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The Regularized Chapman-Enskog Expansion for Scalar Conservation Laws

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Abstract

ROSENAU [R] has recently proposed a regularized version of the Chapman-Enskog expansion of hydrodynamics. This regularized expansion resembles the usual Navier-Stokes viscosity terms at low wave numbers, but unlike the latter, it has the advantage of being a bounded macroscopic approximation to the linearized collision operator.

In this paper we study the behavior of the Rosenau regularization of the Chapman-Enskog expansion (R-C-E) in the context of scalar conservation laws. We show that this R-C-E model retains the essential properties of the usual viscosity approximation, e.g., existence of travelling waves, monotonicity, upper-Lipschitz continuity, etc., and at the same time, it *sharpens* the standard viscous shock layers. We prove that the regularized R-C-E approximation converges to the underlying inviscid entropy solution as its mean-free-path $\varepsilon \downarrow 0$, and we estimate the convergence rate.

1. Introduction

ROSENAU [R] has recently proposed the scalar equation

$$u_t + f(u)_x = \left[\frac{-\varepsilon k^2}{1 + m^2\varepsilon^2 k^2}\,\hat{u}(k)\right]^{\vee} = \varepsilon\left[\frac{1}{1 + m^2\varepsilon^2 k^2}\,\hat{u}(k)\right]^{\vee}_{xx} \tag{1.1}$$

as a model for his regularized version of the Chapman-Enskog expansion for hydrodynamics. Here the superscript " $\wedge$ " denotes the Fourier transform and the superscript " $\vee$ " denotes the inverse Fourier transform. The operator on the right side looks like the usual viscosity term εu_{xx} at low wave numbers k, while for higher wave numbers it is intended to model a bounded approximation of a linearized collision operator, thereby avoiding the artificial instabilities that occur when the Chapman-Enskog expansion for such an operator is truncated after a finite number of terms [R].

In this article we shall compare the behavior of solutions of the Rosenau regularization for the Chapman-Enskog expansion, abbreviated hereafter as the R-C-E equation, (1.1) with those of the conservation law with viscosity

$$u_t + f(u)_x = \varepsilon u_{xx}, \tag{1.2}$$

towards which (1.1) tends as $m \to 0$, and with the conservation law with absorption

$$u_t + f(u)_x = -\frac{u}{\varepsilon m^2}. \tag{1.3}$$

Since the right side of (1.1) tends to that of (1.3) in the limit of large k, it is not surprising that the smoothness properties of solutions of the former resemble those of the latter. In particular, the R-C-E equation does not smooth out initial discontinuities, but as shown in § 2, it does preserve the smoothness of smooth small initial data. On the other hand, the right side of (1.1) also resembles that of (1.2) in that both are second derivatives. Consequently, it is plausible that the regularized R-C-E equation (1.1), like the ordinary viscosity equation (1.2), should have travelling wave solutions connecting shock states of the underlying conservation law

$$u_t + f(u)_x = 0. \tag{1.4}$$

In § 3 we show that when $f'' > 0$, such solutions exist if and only if m is sufficiently small.

At the same time, solutions of the R-C-E equation (1.1) also resemble those of the conservation law (1.4) in that both admit unique entropy solutions which share similar properties. In § 4 we show that the R-C-E solution operator associated with (1.1), like the entropy solution operator of (1.4), is L^1-contractive, monotone, and BV-bounded. Furthermore, the R-C-E solution of (1.1) tends to the entropy solution of (1.4) as the 'mean-free-path' $\varepsilon \downarrow 0$. Finally, if $f'' > 0$, the R-C-E entropy solution of (1.1) is also upper-Lipschitz continuous, in agreement with Oleinik's E-condition, which characterizes the entropy solution of (1.4). In § 5 we estimate the convergence rate of the R-C-E solution to the entropy solution as $\varepsilon \downarrow 0$.

2. Smoothness

It is well known that solutions of (1.2) are smooth for $t > 0$; i.e., initial discontinuities are smoothed out at positive times. In contrast, by looking at piece-wise constant initial data or at the linear case $F(u) = u$, one sees that initial discontinuities of solutions of (1.3) are merely attenuated, not smoothed out, at positive times. Since the damping of (1.1) is less than that of (1.3), it is clear that (1.1) also does not smooth out initial discontinuities. On the other hand, if the (ε-independent) initial data for (1.3) is smooth, then the solution will remain so provided that m is sufficiently small (see below). The next theorem tells us that the same holds for the R-C-E equation (1.1).

Theorem 2.1. *The solution of the R-C-E equation* (1.1) *remains as smooth as its initial data,*

$$u(x, 0) = u_0(x), \tag{2.1}$$

provided the initial data u_0 *are sufficiently small so that*

$$2\{m\|u_0\|_{L^\infty}\|f''(u_0)\|_{L^\infty}\}^{1/2} + m^2\varepsilon\|f''(u_0)\|_{L^\infty}\|u_0'\|_{L^\infty} < 1. \tag{2.2}$$

Remark. Since we can ensure that (2.2) is satisfied for any fixed initial data by making m sufficiently small, Theorem 2.1 can also be viewed as showing how the smoothness properties of the R-C-E equation (1.1) approach those of equation (1.2) as $m \to 0$.

Proof. We show formally that (2.2) implies a bound on the L^∞-norms of u and u_x. Estimates for higher derivatives then follow in standard fashion; see [M]. Furthermore, this fact ensures that the formal estimates can be justified either by smoothing the initial data or by applying a further regularization with vanishing viscosity.

The first step towards obtaining the desired bounds is to note that the right side of (1.1) can be written as

$$\left[\frac{-\varepsilon k^2}{1+m^2\varepsilon^2k^2}\hat{u}(k)\right]^\vee = \frac{-1}{m^2\varepsilon}\left\{u - \left[\frac{1}{1+m^2\varepsilon^2k^2}\hat{u}(k)\right]^\vee\right\} \tag{2.3}$$

$$= \frac{-1}{m^2\varepsilon}\{u - Q_{m\varepsilon} * u\},$$

where $*$ denotes convolution, and

$$Q_\varepsilon(x) \equiv \frac{1}{2\varepsilon}e^{-|x|/\varepsilon} \tag{2.4}$$

satisfies

$$\|Q_\varepsilon\|_{L^1} = 1. \tag{2.5}$$

To obtain a uniform bound on u, multiply (1.1) by $|u^{p-2}|\,u$ and integrate over x; since $|u^{p-2}|\,uf(u)_x$ is an exact derivative, its integral vanishes, while the contribution of the right side (2.3) is nonpositive, for by (2.5),

$$\int -\frac{|u^{p-2}|}{m^2\varepsilon}u\{u - Q_{m\varepsilon} * u\}\,dx \leqq \frac{-1}{m^2\varepsilon}\{\|u\|_{L^p}^p - \|u\|_{L^p}^{p-1}\|Q_{m\varepsilon} * u\|_{L^p}\} \tag{2.6}$$

$$\leqq \frac{-1}{m^2\varepsilon}\|u\|_{L^p}^p\{1 - \|Q_{m\varepsilon}\|_{L^1}\} = 0.$$

Dividing the remaining inequality by $(p-1)\|u\|_{L^p}^{p-1}$ and integrating over t from 0 to T, we obtain

$$\|u(T)\|_{L^p} \leqq \|u_0\|_{L^p}, \tag{2.7}$$

and the boundedness of $\|u(T)\|_{L^\infty}$ follows by letting $p \uparrow \infty$.

In order to estimate in similar fashion the L^∞ norm of u_x, we differentiate (1.1), obtaining

$$u_{xt} + f(u)_{xx} = \frac{-1}{m^2\varepsilon}\{u_x - Q'_{m\varepsilon} * u\}, \tag{2.8}$$

and as before, we multiply (2.8) by $|u_x|^{p-2}u_x$ and integrate over x. Integrating by parts where necessary in the term containing f, noting that by (2.4)

$$\| Q'_{m\varepsilon} \|_{L^1} = \frac{1}{m\varepsilon}, \tag{2.9}$$

and factoring out $(p-1)\, \|u_x\|_{L^p}^{p-1}$, we obtain

$$\frac{d}{dt}\|u_x\|_{L^p} + \frac{1}{m^2\varepsilon}\{1 - m^2\varepsilon \|f''(u)\|_{L^\infty(t,x)} \|u_x\|_{L^\infty(t,x)}\} \|u_x\|_{L^p} \tag{2.10}$$
$$\leqq \frac{1}{m^3\varepsilon^2}\|u\|_{L^p} \leqq \frac{1}{m^3\varepsilon^2}\|u_0\|_{L^p}.$$

Next we denote

$$Y(T) \equiv m^2\varepsilon \|f''(u_0)\|_{L^\infty} \|u_x(T)\|_{L^\infty}. \tag{2.11}$$

Applying Gronwall's Lemma to (2.10) and letting $p \uparrow \infty$, we obtain that $Y \equiv \sup_T Y(T)$ does not exceed

(2.12)
$$Y(T) \leqq Y \leqq e^{-T(1-Y)/m^2\varepsilon} Y(0) + \frac{m}{1-Y}\{1 - e^{-T(1-Y)/m^2\varepsilon}\} \|u_0\|_{L^\infty} \|f''(u_0)\|_{L^\infty}$$
$$\leqq Y(0) + \frac{m}{1-Y}\|u_0\|_{L^\infty} \|f''(u_0)\|_{L^\infty},$$

as long as $Y < 1$. Estimate (2.12) is a quadratic inequality for Y, for which the roots of the corresponding equation are

$$Y = \tfrac{1}{2}\{1 + Y(0) \pm \sqrt{(1 - Y(0))^2 - 4m\|u_0\|_{L^\infty} \|f''(u_0)\|_{L^\infty}}\}. \tag{2.13}$$

Our assumption (2.2) tells us that the expression under the square root on the right is positive. Since $Y(0)$ is bounded by the smaller root in (2.13), it follows from (2.12) and from the continuity of $Y(T)$ that $Y(T)$ remains bounded by this root. This in turn confirms that indeed $Y < 1$, and the uniform bound of $\|u_x(T)\|_{L^\infty}$ follows. □

Remark. Arguing along the above lines for the conservation law with absorption (1.3), one arrives at the inequality $Y(T) \leqq e^{-T(1-Y)/m^2\varepsilon} Y(0)$, analogous to (2.12). This shows that if $Y(0) < 1$, then $Y(T)$ remains < 1. Consequently, if $Y(0) < 1$, then $Y(T)$ and hence $\|u_x(T)\|_{L^\infty}$ satisfy a maximum principle in this case.

3. Shock Profiles

LAX's generalized entropy conditions [L] for "legitimate" shock-wave solutions of the conservation law (1.4) can be interpreted as the requirement that these shocks can be realized as the limit of travelling wave solutions of (1.2). If the flux function f is convex, these conditions reduce to the shock inequalities [L]

$$f'(u_-) > s > f'(u_+), \tag{3.1}$$

where s is the speed of the shock joining u_- on the left to u_+ on the right. In this section we show the analogous result for the (convex) R-C-E equation (1.1): It admits travelling wave solutions whose limit as $\varepsilon \downarrow 0$ are shock wave solutions of (1.4), if and only if (3.1) holds and m is sufficiently small.

Theorem 3.1. *Assume* $f'' > 0$. *Then* (3.1) *and the Rankine-Hugoniot shock condition*

$$H(u_+) = 0, \quad H(u) \equiv -s\{u - u_-\} + \{f(u) - f(u_-)\} \tag{3.2}$$

are necessary conditions for the existence of a travelling wave solution

$$u\left(z \equiv \frac{x - st}{\varepsilon}\right), \qquad \lim_{z\to\pm\infty} u(z) = u_\pm,$$

for (1.1).

Conversely, if (3.1), (3.2) *hold, then a sufficient condition on* m *for the existence of such a travelling wave is*

$$4m^2 \sup_{u_+<u<u_-} \{-f''(u)\, H(u)\} \leqq 1, \tag{3.3}$$

and a necessary condition is

$$4m^2\{-f''(u_*)\, H(u_*)\} \leqq 1. \tag{3.4}$$

Here u_* *is defined by*

$$f'(u_*) = s. \tag{3.5}$$

Proof. Define $z = \dfrac{x - st}{\varepsilon}$ and let $'$ denote $\dfrac{d}{dz}$. Using (2.4) we find that a solution of (1.1) of the form $u = u(z)$ satisfies

$$-su' + f(u)' = \{Q_m * u\}'', \tag{3.6}$$

where the convolution is now taken with respect to the variable z. The condition $\lim_{z\to\pm\infty} u \to u_\pm$ implies that also $Q_m * u \to u_\pm$ as $z \to \pm\infty$, so there exists a sequence of values $z_j^\pm$ tending to $\pm\infty$ on which $(Q_m * u)'$ tends to zero. Hence, integrating (3.6) from z_j^- to z and letting $j \to \infty$, we obtain

$$H(u) \equiv -s\{u - u_-\} + \{f(u) - f(u_-)\} = \{Q_m * u\}'. \tag{3.7}$$

Now letting z tend to $+\infty$ along the sequence z_j^+, we find from (3.7) that $H(u_+) = 0$, i.e., (3.2) holds.

Noting that $H'' = f'' > 0$, we see that $H(u_-) = 0 = H(u_+)$ implies $H' = f' - s < 0$ at the smaller of $u_\pm$, and $H' > 0$ at the greater of the two. Hence, if $u_+ < u_-$, then (3.1) holds, while if this inequality is reversed, then so are those of (3.1), i.e., we can replace (3.1) by the condition

$$u_- > u_+. \tag{3.8}$$

Next, we apply to (3.7) the operator $1 - m^2 \frac{d^2}{dz^2}$ (the inverse of the operator of convolution with Q_m), to obtain

$$u' = \left\{1 - m^2 \frac{d^2}{dz^2}\right\} H(u) = H(u) - m^2\{H'(u)\, u'' + H''(u)\, (u')^2\}. \tag{3.9}$$

We note that since all nonzero solutions g of $\left\{1 - m^2 \frac{d^2}{dz^2}\right\} g = 0$ are unbounded on R, the solution of (3.9), with bounded u and u', which we construct below, also satisfies (3.7). To construct such a solution we introduce the auxiliary variable

$$v = u', \tag{3.10}$$

which enables us to rewrite (3.9) as the 2×2 system

$$u' = v, \tag{3.11}$$

$$m^2 H'(u)\, v' = H(u) - v - m^2 H''(u)\, v^2. \tag{3.12}$$

The convexity of $H(u)$ together with the Rankine-Hugoniot condition (3.2) imply that the only critical points of system (3.11), (3.12) are $(u_-, 0)$ and $(u_+, 0)$.

We remark that the linearization of (3.11), (3.12) near the critical points $(u_-, 0)$ and $(u_+, 0)$ shows that they are both saddles, so that topological methods (see, e.g., [S]) cannot be applied; one might even be tempted to conclude that the existence of a trajectory joining these saddle points is unlikely. What saves the day, however, is that the system is singular on the line $u = u_*$, i.e., that the coefficient $H'(u)$ on the left of (3.12) vanishes at u_*, which by (3.1) lies between u_- and u_+.

The key to finding a trajectory joining the two critical points is to note that solutions of (3.11), (3.12) can cross the line $u = u_*$ only at points (u_*, v_*) which make the right side of (3.12) vanish: Equation (3.11) implies that $H'(u(z))$ is $\mathcal{O}(z - z_*)$ near the value z_* for which $u(z_*) = u_*$, and hence (3.12) shows that $|v| \to \infty$ as $z \to z_*$, unless the right side of (3.12) tends to zero. Also, since the right side of (3.12) is quadratic in v, a comparison of (3.12) with the equations $zv' = \pm v^2$ shows that in fact $|v|$ reaches infinity before u reaches u_*.

In order to obtain a trajectory joining u_- at $z = -\infty$ to u_+ at $z = +\infty$, it is therefore necessary and sufficient to find trajectories joining $(u_-, 0)$ at $z = -\infty$ and $(u_+, 0)$ at $z = +\infty$ to (u_*, v_*) at some finite values of z; we can always arrange for the two values of z to coincide because the system is autonomous. Since trajectories through (u_*, v_*) are not unique, the existence of our desired trajectories, which when put together join u_- to u_+, no longer seems so unlikely.

Now the right side of (3.12) is a quadratic experession in v, whose roots are

$$v_{\pm}(u) = \frac{-1 \pm \sqrt{1 + 4m^2 H''(u)\, H(u)}}{2m^2 H''(u)}. \tag{3.13}$$

If the argument under the square root is negative at $u = u_*$, then clearly no such v_* exists; this gives the necessity of (3.4) for the existence of travelling wave solutions. We now turn to discuss the sufficiency of condition (3.3): it says that $v_{\pm}$ exist for all u between u_- and u_+. We want to show that when this happens, then the trajectories mentioned above exist if and only if (3.8) holds, i.e., that these trajectories exist if we replace u_- and u_+ by

$$u^- = \max\{u_-, u_+\} \quad \text{and} \quad u^+ = \min\{u_-, u_+\}, \tag{3.14}$$

respectively, but not if we replace u_- by u^+ and u_+ by u^-.

The linearization of our system around the two critical points has the form

$$\begin{pmatrix} U \\ V \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \dfrac{1}{m^2} & -\dfrac{1}{m^2 H'_{\pm}} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{3.15}$$

Since the determinant of the matrix on the right of (3.15) is negative, both critical points are saddles, as asserted. Now, it is not hard to calculate directly the asymptotic directions of the solutions that approach each critical point as z tends to $\pm\infty$, as these are simply the eigenvectors of the matrix in (3.15), but in any case we have to determine from (3.11), (3.12) the signs of u' and v' in various regions, and this information suffices to determine in which regions the various asymptotic directions lie. In this way we obtain the phase-plane Diagram 1 for the case when m satisfies (3.3).

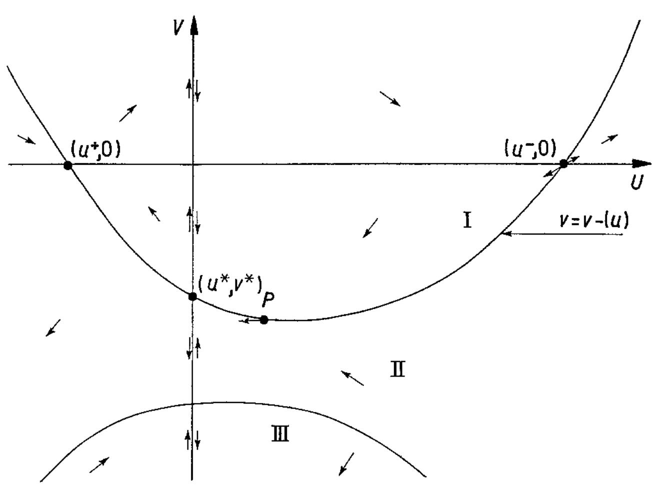

Diagram 1. Phase-plane for small m.

Based on Diagram 1, the existence of a travelling wave solution is argued as follows. There is a trajectory that leaves the critical point $(u^-, 0)$ and enters region I. If this trajectory remains in region I until u reaches the value u_*, then by the above analysis it reaches the point (u_*, v_*); in this case $u' \equiv v$ as well as u are monotonic on this semi-trajectory. The only way that the trajectory can leave region I before reaching the line $u = u_*$ is by entering region II; but $v' > 0$ in this region, so "clearly" the trajectory still reaches (u_*, v_*). A similar analysis backwards in the "time" z shows that there is a semi-trajectory from (u_*, v_*) to $(u^+, 0)$. By checking the other trajectories leaving and entering each critical point we see that no trajectory joins u^+ to u^- or either point to itself.

Although the above argument is sound provided that Diagram 1 is accurate, we have yet to verify one crucial feature of that diagram. Specifically, the argument assumes that if a trajectory enters region II at the point P and travels within this region keeping $v' > 0$ and $u' < 0$, then it cannot reach region III. (Clearly, no problem arises from the possibility of re-entering region I.) Thus we have to show that the situation illustrated in Diagram 2 is impossible:

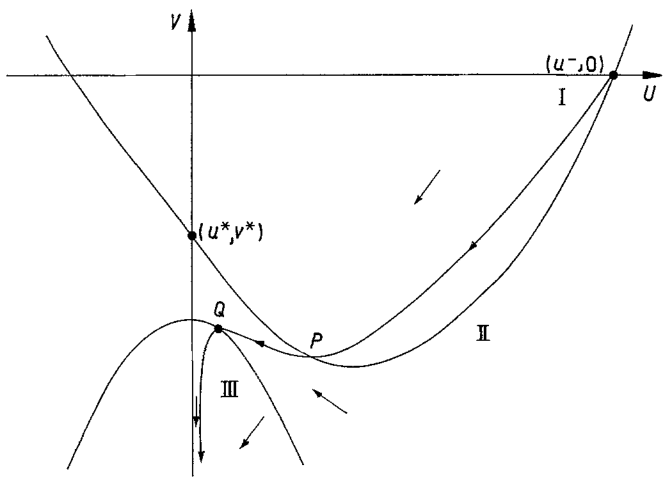

Diagram 2. We show that this phase plane diagram cannot occur because no such point Q exists.

Analytically, we must show that

$$v_-(u_1) \leqq v_+(u_2) \quad \text{for } u_* < u_1 < u_2. \tag{3.16}$$

Defining $H_i = -H(u_i)$ and $A_i = -4m^2 H''(u_i)\, H(u_i)$, we reduce (3.16) to

$$-2H_1\{1 + \sqrt{1 - A_1}\}/A_1 \leqq -2H_2\{1 - \sqrt{1 - A_2}\}/A_2. \tag{3.17}$$

Now, the convexity of f (and hence of H) together with the fact that H vanishes at $u_\pm$ imply that $H_1 > H_2 > 0$; these facts together with our assumption (3.3) imply that

(3.18) $$1 \geqq A_i > 0.$$

Therefore, a sufficient condition for (3.17) to hold is that

(3.19) $$\{1 + \sqrt{1 - A_1}\}/A_1 \geqq \{1 - \sqrt{1 - A_2}\}/A_2$$

for all A_i satisfying (3.18). A little algebraic manipulation shows that this is indeed the case. Consequently, (3.16) holds, i.e., no point such as the point Q in Diagram 2 can exist, and so the argument based on Diagram 1 is valid.

As m increases past the value that makes equality hold in (3.3), we obtain the situation of Diagram 3. Namely, a gap appears in region II, through which our trajectory might possibly plunge into the abyss of region III. Hence we cannot say whether a trajectory joining u^- to u^+ exists or not. Finally, when m increases past the value that makes equality hold in (3.4), then the phase-plane looks like Diagram 4, and the descent of our trajectory to $-\infty$ becomes a certainty. □

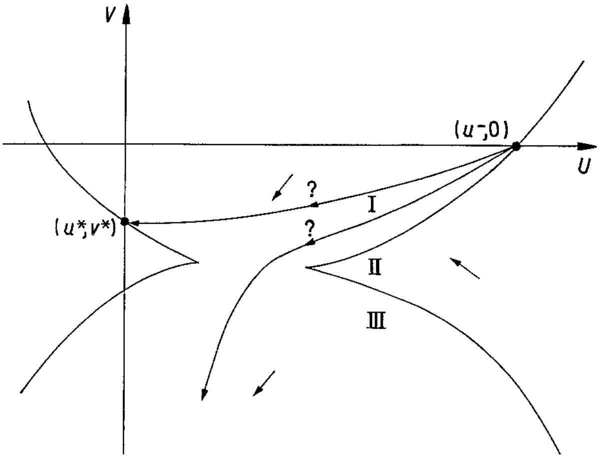

Diagram 3. Phase plane for intermediate m.

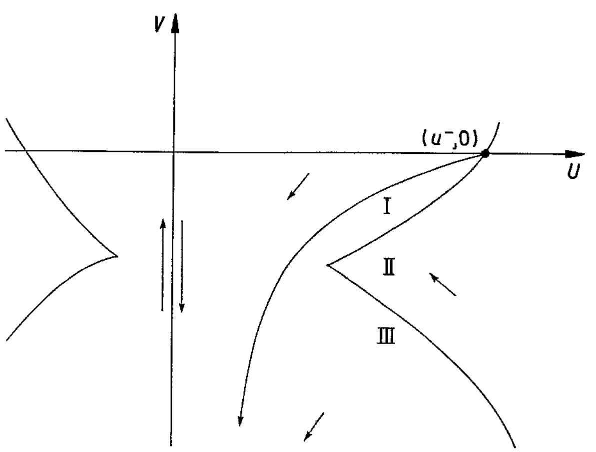

Diagram 4. Phase plane for large m.

We close this section by quantifying ROSENAU's [R] statement that the travelling-wave solutions of the R-C-E equation (1.1) give narrower shock layers than those of the viscosity equation (1.4). As our measure of shock width, we adopt $w \equiv (u_- - u_+)/|u'_*|$ with u'_* evaluated at the point u_* at which $H'(u) = 0$. (It should be noted, however, that although this value of u'_* is always maximum for (1.4), it is maximum for (1.1) only if the trajectory does not enter region II of Diagram 1; this last condition is guaranteed by our analysis only when the curve $v = v_+(u)$ has its unique local minimum at u_*.) Since the relevant value of u'_* for (1.4) is given by $u'_* = -H(u_*)$, while the value of u'_* for (1.1) is $v_+(u_*)$, the estimate

$$\tag{3.20} \tfrac{1}{2} \leqq \frac{w_{\text{Chapman—Enskog}}}{w_{\text{viscous}}} \leqq 1$$

follows from the simple lemma: *If a quadratic equation has real roots of the same sign, then the root closer to zero lies between r and 2r, where r is the root of its linear part.*

4. Entropy Solutions and the Zero Mean-Free-Path Limit

The parameter m does not play a role in our analysis in this section, and so will be set equal to 1 for convenience.

Since solutions of the R-C-E equation (1.1) may contain singularities, weak solutions must be admitted. Since the latter need not be unique, we single out an "entropy" solution of the R-C-E equation (1.1) as the one satisfying the KRUZHKOV-like [K] inequality

$$\tag{4.1} \begin{aligned} &\partial_t |u_\varepsilon - c| + \partial_x[\operatorname{sgn}(u_\varepsilon - c)\{f(u_\varepsilon) - f(c)\}] \\ &\qquad \leqq -\frac{1}{\varepsilon}\{|u_\varepsilon - c| - \operatorname{sgn}(u_\varepsilon - c)\, Q_\varepsilon * (u_\varepsilon - c)\}, \end{aligned}$$

for all real c's. In particular, by choosing $c = +\sup|u_\varepsilon|$ or $c = -\sup|u_\varepsilon|$, we obtain from (4.1) that u_ε is respectively a supersolution or a subsolution of (1.1), and hence (1.1) is satisfied in the sense of distributions. We turn to show that (4.1) admits a unique solution u_ε, and that this solution converges to the unique entropy solution of (1.4) as ε goes to zero.

Theorem 4.1. *For any u_0 in* BV *there exists a unique solution u_ε of the R-C-E equation* (4.1), *and as $\varepsilon \downarrow 0$, u_ε converges in L^1 to the unique entropy solution of the conservation law* (1.4).

Proof. Add the artifical viscosity term δu_{xx} to the right side of (1.1); the resulting equation has a unique smooth solution u_ε^δ. By a straightforward adaptation of KRUZHKOV's proof [K, Section 4] for the artificial-viscosity method for (1.4), we obtain that the set $\{u_\varepsilon^\delta\}_{\delta>0}$ is bounded in BV (uniformly in ε and δ) and precompact in L^1, and hence that a subsequence converges as $\delta \to 0$ to a solution u_ε of (4.1). Similarly, by the argument on pages 224, 225 of [K] we obtain from (4.1) the consequence that

$$\int_0^T \int_{-\infty}^{\infty} \{|u_\varepsilon - v_\varepsilon| \, \Phi_t + \operatorname{sgn}(u_\varepsilon - v_\varepsilon) [f(u_\varepsilon) - f(v_\varepsilon)] \, \Phi_x\} \, dx \, dt \tag{4.2}$$

$$\geqq \int_0^T \int_{-\infty}^{\infty} \frac{1}{\varepsilon} \{|u_\varepsilon - v_\varepsilon| - \operatorname{sgn}(u_\varepsilon - v_\varepsilon) \, Q_\varepsilon * (u_\varepsilon - v_\varepsilon)\} \, \Phi \, dx \, dt$$

where Φ is an arbitrary nonnegative test function.

Next, we remark that the expression inside the braces on the right of (4.2) need not be positive, but in view of (2.5), its spatial integral is. Therefore, by choosing $\Phi(t, x) = \Phi_1(t) \, \Phi_2(x)$ and letting Φ_2 tend to the function that is identically one, we obtain

$$\int_0^T \int_{-\infty}^{\infty} |u_\varepsilon - v_\varepsilon| \, \Phi_{1t} \geqq 0. \tag{4.3}$$

Continuing as in [K], we let Φ_1 approach the indicator function of the interval $[0, t]$ to conclude that

$$\|u_\varepsilon(t) - v_\varepsilon(t)\|_{L^1} \leqq \|u_\varepsilon(0) - v_\varepsilon(0)\|_{L^1}. \tag{4.4}$$

In particular, this shows that the solution of (4.1) is unique.

The solutions $\{u_\varepsilon\}$ of (4.1) inherit the BV-bound of the $\{u_\varepsilon^\delta\}$, and the argument of Section 4 of [K] shows that this bound implies precompactness in L^1. Hence as $\varepsilon \to 0$ a subsequence converges to a weak solution u of (1.4). Because the right side of (4.2) is known to be positive only when Φ has no dependence on x, we cannot use the entropies of (1.4) (as in [Ta]) to conclude that u is the entropy solution of (1.4). However, (4.4) implies the corresponding estimate for the weak solutions u and v obtained in the limit as ε goes to zero, and by an argument of LAX [L] this suffices to show that we obtain the entropy solution: It is not hard to see that when (1.4) has a smooth solution, then our scheme must converge to that solution. Hence by the corollary to Theorem (3.5) of [L], any solution u of (1.4) obtained in the limit $\varepsilon \to 0$ from (4.1) has the property that all of its discontinuities satisfy the generalized Lax shock inequalities. By Theorem (3.5) of [L], this implies that u is the unique entropy solution of (1.4). Finally, since any sequence of ε's tending to zero has a convergent subsequence, the uniqueness of the limit shows that convergence holds without passing to a sequence. □

5. The Convergence Rate of the Zero Mean-Free-Path Limit

Theorem 4.1 shows that the R-C-E equation (1.1) retains several properties of the conservation law with viscosity (1.2). In particular, (4.4) asserts that the solution operator is an L^1-contraction, and hence by conservation and translation invariance it is monotone [CM, Lemma 3.2], and by translation invariance it is BV-bounded:

$$\|u_\varepsilon(t)\|_{\mathrm{BV}} \leqq \|u_\varepsilon(0)\|_{\mathrm{BV}}. \tag{5.1}$$

Next we show that the nonlinear R-C-E equation (1.1) also satisfies Oleinik's E-entropy condition (cf., e.g., [Sm, T]).

Theorem 5.1. *Assume $f'' \geqq \alpha > 0$. Then the following a priori estimate holds*[1]

$$\|u_\varepsilon(t)\|_{\mathrm{Lip}^+} \leqq \frac{1}{\|u_\varepsilon(0)\|_{\mathrm{Lip}^+}^{-1} + \alpha t}, \qquad t \geqq 0. \tag{5.2}$$

Remark. The inequality (5.2) implies that the positive variation and hence the total variation of $u_\varepsilon(t)$ decay in time. Furthermore, this gives us another proof of the zero mean-free-path convergence to the entropy solution of (1.4) for any initial data u_0 in L^∞_{loc} (cf. Corollary 5.2).

Proof. We add the artifical viscosity term δu_{xx} to regularize (1.1), obtaining

$$\partial_t u_\varepsilon^\delta + \partial_x f(u_\varepsilon^\delta) = \frac{1}{m^2\varepsilon}\{u_\varepsilon^\delta - Q_{m\varepsilon} * u_\varepsilon^\delta\} + \delta\partial_x^2 u_\varepsilon^\delta. \tag{5.3}$$

Differentiation of (5.3) yields for $w \equiv \partial_x u_\varepsilon^\delta$,

$$\partial_t w + f'(u_\varepsilon^\delta)\,\partial_x w + f''(u_\varepsilon^\delta)\,w^2 = -\frac{1}{m^2\varepsilon}\{w - Q_{m\varepsilon} * w\} + \delta\partial_x^2 w. \tag{5.4}$$

Hence, since $f'' > \alpha > 0$, it follows that $W(t) \equiv \max_x w(t)$ is governed by the differential inequality

$$\dot W(t) + \alpha W^2(t) \leqq \frac{1}{m^2\varepsilon}\{W(t) - Q_{m\varepsilon} * W\} \leqq 0 \tag{5.5}$$

and (5.2) follows by letting $\delta \downarrow 0$. □

Theorem 5.1 shows that solutions of the R-C-E equation (1.1) are Lip$^+$-stable. Moreover, (5.1) implies that the Lip$'$-size of their truncation is of order $\mathcal{O}(\varepsilon)$, for

(5.6)

$$\|\partial_t u_\varepsilon + \partial_x f(u_\varepsilon)\|_{\mathrm{Lip}'} = \varepsilon\|Q_{m\varepsilon} * \partial_x u_\varepsilon\|_{L^1} \leqq \varepsilon\|Q_{m\varepsilon}\|_{L^1}\,\|u_\varepsilon(t)\|_{\mathrm{BV}} \leqq \varepsilon\|u_\varepsilon(0)\|_{\mathrm{BV}}.$$

Using the result of [T] we conclude that the Lip$'$-convergence *rate* of the R-C-E solutions to the corresponding entropy solution is also of order $\mathcal{O}(\varepsilon)$.

Corollary 5.2. *Let $f'' \geqq \alpha > 0$ and let u_ε be the unique R-C-E solution of* (4.1) *subject to C^1 initial conditions $u_\varepsilon(0) \equiv u(0)$. Then u_ε converges to the unique entropy solution of* (1.4), *and the following error estimates hold*:

$$\|u_\varepsilon(t) - u(t)\|_{W^{-s,p}} \leqq \mathrm{Const}\cdot\varepsilon^{(sp+1)/2p}, \quad 1 \leqq p < \infty, \quad s = 0, 1.$$

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[1] We let $\|\phi\|_{\mathrm{Lip}}$, $\|\phi\|_{\mathrm{Lip}^+}$ and $\|\phi\|_{\mathrm{Lip}'}$ denote respectively $\operatorname{ess\,sup}_{x\neq y}\left|\dfrac{\phi(x)-\phi(y)}{x-y}\right|$, $\operatorname{ess\,sup}_{x\neq y}\left[\dfrac{\phi(x)-\phi(y)}{x-y}\right]_+$, and $\sup_\psi \dfrac{(\phi - \hat\phi(0), \psi)}{\|\psi\|_{\mathrm{Lip}}}$.

Remark. The choice $(s, p) = (1, 1)$ corresponds to a Lip$'$-convergence rate of order $\mathcal{O}(\varepsilon)$. The choice $(s, p) = (0, 1)$ corresponds to the usual L^1-convergence rate of order $\mathcal{O}(\varepsilon^{1/2})$ for problems with viscosity.

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