}^{s} |f|_{r,\infty} M^{r-1}\right) \|D^s u\|_\omega. \tag{3.14}$$

Proof. We refer to [CDT] for the result with $\omega \equiv 1$. Let α_j $(j \geq 1)$ be positive integers and $|\alpha|_0 = 0, |\alpha|_r = \sum_{j=1}^{r}\alpha_j$. According to the chain rule we have

$$\begin{aligned} D^s f(u) = D^{s-1}[f'(u)Du] &= \sum_{l=0}^{s-1} C^l_{s-1} D^{s-1-l} f'(u)\, D^{l+1}u \\ &= \sum_{\alpha_1=1}^{s} C^{\alpha_1-1}_{s-1} D^{s-\alpha_1} f'(u)\, D^{\alpha_1}u \\ &= \sum_{\alpha_1=1}^{s} C^{\alpha_1-1}_{s-1} \sum_{\alpha_2=1}^{s-\alpha_1} C^{\alpha_2-1}_{s-\alpha_1-1} D^{s-\alpha_1-\alpha_2} f^{(2)}(u)\, D^{\alpha_1}u\, D^{\alpha_2}u \\ &= \cdots = \sum_{r=1}^{s}\sum_{|\alpha|_r=s}\left(\prod_{j=1}^{r} C^{\alpha_j-1}_{s-|\alpha|_{j-1}-1}\right) f^{(r)}(u) \prod_{j=1}^{r} D^{\alpha_j}u. \end{aligned} \tag{3.15}$$

By the Hölder inequality,

$$\|\prod_{j=1}^{r} D^{\alpha_j}u\|_\omega \leq \prod_{j=1}^{r} \|D^{\alpha_j}u\|_{L^{2s/\alpha_j}_\omega}. \tag{3.16}$$

Under the transformation $x = \cos\theta$, the Gagliardo–Nirenberg (GN) inequality [KZ] gives us

$$\|D^l u\|_{L^q_\omega} \le C\|u\|_{L^\infty}^{1-\lambda}\|D^s u\|_{L^p_\omega}^{\lambda}, \qquad \lambda = \frac{l - q^{-1}}{s - p^{-1}}. \tag{3.17}$$

Let $l = \alpha_j, q = 2s/\alpha_j$, and $p = 2$ in (3.17). Then $\lambda = \alpha_j/s$ and we have from (3.15) and (3.16) that

$$\begin{aligned} \|D^s f(u)\|_\omega &\le C_s \sum_{r=1}^{s} |f|_{r,\infty} \prod_{j=1}^{r} \|u\|_{L^\infty}^{1-\alpha_j/s}\|D^s u\|_\omega^{\alpha_j/s} \\ &= C_s \sum_{r=1}^{s} |f|_{r,\infty} M^{r-1}\|D^s u\|_\omega. \qquad \square \end{aligned} \tag{3.18}$$

We note that C_s usually grows exponentially with s. In fact, if $f(u) = u^2$ and $u = T_1$, then $C_s \sim 2^s$. We should keep this in mind when we choose s.

4. A priori estimates. This section is devoted to some a priori estimates of the approximation solution of (2.1), which will be needed in the proof of convergence. We assume that the approximation solution is uniformly bounded.

ASSUMPTION (L^∞-boundedness). *There exists a finite constant M such that*

$$\|u_N\|_{L^\infty(\bar{I}\times[0,T])} \le M. \tag{4.1}$$

We will denote by C_M the constant dependent only on the bound M and the flux function f. To simplify the presentation, we only consider the case where $x = -1$ is an inflow boundary prescribed with the data $g(t) \in H^1(0,T)$, while $x = 1$ is an outflow one. The boundary term $B(u_N)$ is now of the form

$$B(u_N) = \tau e(t) R_N^0(x), \qquad e(t) \equiv u_N(-1,t) - g(t).$$

We begin with an $L^2(I)$-estimate. Let $\|\cdot\|_{\omega;0} \equiv \|\cdot\|_{L^2(0,T;L^2_\omega(I))}$. We first quote the Sobolev inequality and the GN inequality (3.17) with $p = q = 2$, which will be used frequently.

$$\|u\|_{L^\infty} \le C\|u\|_\omega^{1/2}\|u\|_{1,D}^{1/2} \qquad \forall u \in H^1_D(I), \tag{4.2}$$

$$\|D^l u\|_\omega \le C\|u\|_{L^\infty}^{\frac{2(s-l)}{2s-1}}\|D^s u\|_\omega^{\frac{2l-1}{2s-1}}, \qquad 1 \le l \le s \quad \forall u \in H^s_D(I). \tag{4.3}$$

LEMMA 4.1. *Assume that* (2.2) *holds and*

$$\varepsilon_N \ge \ 4C_M \frac{C_s \ln^2 N}{N^{2s-1}}, \tag{4.4}$$

where C_s is the constant appearing in (3.14). *We have that*

$$\|u_N(T)\|^2 + \varepsilon_N\|D^s u_N\|_{\omega;0}^2 + \tau\|e\|_{L^2}^2 \le C_M(1 + \|g\|_{H^1}^2). \tag{4.5}$$

Proof. Let $F(u) = \int^u \xi f'(\xi)\, d\xi$. We get from the scheme (2.1) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + F(u_N)|_{-1}^{+1} + \varepsilon_N\|D^s u_N\|_\omega^2 + \tau e^2(t) + \tau e(t)g(t) \\ &\quad = (\partial_x(I - P_N)f(u_N), u_N) \equiv I(P_N). \end{aligned} \tag{4.6}$$

We estimate $I(P_N)$ in different cases as follows.

1. By the Sobolev inequality (4.2), (3.11), the GN inequality (4.3), (3.14), and the Hölder inequality,

$$
\begin{aligned}
(4.7)\qquad |I(P_N^C)| &= |\ [(I-P_N^C)f(u_N)u_N]|_{-1}^{+1} - ((I-P_N^C)f(u_N), \partial_x u_N)\ | \\
&\le C_M \|(I-P_N^C)f(u_N)\|_{L^\infty(I)} + ((I-P_N^C)f(u_N), Du_N)_\omega \\
&\le C_M \|(I-P_N^C)f(u_N)\|_\omega^{1/2} \|D(I-P_N^C)f(u_N)\|_\omega^{1/2} \\
&\quad + \frac{C_M}{N^s}\|D^s f(u_N)\|_\omega \|D^s u_N\|_\omega^{\frac{1}{2s-1}} \\
&\le \frac{C_M}{N^{s-1/2}}\|D^s f(u_N)\|_\omega + C_M \frac{C_s}{N^s}\|D^s u_N\|_\omega^{\frac{2s}{2s-1}} \\
&\le C_M\left(1 + \frac{C_s}{N^{2s-1}}\|D^s u_N\|_\omega^2\right).
\end{aligned}
$$

2. By (3.12), the GN inequality (4.3), and the Hölder inequality,

$$
\begin{aligned}
(4.8)\qquad |I(I_N^C)| &= |((I-I_N^C)f(u_N), \partial_x u_N)| \le \|(I-I_N^C)f(u_N)\|_\omega \|Du_N\|_\omega \\
&\le \frac{C}{N^s}\|D^s f(u_N)\|_\omega \|Du_N\|_\omega \le C_M \frac{C_s}{N^s}\|D^s u_N\|_\omega^{\frac{2s}{2s-1}} \\
&\le C_M\left(1 + \frac{C_s}{N^{2s-1}}\|D^s u_N\|_\omega^2\right).
\end{aligned}
$$

3. By the Sobolev inequality (4.2) and (3.13),

$$
\begin{aligned}
(4.9)\qquad |I(P_N^L)| &= |\ [(I-P_N^L)f(u_N)u_N]|_{-1}^{+1}\ | \\
&\le C_M \|(I-P_N^L)f(u_N)\|_\omega^{1/2} \|D(I-P_N^L)f(u_N)\|_\omega^{1/2} \\
&\le C_M \frac{\ln N}{N^{s-1/2}}\|D^s f(u_N)\|_\omega \le C_M\left(1 + \frac{C_s \ln^2 N}{N^{2s-1}}\|D^s u_N\|_\omega^2\right).
\end{aligned}
$$

Thus we obtain from (4.6) that

$$
\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \left(\varepsilon_N - C_M \frac{C_s \ln^2 N}{N^{2s-1}}\right)\|D^s u_N\|_\omega^2 + \tau e^2(t) \le C_M + |F|_\infty - \tau e(t) g(t),
$$

where $|F|_\infty \equiv \max_{|\xi|\le M}|F(\xi)|$. To bound $\tau e(t)g(t)$, we use the scheme (2.1) to get

$$
(4.10)\qquad \frac{d}{dt}(u_N, 1) + f(u_N)|_{-1}^{+1} + \tau e(t) = [(I-P_N)f(u_N)]|_{-1}^{+1} \equiv J(P_N),
$$

where $J(I_N^C) = 0$, and we can see from (4.7) and (4.9) that

$$
(4.11)\qquad |J(P_N^C)| \le C_M \frac{C_s}{N^{s-1/2}}\|D^s u_N\|_\omega,
$$

$$
(4.12)\qquad |J(P_N^L)| \le C_M \frac{C_s \ln N}{N^{s-1/2}}\|D^s u_N\|_\omega.
$$

Thus, we have from (4.10) that for $E(T) \equiv \tau\int_0^T e(t)\,dt$,

$$
(4.13)\qquad |E(T)| \le C_M + C_M \frac{C_s \ln N}{N^{s-1/2}}\|D^s u_N\|_{\omega;0}
$$

and

$$
\begin{aligned}
(4.14)\qquad \left|\int_0^T \tau e(t) g(t)\,dt\right| &= \left|E(t)g(t)|_0^T - \int_0^T E(t)\frac{d}{dt}g(t)\,dt\right| \\
&\le C_M\left(\|g\|_{H^1}^2 + \frac{C_s \ln^2 N}{N^{2s-1}}\|D^s u_N\|_{\omega;0}^2\right).
\end{aligned}
$$

The proof is completed by temporal integration of (4.6) and the use of (4.14). ∎

Next, we work on an $H^1(I)$-estimate.

LEMMA 4.2. *Assume that* (2.2), (4.4) *hold and*

$$\|Du_N(0)\|^2 \leq C\sqrt{\tau}N^{2(1-\frac{\theta}{2s-1})}. \tag{4.15}$$

We have that

$$\|Du_N(T)\|^2 + \varepsilon_N\|D^{s+1}u_N\|^2_{\omega;0} \leq C_M(1+\|g\|^2_{H^1})\sqrt{\tau}N^{2(1-\frac{\theta}{2s-1})}. \tag{4.16}$$

Proof. Let $D_L^2 = \partial_x\omega^{-2}\partial_x$. We have from the scheme (2.1) that

$$(\partial_t u_N + \partial_x P_N f(u_N), D_L^2 u_N) = (-1)^{s+1}\varepsilon_N(D^{2s}u_N, D_L^2 u_N)_\omega - (B(u_N), D_L^2 u_N).$$

This gives us

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Du_N\|^2 + \varepsilon_N(D^{s+1}u_N, D^{s-1}D_L^2 u_N)_\omega \\ &= (DP_N f(u_N), D_L^2 u_N)_\omega + \tau e(t)(D_L^2 u_N)(-1,t).\end{aligned}$$

Thus, by the coercive property (3.9), the Sobolev inequality (4.2), and (3.8)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Du_N\|^2 + \varepsilon_N\|D^{s+1}u_N\|^2_\omega \\ &\leq \|Df(u_N)\|_\omega\|D_L^2 u_N\|_\omega + C\tau|e|\ \|D_L^2 u_N\|^{1/2}_\omega\|DD_L^2 u_N\|^{1/2}_\omega \\ &\leq C_M\|Du_N\|_\omega\|D^2 u_N\|_\omega + C\tau|e|\ \|D^2 u_N\|^{1/2}_\omega\|D^3 u_N\|^{1/2}_\omega.\end{aligned} \tag{4.17}$$

The temporal integration of (4.17) and the use of the Hölder inequality and the GN inequality (4.3) yield

$$\begin{aligned}&\|Du_N(T)\|^2 + \varepsilon_N\|D^{s+1}u_N\|^2_{\omega;0} \\ &\leq \|Du_N(0)\|^2 + C_M\|Du_N\|_{\omega;0}\|D^2u_N\|_{\omega;0} + C\tau\|e\|_{L^2}\|D^2u_N\|^{1/2}_{\omega;0}\|D^3u_N\|^{1/2}_{\omega;0} \\ &\leq \|Du_N(0)\|^2 + C_M\|D^s u_N\|^{\frac{1}{2s-1}}_{\omega;0}\|D^s u_N\|^{\frac{3}{2s-1}}_{\omega;0} + C\tau\|e\|_{L^2}\|D^s u_N\|^{\frac{3}{4s-2}}_{\omega;0}\|D^s u_N\|^{\frac{5}{4s-2}}_{\omega;0} \\ &\leq \|Du_N(0)\|^2 + C_M\sqrt{\tau}(\sqrt{\tau}\|e\|_{L^2})\varepsilon_N^{-\frac{2}{2s-1}}(\varepsilon_N\|D^s u_N\|^2_\omega)^{\frac{2}{2s-1}},\end{aligned}$$

which completes the proof. ∎

5. The convergence of the SSV method. In this section, we prove the convergence of the SSV approximation (2.1) by compensated compactness arguments. Based on the framework of [Tr], we follow [Ta3] to prove that both $\partial_t u_N + \partial_x f(u_N)$ and $\partial_t U(u_N) + \partial_x F(u_N)$ for the quadratic entropy function flux pair can be expressed as a sum of two terms such that one belongs to a compact subset of $H^{-1}_{loc}(\Omega)$ and the other is bounded in $L^1_{loc}(\Omega)$, where $\Omega = (-1,1)\times(0,T)$. We will simplify $C_M(1+\|g\|^2_{H^1})$ as C_M, and also use the following notations:

$$(\cdot,\cdot) \equiv (\cdot,\cdot)_{L^2(\Omega)}, \qquad \|\cdot\| \equiv \|\cdot\|_{L^2(\Omega)}, \qquad \|\cdot\|_\infty \equiv \|\cdot\|_{L^\infty(\Omega)},$$

$$(\cdot,\cdot)_\omega \equiv (\cdot,\cdot)_{L^2(0,T;L^2_\omega(I))}, \qquad \|\cdot\|_\omega \equiv \|\cdot\|_{L^2(0,T;L^2_\omega(I))}.$$

For any $\varphi \in H^1_0(\Omega)$, we have

$$\begin{aligned}&(\partial_t u_N + \partial_x f(u_N), \varphi) \\ &= (-1)^{s+1}\varepsilon_N(D^{2s}u_N, P_N^L\varphi)_\omega + (\partial_x(I-P_N)f(u_N),\varphi) - (B(u_N),\varphi) \equiv \sum_{j=1}^{3} I_j(\varphi).\end{aligned} \tag{5.1}$$

By Lemma 4.1, (3.13), and the following inverse property:

$$\|D^\sigma u\|_\omega \le CN^{\sigma-\mu}\|D^\mu u\|_\omega, \qquad 0\le \mu\le\sigma \quad \forall u\in \mathcal{P}_N, \tag{5.2}$$

the first term can be bounded as

$$\begin{aligned}|I_1(\varphi)| &= \varepsilon_N|(D^{2s}u_N,\varphi)_\omega + (D^{2s}u_N,(P_N^L-I)\varphi)_\omega| \\ &\le C\varepsilon_N(\|D^{2s-1}u_N\|_\omega\|D\varphi\|_\omega + \|D^{2s}u_N\|_\omega\|(P_N^L-I)\varphi\|_\omega) \\ &\le C\varepsilon_N N^{s-1}(1+\ln N)\|D^s u_N\|_\omega\|D\varphi\|_\omega \\ &\le C_M N^{-(1-\theta)/2}\ln N\|\partial_x\varphi\| \to 0.\end{aligned} \tag{5.3}$$

We use Lemma 3.2 to estimate the second term:

$$\begin{aligned}|I_2(\varphi)| &= |((I-P_N)f(u_N),\partial_x\varphi)| \le \|(I-P_N)f(u_N)\|\;\|\partial_x\varphi\| \\ &\le C_M\frac{C_s}{N^s}\|D^s u_N\|_\omega\|D\varphi\|_\omega \le C_M\frac{C_s}{N^{(1+\theta)/2}}\|\partial_x\varphi\|_\omega \to 0.\end{aligned} \tag{5.4}$$

For the third term we use (4.5), the inverse property

$$\|u\|_{L^\infty(I)} \le C\sqrt{N}\|u\|_{L^2_\omega(I)} \qquad \forall u\in\mathcal{P}_N, \tag{5.5}$$

and Lemma 3.2 to obtain

$$\begin{aligned}|I_3(\varphi)| &= \left|\int_0^T \tau e(t)[(P_N^L-I_N^C)\varphi](-1,t)\,dt\right| \\ &\le C\tau\|e\|_{L^2}\cdot\sqrt{N}\|(P_N^L-I_N^C)\varphi\|_\omega \le C_M\frac{\sqrt{\tau}\ln N}{\sqrt{N}}\|\partial_x\varphi\| \to 0.\end{aligned} \tag{5.6}$$

Thus, we have shown that $\partial_t u_N + \partial_x f(u_N)$ belongs to a compact subset of $H^{-1}_{loc}(\Omega)$.

Next we consider the quadratic entropy function. We have

$$\begin{aligned}&\frac{1}{2}\partial_t u_N^2 + \partial_x\int^{u_N}\xi f'(\xi)\,d\xi \\ &= (-1)^{s+1}\varepsilon_N u_N P_N^L(\omega D^{2s}u_N) + u_N\partial_x(I-P_N)f(u_N) - u_N B(u_N) \equiv \sum_{j=1}^3 I_j.\end{aligned} \tag{5.7}$$

We want to rewrite the right-hand side of (5.7) explicitly as two parts such that one belongs to a compact subset of $H^{-1}_{loc}(\Omega)$ and the other is bounded in $L^1_{loc}(\Omega)$.

For $s\ge 0$, as in (4.6a) of [Ta3], we have the following identity:

$$u\,D^{2s}v = (-1)^s D^s u\,D^s v + \sum_{\substack{l+m=2s-1\\ 0\le l<s}}(-1)^l D(D^l u\,D^m v). \tag{5.8}$$

Let $P_N^* u = \omega^{-1}P_N^L(\omega u)$ so that

$$(P_N^* u, v)_\omega = (P_N^L[\omega u], v) = (u, P_N^L v)_\omega \qquad \forall u\in L^2_\omega(I), \quad v\in L^2(I). \tag{5.9}$$

Using (5.8) and (3.5), noting $\mu_\omega((P_N^*-I)D^{2s}u_N)=0$, we have

$$\begin{aligned}I_1 &= (-1)^{s+1}\varepsilon_N\omega\,[u_N D^{2s}u_N + u_N D^{2s}D^{-2s}(P_N^*-I)D^{2s}u_N] \\ &= -\varepsilon_N\omega\,[(D^s u_N)^2 + D^s u_N D^{-s}(P_N^*-I)D^{2s}u_N] \\ &\quad + \varepsilon_N\omega\sum_{\substack{l+m=2s-1\\ 0\le l<s}}(-1)^{s+l+1}D[D^l u_N\,D^m u_N + D^l u_N\,D^{m-2s}(P_N^*-I)D^{2s}u_N] \\ &\equiv I_{11}+I_{12}.\end{aligned} \tag{5.10}$$

For any $\varphi \in H_0^1(\Omega)$, we get from (4.5), the inverse property (5.2), (3.13), and (4.16) that

$$
\begin{aligned}
(5.11) \qquad |(I_{11},\varphi)| &\le \varepsilon_N \,|(D^s u_N, D^s u_N\varphi)_\omega + (D^{2s}u_N, (P_N^L - I)(D^{-s})^*[D^s u_N\varphi])_\omega| \\
&\le \varepsilon_N \left(\|D^s u_N\|_\omega^2 \|\varphi\|_\infty + \frac{C\ln N}{N}\|D^{s+1}u_N\|_\omega \|D^s u_N\varphi\|_\omega \right) \\
&\le C_M \left(1 + \frac{\tau^{1/4}\ln N}{N^{\theta/(2s-1)}}\right)\|\varphi\|_\infty.
\end{aligned}
$$

By the same argument, but also using the inverse property

$$
(5.12) \qquad \|D^\sigma u\|_\infty \le CN^\sigma \|u\|_\infty, \qquad \sigma \ge 0 \quad \forall u \in \mathcal{P}_N,
$$

we have that

$$
\begin{aligned}
&|(I_{12},\varphi)| \\
&\le \varepsilon_N \sum_{\substack{l+m=2s-1\\0\le l<s}} |(D^m u_N, D^l u_N D\varphi)_\omega + (D^{2s}u_N, (P_N^L - I)(D^{m-2s})^*[D^l u_N D\varphi])_\omega| \\
&\le C\varepsilon_N \sum_{\substack{l+m=2s-1\\0\le l<s}} (1+\ln N)N^{m-s+l}\,\|D^s u_N\|_\omega \|u_N\|_\infty \|D\varphi\|_\omega \\
&\le C_M s\sqrt{\varepsilon_N}N^{s-1}\ln N\|D\varphi\|_\omega \le C_M \frac{s\ln N}{N^{(1-\theta)/2}}\|\partial_x\varphi\| \to 0.
\end{aligned}
$$

For the second term we rewrite it as

$$
(5.13) \qquad I_2 = \partial_x[u_N(I-P_N)f(u_N)] - \partial_x u_N(I-P_N)f(u_N) \equiv I_{21} + I_{22}.
$$

From Lemma 3.2 and (3.14), it is easy to see that

$$
\begin{aligned}
|(I_{21},\varphi)| &\le |((I-P_N)f(u_N), u_N\partial_x\varphi)_\omega| \le \|(I-P_N)f(u_N)\|\ \|u_N\|_\infty \|\partial_x\varphi\| \\
&\le C_M \frac{C_s}{N^s}\|D^s u_N\|_\omega \|\partial_x\varphi\| \le C_M \frac{C_s}{N^{(1+\theta)/2}}\|\partial_x\varphi\| \to 0.
\end{aligned}
$$

For I_{22} we use the GN inequality (4.3) in addition and get

$$
\begin{aligned}
|(I_{22},\varphi)| &\le |((I-P_N)f(u_N), Du_N\varphi)_\omega| \le \frac{C_s\ln N}{N^s}\|D^s u_N\|_\omega \|Du_N\|_\omega \|\varphi\|_\infty \\
&\le C_M C_s \ln N\ (N\varepsilon_N^{\frac{1}{2s-1}})^{-s}(\varepsilon_N\|D^s u_N\|_\omega^2)^{\frac{s}{2s-1}}\|\varphi\|_\infty \\
&\le C_M C_s N^{-\frac{s\theta}{2s-1}}\ln N\|\varphi\|_\infty \to 0.
\end{aligned}
$$

To deal with the third term, we first introduce an adjoint operator of I_N^C. Let $u \in L^2(I)$. We define $I_N^* u \in H^{-1}(I)$ such that

$$
(5.14) \qquad (I_N^* u, v)_{L^2(I)} = (u, I_N^C v)_{L^2(I)} \qquad \forall v \in H^1(I).
$$

It is not difficult to give an explicit expression of $I_N^* u$, but we do not need it. We note that

$$
\begin{aligned}
(5.15) \qquad \mu_\omega(\omega^{-1}(I_N^* - I)B(u_N)) &= \frac{1}{\pi}((I_N^* - I)B(u_N), 1)_{L^2(I)} \\
&= \frac{1}{\pi}(B(u_N), (I_N^C - I)1)_{L^2(I)} = 0;
\end{aligned}
$$

then by (5.8), I_3 can be expressed as

$$\begin{aligned}I_3 &= \omega u_N\, D^{2s} D^{-2s}[\omega^{-1}(I_N^* - I)B(u_N)] - u_N I_N^* B(u_N)\\ &= (-1)^s \omega\, D^s u_N\, D^{-s}[\omega^{-1}(I_N^* - I)B(u_N)]\\ &\quad + \omega \sum_{\substack{l+m=2s-1\\ 0\le l<s}} (-1)^l D\{D^l u_N D^{m-2s}[\omega^{-1}(I_N^* - I)B(u_N)]\} - u_N I_N^* B(u_N)\\ &\equiv I_{31} + I_{32} + I_{33}.\end{aligned}$$

For any $\varphi \in H_0^1(\Omega)$, as in (5.6), we have that

$$\begin{aligned}|(I_{31},\varphi)| &\le |(B(u_N), (I_N^C - I)(D^{-s})^*[D^s u_N \varphi])|\\ &\le C\tau \|e\|_{L^2} \sqrt{N} \ln N\, \|(I_N^C - I)(D^{-s})^*[D^s u_N\varphi]\|_\omega\\ &\le C_M \frac{\sqrt{\tau}\ln N}{N^{s-1/2}} \|D^s u_N\|_\omega \|\varphi\|_\infty \le C_M \frac{\sqrt{\tau}\ln N}{N^{\theta/2}} \|\varphi\|_\infty \to 0.\end{aligned} \tag{5.16}$$

Similarly, but also using (5.12), we get

$$\begin{aligned}|(I_{32},\varphi)| &\le \sum_{\substack{l+m=2s-1\\ 0\le l<s}} |(B(u_N), (I_N^C - I)(D^{m-2s})^*[D^l u_N D\varphi])|\\ &\le C\tau\|e\|_{L^2}\sqrt{N}\ln N \sum_{\substack{l+m=2s-1\\ 0\le l<s}} \|(I_N^C - I)(D^{m-2s})^*[D^l u_N D\varphi]\|_\omega\\ &\le C_M\sqrt{\tau}\,\sqrt{N}\ln N \sum_{\substack{l+m=2s-1\\ 0\le l<s}} N^{m-2s+l}\, \|u_N\|_\infty \|D\varphi\|_\omega\\ &\le C_M \frac{s\sqrt{\tau}\ln N}{\sqrt{N}} \|\partial_x \varphi\| \to 0.\end{aligned}$$

Finally, I_{33} is just a null functional on $H_0^1(\Omega)$ such that

$$(I_{33},\varphi) = -(B(u_N), I_N^C[u_N\varphi]) = 0. \tag{5.17}$$

Thus we have shown that, by Murat's lemma, the right-hand side of (5.7) belongs to a compact subset of $H_{loc}^{-1}(\Omega)$. Furthermore, we can see that it tends weakly to a negative measure. In fact, for any $\varphi \in H_0^1(\Omega)$ and $\varphi \ge 0$, we have as in (5.11) that

$$\begin{aligned}(I_{11},\varphi) &\le -\varepsilon_N(D^s u_N, D^s u_N\varphi)_\omega + \varepsilon_N |(D^{2s}u_N, (P_N^L - I)(D^{-s})^*[D^s u_N\varphi])_\omega|\\ &< C_M \frac{\tau^{1/4}\ln N}{N^{\theta/(2s-1)}} \|\varphi\|_\infty \to 0.\end{aligned}$$

Thanks to the div-curl lemma [Tr], we arrive at the following convergence result.

THEOREM 5.1. *Assume that the conditions of Lemmas* 4.1 *and* 4.2 *hold. Then the bounded solution* $u_N(x,t)$ *of the SSV scheme* (2.1) *converges strongly in* $L^p_{loc}(\Omega)$ $(P < \infty)$ *to the unique entropy solution of* (1.1).

Remark 1. *Exponential filter* [GS4]. We give a brief description of how the SSV method is related to the currently used exponential filter spectral method. To concentrate on the basic idea, we temporarily drop the factor ω in the SSV term. Thus the SSV scheme (2.1) reads

$$\partial_t u_N + \partial_x P_N f(u_N) = (-1)^{s+1}\varepsilon_N D^{2s} u_N - B(u_N). \tag{5.18}$$

Now suppose that we solve (5.18) by a splitting method such that

$$\left\{ \begin{array}{ll} \partial_t w_N^n = (-1)^{s+1}\varepsilon_N D^{2s} w_N^n, & t_{n-1} \le t \le t_n, \\ w_N^n(t_{n-1}) = u_N^{n-1}(t_{n-1}), & \end{array} \right. \tag{5.19}$$

$$\left\{ \begin{array}{ll} \partial_t u_N^n + \partial_x P_N f(u_N^n) = -B(u_N^n), & t_{n-1} \le t \le t_n, \\ u_N^n(t_{n-1}) = w_N^n(t_n), & \end{array} \right. \tag{5.20}$$

where $t_n = n\Delta t$. It is easy to see that (5.19) can be solved exactly so that if $u_N^{n-1}(t_{n-1}) = \sum_{k=0}^N \hat{u}_k^{n-1} T_k$, then

$$u_N^n(t_{n-1}) = w_N^n(t_n) = \sum_{k=0}^N e^{-\varepsilon_N k^{2s}\Delta t} \hat{u}_k^{n-1} T_k = \sum_{k=0}^N \sigma\left(\frac{k}{N}\right) \hat{u}_k^{n-1} T_k, \tag{5.21}$$

where $\sigma(\xi) = e^{-\alpha|\xi|^\gamma}, \alpha = N^{1+\theta}\Delta t$, and $\gamma = 2s$. Therefore, the procedure is that we use the usual spectral method (5.20) solving the nonlinear conservation law, but at each time step, the numerical solution is filtered by (5.21). Readers are referred to [SW], [D] for details.

Remark 2. We note that $s = s(N)$ may increase with N, but it should not spoil the conditions we have used in the proof of convergence. The most strict ones come from (4.4) and (5.11) such that

$$C_s \le \frac{N^\theta}{4C_M \ln^2 N}, \qquad (\tau^{1/4} \ln N)^{2s-1} \le N^\theta,$$

which suggest that $s \le \mathcal{O}(\ln N)$ (we have mentioned that $C_s \sim 2^s$). In practice, a little larger s is used [SW], [D], and it should be dependent on the problem we solve as pointed out in [D].

Remark 3. It seems that the factor ω in the SSV term is needed not simply because we can integrate by parts easy. It really helps control the L^∞ bound, which can be seen from the fact [CHQZ, pp. 288, 295] that

$$\|u\|_\infty \le CN\|u\|, \qquad \|u\|_\infty \le C\sqrt{N}\|u\|_\omega \qquad \forall u \in \mathcal{P}_N.$$

In fact, even if we replace the Chebyshev differential operator D^2 with the Legendre differential operator D_L^2 in the SSV term so that we can integrate by parts without the ω, we may still need the viscosity a little more strong to obtain an a priori estimate. On the other hand, we can let the viscosity be even weaker by cutting its effect on the lower modes as in [Ta3].

6. Conclusion. We have shown that the SSV is much weaker but good enough to guarantee the convergence of the bounded solution of the Chebyshev–Legendre SSV method. As mentioned in [M], the approximate solution u_N should be expanded in Legendre polynomials, and a suitable postprocessing procedure is needed. The most difficult problem of the SSV method may be the boundedness of the solution, which remains an open question even for the periodic problem [Ta3].

Appendix. The proof of Lemma 3.1. According to the definition,

$$D_L^2 = \omega^{-2}\partial_x^2 - 2x\partial_x = D^2 - x\partial_x. \tag{A.1}$$

Let

$$a_s(u) = (-1)^s (D^{2s+2}u, D_L^2 u)_\omega = (D^{s+2}u, D^s D_L^2 u)_\omega, \tag{A.2}$$

$$b_s(u) = (-1)^{s+1} (D^{2s+2}u, x\partial_x u)_\omega = -(D^{s+2}u, D^s(x\partial_x u))_\omega. \tag{A.3}$$

By (A.1), we have

$$(\text{A.4})\qquad a_s(u) = \|D^{s+2}u\|_\omega^2 + b_s(u),$$

$$(\text{A.5})\qquad \begin{aligned} a_s(u) &= \|D^s D_L^2 u\|_\omega^2 + (D^s(x\partial_x u), D^s D_L^2 u)_\omega \\ &= \|D^s D_L^2 u\|_\omega^2 - b_s(u) - \|D^s(x\partial_x u)\|_\omega^2, \end{aligned}$$

which also give us

$$(\text{A.6})\qquad \|D^{s+2}u\|_\omega^2 + \|D^s(x\partial_x u)\|_\omega^2 + 2b_s(u) = \|D^s D_L^2 u\|_\omega^2.$$

For $s = 0$, we have

$$(\text{A.7})\qquad \begin{aligned} b_0(u) &= -\int_I x\omega Du\partial_x Du\,dx = -\frac{1}{2}\int_I x\omega\partial_x(Du)^2\,dx \\ &= \frac{1}{2}\|\partial_x u\|_\omega^2 = \frac{1}{2}(\|x\partial_x u\|_\omega^2 + \|Du\|_\omega^2), \end{aligned}$$

which plays important roles in obtaining the coercive properties.

For $s \geq 1$, it seems troublesome to obtain similar results by integrals. Fortunately, these quantities can be expressed by some simple matrices, which enable us to show that

$$(\text{A.8})\qquad b_s(u) \geq \frac{2s+1}{2}\|D^s\partial_x u\|_\omega^2,$$

$$(\text{A.9})\qquad b_s(u) \geq \frac{1}{2}(\|D^s(x\partial_x u)\|_\omega^2 + \|D^{s+1}u\|_\omega^2).$$

To simplify the presentation, we first introduce a special symmetric matrix $A = (a_{ij})$ with elements of the form

$$(\text{A.10})\qquad a_{ij} = a_{ji} = \begin{cases} a_i b_j, & i+j \text{ even}, \quad i \leq j, \\ 0, & i+j \text{ odd}, \quad i < j, \end{cases}$$

which will be called a symmetric proportional splitting matrix and denoted by $A = \mathrm{SPS}(a_i, b_j)$. We first give the following result.

LEMMA A.1. *Let* $A = \mathrm{SPS}(a_i, b_j)$ $(i, j \geq 1)$. *If* $a_1b_1, a_2b_2 > 0$ *and*

$$(\text{A.11})\qquad \Delta_n \equiv b_n\left(a_n - \frac{b_n}{b_{n-2}}a_{n-2}\right) > 0 \qquad \forall n \geq 3,$$

then A *is positive definite.*

Proof. Let A_n be the leading principal submatrix of A of order n. Adding $-b_n/b_{n-2}$ times $(n-2)$th column to the nth column of A_n, we find that the nth column vanishes except for its last element Δ_n. Therefore, it is easy to see that the determinants satisfy

$$(\text{A.12})\qquad \mathrm{Det}(A_n) = \Delta_n \mathrm{Det}(A_{n-1}) = \cdots = a_1b_1a_2b_2\prod_{l=3}^{n}\Delta_l > 0, \qquad n > 2.$$

The desired result follows from the well-known theorem of matrices. In fact, if we do the same transformations on the rows, we can see that A is congruent to a diagonal matrix such that

$$PAP^T = \mathrm{diag}(a_1b_1, a_2b_2, \Delta_3, \ldots, \Delta_n, \ldots),$$

where P is nonsingular. $\square$

Now we prove (A.8) and (A.9). We suppose that u is a polynomial. For $u \in H_D^{s+2}(I)$ the results can be justified through a limit procedure. Let

$$T = (T_0, T_1, \ldots, T_j, \ldots), \qquad \hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_j, \ldots)^T. \tag{A.13}$$

The following relations,

$$\partial_x T_j = 2j \sum_{\substack{l=0 \\ l+j \text{ odd}}}^{j-1} \frac{T_l}{c_l}, \qquad c_0 = 2,\ c_l = 1\ (l \geq 0), \tag{A.14}$$

$$xT_0 = T_1, \qquad 2xT_j = T_{j-1} + T_{j+1}, \qquad j \geq 1, \tag{A.15}$$

can be expressed as

$$\partial_x T = T D_x, \qquad (D_x)_{ij} = \begin{cases} 2j/c_i, & i < j,\quad i+j \text{ even}, \\ 0, & \text{otherwise}, \end{cases} \tag{A.16}$$

$$xT = TM, \qquad (M)_{ij} = \begin{cases} c_j/2, & |i-j| = 1, \\ 0, & \text{otherwise}. \end{cases} \tag{A.17}$$

Let $\Lambda = \operatorname{diag}(0, 1, \ldots, j, \ldots)$. We have

$$D^{2s+2} u = D^{2s+2} T \hat{u} = (-1)^{s+1} T \Lambda^{2s+2} \hat{u}, \tag{A.18}$$

$$x \partial_x u = x \partial_x T \hat{u} = T M D_x \hat{u}. \tag{A.19}$$

Thus, we get that

$$b_s(u) = \frac{\pi}{2} \hat{u}^T \Lambda^{2s+2} M D_x \hat{u} = \frac{\pi}{2} \hat{u}^T B \hat{u}, \tag{A.20}$$

where the elements of $\Lambda^{2s+2} M D_x$ are

$$(\Lambda^{2s+2} M D_x)_{ij} = \begin{cases} 2 i^{2s+2} j, & i < j,\ i+j \text{ even}, \\ i^{2s+2} j, & i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{A.21}$$

and its symmetric form is

$$B \equiv \frac{1}{2}(\Lambda^{2s+2} M D_x + (\Lambda^{2s+2} M D_x)^T) = \mathrm{SPS}(i^{2s+2}, j)\ (i, j \geq 0). \tag{A.22}$$

In the same way, for $s \geq 1$ we have that

$$\|D^s \partial_x u\|_\omega^2 = \frac{\pi}{2} \hat{u}^T D_x^T \Lambda^{2s} D_x \hat{u} \equiv \frac{\pi}{2} \hat{u}^T C \hat{u},$$

where $C = \mathrm{SPS}(c_i, d_j)$ with

$$c_i = 4i \sum_{\substack{l=0 \\ l+i \text{ odd}}}^{i-1} l^{2s}, \qquad d_j = j.$$

Therefore, by denoting $\sigma_s = (2s+1)/2$ we find that

$$b_s(u) - \sigma_s \|D^s \partial_x u\|_\omega^2 = \frac{\pi}{2} \hat{u}^T (B - \sigma_s C) \hat{u} \equiv \frac{\pi}{2} \hat{u}^T A \hat{u},$$

where $A = \mathrm{SPS}(a_i, b_j)$ with

$$a_i = i^{2s+2} - 4\sigma_s i \sum_{\substack{l=0 \\ l+i \text{ odd}}}^{i-1} l^{2s}, \qquad b_j = j.$$

Since the first row and column of A are zeros, we drop them and still denote the matrix by A. It is easy to see that $a_1 b_1 = 1,\ a_2 b_2 = 2(4^{s+1} - 8\sigma_s) > 0$, and

$$\begin{aligned}
\Delta_{n+1} &\equiv b_{n+1}\left(a_{n+1} - \frac{b_{n+1}}{b_{n-1}} a_{n-1}\right) \\
&= (n+1)^2 \left[(n+1)^{2s+1} - (n-1)^{2s+1} - 4\sigma_s n^{2s}\right] \\
&= (n+1)^2 n^{2s+1} \left[\left(1 + \frac{1}{n}\right)^{2s+1} - \left(1 - \frac{1}{n}\right)^{2s+1} - 4\sigma_s \frac{1}{n}\right] \\
&= 2(n+1)^2 n^{2s+1} \sum_{\substack{l=3 \\ l \text{ odd}}}^{2s+1} C_{2s+1}^{l} \left(\frac{1}{n}\right)^{l} > 0 \qquad \forall n \geq 2.
\end{aligned} \tag{A.23}$$

From Lemma A.1 we know that A is positive definite and (A.8) follows.

Next, we consider (A.9). We have

$$\|D^s(x\partial_x u)\|_\omega^2 = \frac{\pi}{2} \hat{u}^T (MD_x)^T \Lambda^{2s} (MD_x) \hat{u} \equiv \frac{\pi}{2} \hat{u}^T C \hat{u}.$$

We find that $C + \Lambda^{2s+2} = \mathrm{SPS}(c_i, d_j)$ $(i, j \geq 0)$ with

$$c_i = 2i^{2s+1} + 4i \sum_{\substack{l=0 \\ l+i \text{ even}}}^{i-2} l^{2s}, \qquad d_j = j.$$

Therefore, we get

$$b_s(u) - \frac{1}{2}(\|D^s(x\partial_x u)\|_\omega^2 + \|D^{s+1}u\|_\omega^2) = \frac{\pi}{2} \hat{u}^T \left[B - \frac{1}{2}(C + \Lambda^{2s+2})\right] \hat{u} \equiv \frac{\pi}{2} \hat{u}^T A \hat{u},$$

where B is given in (A.22), and hence $A = \mathrm{SPS}(a_i, b_j)$ with

$$a_i = i^{2s+2} - i^{2s+1} - 2i \sum_{\substack{l=0 \\ l+i \text{ even}}}^{i-2} l^{2s}, \qquad b_j = j.$$

In this case, the first two rows and columns of A are zeros, $a_2 b_2 = 2^{2s+2}$, and

$$\begin{aligned}
\Delta_n &\equiv b_n \left(a_n - \frac{b_n}{b_{n-2}} a_{n-2}\right) \\
&= n\left[n^{2s+2} - n^{2s+1} - n(n-2)^{2s+1} + n(n-2)^{2s} - 2n(n-2)^{2s}\right] \\
&= n^2(n-1)\left[n^{2s} - (n-2)^{2s}\right] > 0 \qquad \forall n \geq 3,
\end{aligned} \tag{A.24}$$

which combined with Lemma A.1 gives (A.9).

Now the desired result (3.7) follows immediately from (A.6) and (A.8), and (3.9) follows from (A.4) and (A.8). Also, we have from (A.9) that

$$\begin{aligned}
\|D^s(x\partial_x u)\|_\omega^2 + \|D^{s+1}u\|_\omega^2 &\leq 2b_s(u) \leq 2\|D^{s+2}u\|_\omega \|D^s(x\partial_x u)\|_\omega \\
&\leq 2\|D^{s+2}u\|_\omega^2 + \frac{1}{2}\|D^s(x\partial_x u)\|_\omega^2,
\end{aligned} \tag{A.25}$$

which leads to

$$\|D^s(x\partial_x u)\|_\omega^2 + 2\|D^{s+1}u\|_\omega^2 \leq 4\|D^{s+2}u\|_\omega^2. \tag{A.26}$$

Thus, we have from (A.1) that

$$\begin{aligned} \|D^s D_L^2 u\|_\omega^2 &\leq (\|D^{s+2}u\|_\omega^2 + \|D^s(x\partial_x u)\|_\omega^2)^2 \\ &\leq 3\|D^{s+2}u\|_\omega^2 + \frac{3}{2}\|D^s(x\partial_x u)\|_\omega^2 \\ &\leq 3\|D^{s+2}u\|_\omega^2 + 3(2\|D^{s+2}u\|_\omega^2 - \|D^{s+1}u\|_\omega^2) \\ &= 9\|D^{s+2}u\|_\omega^2 - 3\|D^{s+1}u\|_\omega^2, \end{aligned} \tag{A.27}$$

and (3.8) follows. Finally, combining (A.4) with (A.5) and using (A.9) and (3.8), we get

$$\begin{aligned} 4a_s(u) &= \|D^s D_L^2 u\|_\omega^2 + 3\|D^{s+2}u\|_\omega^2 + 2b_s(u) - \|D^s(x\partial_x u)\|_\omega^2 \\ &\geq \left(1 + \frac{1}{3}\right)\|D^s D_L^2 u\|_\omega^2 + 2\|D^{s+1}u\|_\omega^2, \end{aligned} \tag{A.28}$$

and (3.10) follows.

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