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ON THE ZERO RELAXATION LIMIT FOR A SYSTEM MODELING THE MOTIONS OF A VISCOELASTIC SOLID*

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Abstract. We consider a simple model of the motions of a viscoelastic solid. The model consists of a two-by-two system of conservation laws including a strong relaxation term. We establish the existence of a BV-solution of this system for any positive value of the relaxation parameter. We also show that this solution is stable with respect to the perturbations of the initial data in L^1. By deriving the uniform bounds, with respect to the relaxation parameter, on the total variation of the solution, we obtain the convergence of the solutions of the relaxation system towards the solutions of a scalar conservation law as the relaxation parameter δ goes to zero. Due to the Lip^+ bound on the solutions of the relaxation system, an estimate on the rate of convergence towards equilibrium is derived. In particular, an $\mathcal{O}(\sqrt{\delta})$ bound on the L^1-error is established.

Key words. hyperbolic conservation laws, relaxation terms, nonequilibrium, convergence towards equilibrium, viscoelasticity, finite difference schemes

AMS subject classifications. 35L65, 65M99

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1. Introduction.

In this paper we study the following system of conservation laws:

$$u_t + \sigma_x = 0, \qquad (1.1)$$
$$(\sigma - f(u))_t + \frac{1}{\delta}(\sigma - \mu f(u)) = 0,$$

where the parameters μ and δ satisfy $0 < \mu < 1$ and $0 < \delta \ll 1$. Here μ is a fixed parameter, while we are, in particular, interested in the limit as the relaxation parameter δ tends to zero.

If $\delta \to 0$, we formally obtain the equilibrium relation

$$\bar{\sigma} = \mu f(\bar{u}),$$

and hence the equilibrium model

$$\bar{u}_t + \mu f(\bar{u})_x = 0. \qquad (1.2)$$

The purpose of this paper is to study the limit process rigorously. We will prove that under proper conditions on the initial data, the solutions of the nonequilibrium model converge to the solutions of the equilibrium model in L^1, uniformly in δ at a rate of $\mathcal{O}(\sqrt{\delta})$.

The system (1.1) arises in the modeling of motions of a viscoelastic solid, where the relaxation phenomenon presents the strength of memory. The Riemann problem for the system with $\delta = 1$ is studied by Greenberg and Hsiao [4]. The zero relaxation limit

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of this viscoelasticity model with vanishing memory is analyzed in the fundamental paper of Chen and Liu [1], where nonlinear stability in the zero relaxation limit is established for the model. This is achieved by first deriving energy estimates from proper construction of entropy pairs, and then applying the theory of compensated compactness. More recent results can be found in the paper by Chen, Levermore, and Liu [2]. In this paper, we will establish similar results, but in the BV-framework. For any positive values of the relaxation parameter, we will prove the existence of a BV-solution of the system. The bound on the total variation of the solution, and a proper stability estimate with respect to perturbations of the initial data in L^1, are both independent of the relaxation parameter. Furthermore, a uniform Lip^+ bound, similar to Oleinik's entropy condition (cf. [12]), is obtained. By following the framework of Tadmor, Nessyahu, and Kurganov [15, 11, 6], this bound is used to establish an $\mathcal{O}(\sqrt{\delta})$ estimate for the L^1 difference between the solution of the relaxation system (1.1) and the solution of the equilibrium model (1.2).

Hyperbolic conservation laws with relaxation terms arise in modeling of many physical phenomena, such as chromatography, traffic modeling, water waves, and viscoelasticity (see, e.g., the book of Whitham [17]). General relaxation effect was analyzed by Liu [8], and the convergence was studied by Natalini [10]. For a system modeling chromatography, convergence and rate of convergence towards equilibrium are proved (cf., [13, 16] for the 1D case and [14] for the 2D case). Sharper estimates on the rate of convergence for this model have been recently derived by Kurganov and Tadmor [6]. The approach here resembles the techniques used in [6, 13, 16]. The same model problem is also studied independently by Yong [18] and Luo and Natalini [9]. However, these papers do not derive a rate for the convergence to equilibrium.

The structure of the paper is as follows. In section 2, we give the preliminaries for the model, and we also state the main results of the paper. Then the properties of the finite difference solutions are studied in section 3, where we establish the uniform bound, the TV bound, and the bound on the deviation from the equilibrium state. In section 4, we prove that the limit of the finite difference solution is the entropy solution of the system, and the stability in L^1 is then proved by Kruzkov-type arguments. Finally, the proof of the convergence of the solution of the nonequilibrium model towards the solution of the equilibrium model is given in section 5.

2. Preliminaries and statement of the main results. In this section, we will give the preliminaries of the paper and state the main result. Throughout this paper we will assume that the flux function $f = f(u)$ is a smooth function with the following properties:

$$f(0) = 0, \quad f'(u) > 0, \quad f''(u) \geq 0 \quad \text{for all } u \geq 0.$$

We introduce the variable $v = f(u) - \sigma$ such that $u = g(\sigma + v)$, where the function $g = f^{-1}$. Under the assumption that $u \geq 0$, we obtain a reformulation of the system (1.1):

$$\begin{aligned} g(\sigma + v)_t + \sigma_x &= 0, \\ v_t &= \frac{1}{\delta} R(\sigma, v), \end{aligned} \tag{2.1}$$

where $R(\sigma, v) = ((1-\mu)\sigma - \mu v)$. The associated equilibrium model is

$$g\left(\frac{\bar{\sigma}}{\mu}\right)_t + \bar{\sigma}_x = 0. \tag{2.2}$$

We observe that the “reaction function” R has the monotonicity property

$$R(\sigma, v)(\mathrm{sgn}(\sigma) - \mathrm{sgn}(v)) \geq 0. \tag{2.3}$$

We seek solutions of (2.1) in the *state space*

$$\mathcal{S} = \{(\sigma, v) : 0 \leq \sigma \leq \mu\ , 0 \leq v \leq 1 - \mu\} \tag{2.4}$$

and solutions of (2.2) in $[0, \mu]$. For a scalar function $u(x)$, let $TV(u)$ denote the total variation defined as

$$TV(u) := \sup_{h \neq 0} \int_{\mathbb{R}} \frac{|u(x+h) - u(x)|}{h}\, dx,$$

and the L^1 norm is defined as

$$\|u\|_{L^1} := \int_{\mathbb{R}} |u(x)|\, dx.$$

Furthermore, we define

$$\mathrm{Lip}^+(u) := \max\left(0,\ \operatorname*{ess\ sup}_{x \neq y} \frac{u(x) - u(y)}{x - y}\right).$$

Let $p = R(\sigma, v)$ denote the residual. We assume the initial data (σ^0, v^0) satisfies the following requirements:

$$\begin{aligned}
&\text{i) } (\sigma^0(x), v^0(x)) \in \mathcal{S}, \quad \forall x \in \mathbb{R},\\
&\text{ii) } TV(\sigma^0) + TV(v^0) \leq M,\\
&\text{iii) } \left\|p^0\right\|_{L^1} \leq M\delta,\\
&\text{iv) } \sigma^0(\pm\infty) = v^0(\pm\infty) = 0,\\
&\text{v) } \mathrm{Lip}^+(\sigma^0) \leq M, \quad \mathrm{Lip}^+(v^0) \leq M.
\end{aligned} \tag{2.5}$$

Here, and throughout this paper, M denotes a generic positive finite constant independent of δ and the grid parameters $(\Delta x, \Delta t)$. Let $G = G(\sigma, v, k, q)$ be defined as

$$G(\sigma, v, k, q) = \frac{g(\sigma + v) - g(k + q)}{(\sigma + v) - (k + q)},$$

and for any $T > 0$, let $\mathcal{D}_+(T)$ be the set of all nonnegative $\mathcal{C}^\infty$-functions with compact support in $\mathbb{R} \times [0, T]$. Then the entropy solutions of (2.1) are defined as follows.

DEFINITION 2.1. *Let* (σ^0, v^0) *be the initial data of* (2.1) *which satisfies the assumptions in* (2.5). *Then a pair of functions* (σ, v) *is called the entropy solution of* (2.1) *with initial data* (σ^0, v^0) *if the following requirements are satisfied:*

i) $(\sigma, v) \in \mathcal{S}, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}_0^+$,

ii) $TV(\sigma(\cdot, t)) + TV(v(\cdot, t)) \leq M, \quad \forall t \geq 0$,

iii) $\|\sigma(\cdot, t) - \sigma(\cdot, \tau)\|_{L^1} + \|v(\cdot, t) - v(\cdot, \tau)\|_{L^1} \leq M|t - \tau|, \quad \forall t, \tau \geq 0$,

iv) $Lip^+(\sigma(\cdot, t)) \leq M,\ Lip^+(v(\cdot, t)) \leq M, \quad \forall t \geq 0$,

v) *for any* $(k, q) \in \mathcal{S}$ *and any* $\phi \in \mathcal{D}_+(T)$, *the following inequality is valid for all* $T > 0$*:*

$$
\begin{aligned}
(2.6)\quad & \int_0^T \int_{\mathbb{R}} [G(\sigma, v, k, q)(|\sigma - k| + |v - q|)\phi_t + |\sigma - k|\phi_x] \, dx \, dt \\
& + \int_{\mathbb{R}} G(\sigma^0, v^0, k, q)(\left|\sigma^0 - k\right| + \left|v^0 - q\right|)\phi(x, 0) \, dx \\
& - \int_{\mathbb{R}} G(\sigma(x, T), v(x, T), k, q)(|\sigma(x, T) - k| + |v(x, T) - q|) \; \phi(x, T) \, dx \\
& + M \int_0^T \int_{\mathbb{R}} [|v - q| - (v - q) sgn(\sigma - k)] \, \phi \, dx \, dt \\
& \geq \frac{1}{\delta} \int_0^T \int_{\mathbb{R}} G(\sigma, v, k, q) R(\sigma, v) [sgn(\sigma - k) - sgn(v - q)] \phi \, dx \, dt.
\end{aligned}
$$

Note that the entropy inequality in (2.6) is the weak formulation of an inequality of the form

$$
\mathcal{E}_t + \mathcal{F}_x \leq -\frac{1}{\delta}\mathcal{G} + M\mathcal{H},
$$

where

$$
\begin{aligned}
\mathcal{E} &= [G(\sigma, v, k, q)\big(|\sigma - k| + |v - q|\big)], \\
\mathcal{F} &= |\sigma - k|, \\
\mathcal{G} &= G(\sigma, v, k, q) R(\sigma, v)[\text{sgn}(\sigma - k) - \text{sgn}(v - q)], \\
\mathcal{H} &= |v - q| - (v - q)\text{sgn}(\sigma - k).
\end{aligned}
$$

Remarks. In order to motivate the weak entropy formulation above, let us assume that (σ, v) and $(\bar{\sigma}, \bar{v})$ are two smooth solutions of the system (2.1). The errors, $\sigma - \bar{\sigma}$ and $v - \bar{v}$, will then be governed by the system

$$
\begin{aligned}
[G\big((\sigma - \bar{\sigma}) + (v - \bar{v})\big)]_t + (\sigma - \bar{\sigma})_x &= 0, \\
(v - \bar{v})_t &= \frac{1}{\delta} R,
\end{aligned}
$$

where $G = G(\sigma, v, \bar{\sigma}, \bar{v})$ and $R = R(\sigma - \bar{\sigma}, v - \bar{v})$. The system can also be rewritten as

$$
\begin{aligned}
G_t \; (\sigma - \bar{\sigma}) + G \; (\sigma - \bar{\sigma})_t + (\sigma - \bar{\sigma})_x &= -G_t \; (v - \bar{v}) - \frac{1}{\delta} GR, \\
G(v - \bar{v})_t + G_t \; (v - \bar{v}) &= G_t \; (v - \bar{v}) + \frac{1}{\delta} GR.
\end{aligned}
$$

By multiplying the first equation above by $\text{sgn}(\sigma - \bar{\sigma})$ and the second one by $\text{sgn}(v - \bar{v})$, and summing, we obtain

$$
\begin{aligned}
(2.7)\quad & [G(|\sigma - \bar{\sigma}| + |v - \bar{v}|)]_t + (|\sigma - \bar{\sigma}|)_x \\
& = G_t[|v - \bar{v}| - (v - \bar{v})\text{sgn}(\sigma - \bar{\sigma})] - \frac{1}{\delta} GR(\text{sgn}(\sigma - \bar{\sigma}) - \text{sgn}(v - \bar{v})).
\end{aligned}
$$

If the function $G = G(x, t)$ satisfies a one-sided Lipschitz condition of the form

$$
G_t(x, t) \leq M, \tag{2.8}
$$

then clearly (2.7) implies that

$$\begin{aligned}(2.9)\quad &[G(|\sigma-\bar{\sigma}|+|v-\bar{v}|)]_t+(|\sigma-\bar{\sigma}|)_x\\ &\le M\,[|v-\bar{v}|-(v-\bar{v})\mathrm{sgn}(\sigma-\bar{\sigma})]-\frac{1}{\delta}GR(\mathrm{sgn}\;(\sigma-\bar{\sigma})-\mathrm{sgn}(v-\bar{v})).\end{aligned}$$

The weak entropy formulation above is motivated from this differential inequality. We also note that since $G\ge 0$, it follows from (2.3) and (2.9) that

$$[G(|\sigma-\bar{\sigma}|+|v-\bar{v}|)]_t+(|\sigma-\bar{\sigma}|)_x\le 2M|v-\bar{v}|.$$

This formal inequality indicates the continuous dependence result which will be established rigorously in this paper.

The motivation for the entropy formulation above relies on the one-sided bound (2.8). Since

$$G(\sigma,v,\bar{\sigma},\bar{v})=\int_0^1 g'(\theta(\sigma+v)+(1-\theta)(\bar{\sigma}+\bar{v}))\,d\theta,$$

and

$$(g'(\sigma+v))_t=-\frac{g''(\sigma+v)}{g'(\sigma+v)}\sigma_x\le M\sigma_x,$$

the bound (2.8) will follow from an estimate of the form

$$\mathrm{Lip}^+(\sigma(\cdot,t)),\mathrm{Lip}^+(\bar{\sigma}(\cdot,t))\le M.$$

As we shall see below, this property for solutions of the system (2.1) will essentially follow from the corresponding assumption (2.5v) on the initial data. This ends our discussion on the motivation for the weak entropy formulation.

For the scalar equilibrium equation, the entropy solutions are defined in the sense of Kruzkov [5]. For a given $T>0$, the entropy solutions satisfy the following inequality for any $k\in\mathcal{S}$ and any $\phi\in\mathcal{D}_+(T)$,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left(\left|g\left(\frac{\bar{\sigma}}{\mu}\right)-g\left(\frac{k}{\mu}\right)\right|\phi_t+|\sigma-k|\phi_x\right)\,dx\,dt\\ &\quad+\int_{\mathbb{R}}\left[\left|g\left(\frac{\bar{\sigma}^0}{\mu}\right)-g\left(\frac{k}{\mu}\right)\right|\phi(x,0)-\left|g\left(\frac{\bar{\sigma}(x,T)}{\mu}\right)-g\left(\frac{k}{\mu}\right)\right|\phi(x,T)\right]\,dx\ge 0.\end{aligned}$$

Our main tool in analyzing the system will be a finite difference scheme derived from the formulation (2.1). Let Δt and Δx denote the steplengths in the t and x directions, respectively. We consider a semi-implicit difference scheme of the form

$$\begin{aligned}(2.10)\qquad \frac{g\left(\sigma_j^{n+1}+v_j^{n+1}\right)-g\left(\sigma_j^n+v_j^n\right)}{\Delta t}+\frac{\sigma_j^n-\sigma_{j-1}^n}{\Delta x}&=0,\\ \frac{v_j^{n+1}-v_j^n}{\Delta t}&=\frac{1}{\delta}R\left(\sigma_j^{n+1},v_j^{n+1}\right).\end{aligned}$$

Here σ_j^n and v_j^n denote approximations of $\sigma(x,t)$ and $v(x,t)$ over the gridblocks

$$B_j^n=[x_{j-1/2},x_{j+1/2})\times[t_n,t_{n+1}),$$

where $x_j = j\Delta x$ and $t_n = n\Delta t$. Let $u_m = g(1) > 0$, and let

$$M_f = \max_{u\in[0,u_m]} f'(u) = \left(\min_{\theta\in[0,1]} g'(\theta)\right)^{-1}.$$

Throughout the paper we shall assume that the CFL-condition

$$\lambda M_f \le 1 \tag{2.11}$$

is satisfied, where $\lambda \equiv \Delta t/\Delta x$ is the mesh ratio which we assume to be a constant. The discrete initial data is taken to be the cell averages

$$\sigma_i^0 := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \sigma^0(x)\,dx, \qquad v_i^0 := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} v^0(x)\,dx.$$

The total variation of a grid function u_i is defined as

$$TV(u) := \sum_i |u_i - u_{i-1}|,$$

and the discrete L^1-norm is

$$\|u\|_{L^1} := \Delta x \sum_i |u_i|.$$

We assume that the following requirements are satisfied:

$$\begin{aligned} &\text{i) } (\sigma_i^0, v_i^0) \in \mathcal{S}, \quad \forall j,\\ &\text{ii) } TV(\sigma^0) + TV(v^0) \le M,\\ &\text{iii) } \left\|p^0\right\|_{L^1} \le M\delta,\\ &\text{iv) } \sigma_{\pm\infty}^0 = v_{\pm\infty}^0 = 0,\\ &\text{v) } \sup_j(\sigma_j^0 - \sigma_{j-1}^0) \le M\Delta t,\ \ \sup_j(v_j^0 - v_{j-1}^0) \le M\Delta t, \quad \forall j. \end{aligned} \tag{2.12}$$

Note that the requirement (v) follows directly from the assumption in (2.5v).

The existence of an entropy solution of the Cauchy problem can be obtained based on the properties of the finite different solutions of the scheme (2.10). Furthermore, the well-posedness of the initial value problem, independent of δ, is also proved.

THEOREM 2.2. *Let (σ^0, v^0) be the initial data of* (2.1) *satisfying the conditions* (2.5)*, and let (σ_i^0, v_i^0) be the discrete initial data for scheme* (2.10)*. Let $(\sigma_\Delta, v_\Delta)$ be the piecewise constant representation of the grid data (σ_i^n, v_i^n) generated by scheme* (2.10)*. Then the family $\{(\sigma_\Delta, v_\Delta)\}$ of approximate solutions converge in $(L^1_{loc}(\mathbb{R}\times\mathbb{R}_0^+))^2$ towards a pair of functions (σ, v) as the grid parameters $(\Delta x, \Delta t)$ tend to zero. The limit is the unique entropy solution which satisfies the requirements in Definition* 2.1*, and the following bounds are valid:*

$$\begin{aligned} &\|p(\cdot,t)\|_{L^1} \le M\delta,\\ &Lip^+(\sigma(\cdot,t)) \le M, \quad Lip^+(v(\cdot,t)) \le M. \end{aligned}$$

Moreover, the solution is stable with respect to perturbations in initial data in the following sense: Let $(\bar\sigma, \bar v)$ be another entropy solution of (2.1) *with initial data $(\bar\sigma^0, \bar v^0)$. Then the following bound holds for all $t > 0$:*

$$\|\sigma(\cdot,t) - \bar\sigma(\cdot,t)\|_{L^1} + \|v(\cdot,t) - \bar v(\cdot,t)\|_{L^1} \le \bar M e^{Mt}\left[\left\|\sigma^0 - \bar\sigma^0\right\|_{L^1} + \left\|v^0 - \bar v^0\right\|_{L^1}\right],$$

where $\bar{M}$ and M are finite constants independent of δ.

This theorem eventually leads to the main result of this paper, i.e., the convergence of the solutions of the nonequilibrium system towards the solutions of the equilibrium equation as δ tends to zero, and an estimate of the rate of convergence. The error estimates are derived by following the framework of Tadmor, Nessyahu, and Kurganov [15, 11, 6]. Hence, we estimate the Lip$'$-norm of the error. For any function $\phi \in L^1$ with $\int \phi = 0$, we define

$$\|\phi\|_{\mathrm{Lip}'} := \sup_{\psi} \frac{\int_{\mathbb{R}} \phi\psi dx}{\|\psi\|_{W^{1,\infty}}}.$$

Here the supremum is taken over all smooth functions ψ with compact support and

$$\|\psi\|_{W^{1,\infty}} := \max\left(\|\psi\|_{L^\infty}, ||\psi||_{Lip}\right).$$

The following convergence result will be proved in section 5.

THEOREM 2.3. *Let (σ^0, v^0) and $\bar{\sigma}^0$ be the initial data for (2.1) and (2.2), respectively. We assume that the initial data (σ^0, v^0) for the nonequilibrium system satisfies the requirements in (2.5) and that $\bar{\sigma}^0 = \sigma^0$. Let $(\sigma_\delta, v_\delta)$ be the entropy solution of (2.1) with initial data (σ^0, v^0) and $\bar{\sigma}$ the corresponding entropy solution of (2.2). For each $T > 0$ there is a constant M, independent of δ, such that*

$$\|u_\delta(\cdot, t) - \bar{u}(\cdot, t)\|_{Lip'} \leq M\delta, \quad 0 \leq t \leq T,$$

where $u_\delta = g(\sigma_\delta + v_\delta)$ and $\bar{u} = g(\frac{\bar{\sigma}}{\mu})$.

We note that the variables $(u_\delta, \sigma_\delta)$ and $(\bar{u}, \bar{\sigma})$ in the theorem above correspond to the solutions of the original models (1.1) and (1.2). The following corollary is a consequence of Theorem 2.3.

COROLLARY 2.4. *Let $(u_\delta, \sigma_\delta)$ and $(\bar{u}, \bar{\sigma})$ be as stated in Theorem 2.3. For each $T > 0$ there is a constant M, independent of δ, such that for any $p \in [1, \infty)$*

$$\|u_\delta(\cdot, t) - \bar{u}(\cdot, t)\|_{L^p} \leq M\delta^{\frac{1}{2p}}, \quad 0 \leq t \leq T.$$

Furthermore,

$$\|\sigma_\delta(\cdot, t) - \bar{\sigma}(\cdot, t)\|_{L^1} \leq M\sqrt{\delta}, \quad 0 \leq t \leq T.$$

3. Existence of a weak solution. The purpose of this section is to use the finite difference scheme (2.10) to establish the existence of weak solutions of Cauchy problem for (2.1) (or (1.1)). We first show that the finite difference solution is well defined.

LEMMA 3.1. *Assume that $\{\sigma_j^0\}$ and $\{v_j^0\}$ for $j \in \mathcal{Z}$ are given. Then the solutions $\{\sigma_j^n\}$ and $\{v_j^n\}$ are uniquely determined by (2.10) for all $j \in \mathcal{Z}$ and $n \geq 0$.*

Proof. Assume that $\{\sigma_j^n\}$ and $\{v_j^n\}$ are computed. Let

$$r_j^n = g\left(\sigma_j^n + v_j^n\right) - \lambda\left(\sigma_j^n - \sigma_{j-1}^n\right).$$

The solutions $\{\sigma_j^{n+1}\}$ and $\{v_j^{n+1}\}$ then satisfy the linear system

$$A\begin{pmatrix} \sigma_j^{n+1} \\ v_j^{n+1} \end{pmatrix} = \begin{pmatrix} f\left(r_j^n\right) \\ v_j^n \end{pmatrix},$$

where the 2×2 matrix A is given by

$$A = \begin{pmatrix} 1 & 1 \\ -(1-\mu)\frac{\Delta t}{\delta} & 1 + \mu\frac{\Delta t}{\delta} \end{pmatrix}.$$

Since $\det(A) = 1 + \frac{\Delta t}{\delta} > 0$, the results follows by induction. ∎

The following results show that the state space $\mathcal{S}$ defined in (2.4), is an invariant region for the scheme (2.10).

LEMMA 3.2. *Assume* $(\sigma_j^0, v_j^0) \in \mathcal{S}$ *for all* $j \in \mathcal{Z}$. *Then* $(\sigma_j^n, v_j^n) \in \mathcal{S}$ *for all* $j \in \mathcal{Z}$ *and* $n \geq 0$.

Proof. For given $\bar{\sigma}, \sigma_L$ and $\bar{v}$, let (σ, v) be the unique solution of the system

$$\begin{aligned} g(\sigma + v) &= g(\bar{\sigma} + \bar{v}) - \lambda\left(\bar{\sigma} - \sigma_L\right), \\ \left(1 + \frac{\Delta t}{\delta}\mu\right) v &- \frac{\Delta t}{\delta}(1-\mu)\sigma = \bar{v}. \end{aligned} \tag{3.1}$$

This system defines functions $\sigma = \sigma(\bar{\sigma}, \sigma_L, \bar{v})$ and $v = v(\bar{\sigma}, \sigma_L, \bar{v})$. Furthermore, $\sigma_j^{n+1} = \sigma(\sigma_j^n, \sigma_{j-1}^n, v_j^n)$ and $v_j^{n+1} = v(\sigma_j^n, \sigma_{j-1}^n, v_j^n)$. Hence, the lemma can be established by studying the functions σ and v.

Assume that $(\bar{\sigma}, \bar{v}) \in \mathcal{S}$ and $\sigma_L \in [0, \mu]$. By differentiating the system (3.1) with respect to $\bar{\sigma}$ and by using the CFL-condition (2.11), we obtain

$$\begin{aligned} g'(\sigma + v)\left(\frac{\partial \sigma}{\partial \bar{\sigma}} + \frac{\partial v}{\partial \bar{\sigma}}\right) &= g'(\bar{\sigma} + \bar{v}) - \lambda > 0, \\ \left(1 + \frac{\Delta t \mu}{\delta}\right)\frac{\partial v}{\partial \bar{\sigma}} &= \frac{\Delta t}{\delta}(1-\mu)\frac{\partial \sigma}{\partial \bar{\sigma}}. \end{aligned}$$

From this we easily conclude that $\frac{\partial \sigma}{\partial \bar{\sigma}}, \frac{\partial v}{\partial \bar{\sigma}} > 0$, and by a similar calculation we also obtain $\frac{\partial \sigma}{\partial \sigma_L}, \frac{\partial v}{\partial \sigma_L} > 0$.

Assume now that $\sigma_L = \bar{\sigma}$. Then we obtain from (3.1) that

$$\sigma + v = \bar{\sigma} + \bar{v},$$

and hence

$$\frac{\partial \sigma}{\partial \bar{v}} + \frac{\partial v}{\partial \bar{v}} = 1.$$

Furthermore, from the second equation of (3.1) we have

$$\frac{\Delta t}{\delta}\mu\frac{\partial v}{\partial \bar{v}} = \left(1 + \frac{\Delta t}{\delta}(1-\mu)\right)\frac{\partial \sigma}{\partial \bar{v}},$$

and hence we can conclude that

$$\frac{\partial \sigma}{\partial \bar{v}}(\bar{\sigma}, \bar{\sigma}, \bar{v}) > 0, \quad \frac{\partial v}{\partial \bar{v}}(\bar{\sigma}, \bar{\sigma}, \bar{v}) > 0.$$

From the monotonicity properties derived above we now have for $(\bar{\sigma}, \bar{v}) \in \mathcal{S}$ and $\sigma_L \in [0, \mu]$

$$\sigma(\bar{\sigma}, \sigma_L, \bar{v}) \geq \sigma(0, 0, \bar{v}) \geq \sigma(0, 0, 0) = 0$$

and

$$\sigma(\bar{\sigma}, \sigma_L, \bar{v}) \leq \sigma(\mu, \mu, \bar{v}) \leq \sigma(\mu, \mu, 1-\mu) = \mu.$$

Similarly, we obtain

$$0 \leq v(\bar{\sigma}, \sigma_L, \bar{v}) \leq 1-\mu,$$

and the invariance of $\mathcal{S}$ follows by induction. ∎

We let p_j^n denote the residual, i.e., $p_j^n = (1-\mu)\sigma_j^n - \mu v_j^n$.

LEMMA 3.3. *Assume that* $\|p^0\|_1$,$TV(\sigma^0)$ *and* $TV(v^0)$ *are finite. Then*

$$TV(\sigma^n) + TV(v^n) \leq TV(\sigma^0) + TV(v^0) \ . \tag{3.2}$$

Furthermore, there is a constant M_1*, depending only on* μ*,* g*,* $TV(\sigma^0)$*, and* $TV(v^0)$ *such that*

$$\frac{\|p^n\|_1}{\delta} \leq \max\left(M_1, \frac{\|p^0\|_1}{\delta}\right).$$

Proof. We first establish the total variation estimate. Let

$$a_j^n = \frac{\left(\sigma_j^{n+1} + v_j^{n+1}\right) - \left(\sigma_j^n + v_j^n\right)}{g\left(\sigma_j^{n+1} + v_j^{n+1}\right) - g\left(\sigma_j^n + v_j^n\right)}.$$

It follows from the monotonicity of g and the CFL-condition (2.11) that

$$0 \leq \lambda a_j^n \leq 1.$$

Furthermore, the difference scheme (2.10) can be written in the form

$$\begin{aligned} \sigma_j^{n+1} &= \sigma_j^n - \lambda a_j^n \left(\sigma_j^n - \sigma_{j-1}^n\right) - \frac{\Delta t}{\delta} R\left(\sigma_j^{n+1}, v_j^{n+1}\right), \\ v_j^{n+1} &= v_j^n + \frac{\Delta t}{\delta} R\left(\sigma_j^{n+1}, v_j^{n+1}\right). \end{aligned} \tag{3.3}$$

Hence, if we let

$$\alpha_j^n = \sigma_{j+1}^n - \sigma_j^n, \quad \beta_j^n = v_{j+1}^n - v_j^n,$$

we obtain

$$\begin{aligned} \alpha_j^{n+1} &= \alpha_j^n - \lambda a_{j+1}^n \alpha_j^n + \lambda a_j^n \alpha_{j-1}^n - \frac{\Delta t}{\delta} R\left(\alpha_j^{n+1}, \beta_j^{n+1}\right), \\ \beta_j^{n+1} &= \beta_j^n + \frac{\Delta t}{\delta} R\left(\alpha_j^{n+1}, \beta_j^{n+1}\right). \end{aligned} \tag{3.4}$$

By multiplying the first equation in (3.4) by sgn (α_j^{n+1}), the second equation by sgn (β_j^{n+1}), using the monotonicity property (2.3), and by summation with respect to j, we obtain

$$\sum_j \left(\left|\alpha_j^{n+1}\right| + \left|\beta_j^{n+1}\right|\right) \leq \sum_j \left(\left|\alpha_j^n\right| + \left|\beta_j^n\right|\right),$$

and this is exactly the total variation bound.

From (3.3) it also follows that

$$p_j^{n+1} = p_j^n - (1-\mu)\lambda a_j^n \left(\sigma_j^n - \sigma_{j-1}^n\right) - \frac{\Delta t}{\delta} p_j^{n+1}.$$

Therefore, it follows from the total variation estimate above that

$$\|p^{n+1}\|_1 \le \|p^n\|_1 + M_1 \Delta t - \frac{\Delta t}{\delta}\|p^{n+1}\|_1,$$

and this implies that

$$\frac{\|p^{n+1}\|_1}{\delta} \le \max\left(M_1, \frac{\|p^n\|_1}{\delta}\right).$$

This completes the proof of Lemma 3.3. ☐

We recall that the initial data satisfies

$$\|p^0\|_1 \le M\delta,$$

where M is independent of δ and the grid parameters Δt and Δx. Hence, by induction, we have

$$\|p^n\|_1 \le M\delta \quad \text{for all} \quad n \ge 0. \tag{3.5}$$

From the total variation estimate (3.2) and (3.5), we now obtain

$$\|\sigma^{n+1} - \sigma^n\|_1 + \|v^{n+1} - v^n\|_1 \le M\Delta t,$$

and hence we obtain L^1-Lipschitz continuity with respect to time, i.e.,

$$\|\sigma^n - \sigma^m\|_1 + \|v^n - v^m\|_1 \le M|n-m|\Delta t,$$

where M is independent of δ and the grid parameters.

4. Entropy solutions and stability in L^1. The purpose of this section is to derive bounds for $\text{Lip}^+(\sigma)$ and $\text{Lip}^+(v)$, which can be used to obtain stability results with respect to perturbations of the initial data which are independent of the relaxation parameter δ. The extra regularity results will technically be derived for the finite difference solutions (σ_j^n, v_j^n).

Define coefficients b_j^n by

$$b_j^n = \frac{a_{j+1}^n - a_j^n}{\alpha_j^{n+1} + \beta_j^{n+1} + \alpha_j^n + \beta_j^n},$$

where as above $\alpha_j^n = \sigma_{j+1}^n - \sigma_j^n$ and $\beta_j^n = v_{j+1}^n - v_j^n$. Observe that if we let $u_j^n = g(\sigma_j^n + v_j^n)$, then

$$a_j^n = \int_0^1 f'\left(u_j^n + \theta\left(u_j^{n+1} - u_j^n\right)\right)\, d\theta.$$

Hence, it follows from the monotonicity of f' and f that there is a positive constant M_b such that

$$0 < b_j^n \le M_b.$$

We claim that for sufficiently small Δt and δ, the initial data (σ_j^0, v_j^0) of (2.10) satisfies the following one-side bound:

$$\sup_j \left\{(1-\mu)\alpha_j^0 \;, \mu\beta_j^0\right\} \le (1-\mu)\mu^2 \frac{\Delta t}{2\delta + \mu\Delta t}. \tag{4.1}$$

Indeed, since $\alpha_j^0 \le \mu$ and $\beta_j^0 \le 1-\mu$ for all j, then by (2.12v), there exists a finite constant M^* and a sufficiently small Δt^* satisfying the relation $M^* \cdot \Delta t^* \le 1$ such that

$$\sup_j \left\{\alpha_j^0\right\} \le M^*\Delta t\mu, \quad \sup_j \left\{\beta_j^0\right\} \le M^*\Delta t(1-\mu),$$

for all $\Delta t \le \Delta t^*$. Then it follows that

$$\sup_j \left\{(1-\mu)\alpha_j^0, \mu\beta_j^0\right\} \le (1-\mu)\mu M^*\Delta t,$$

for all $\Delta t \le \Delta t^*$. By choosing δ sufficiently small, i.e.,

$$\delta \le \frac{\mu(1-M^*\Delta t)}{2M^*},$$

the relation (4.1) follows.

In order to derive the proper results for the solution of the finite difference scheme, we will need a strengthened CFL-condition. We will assume throughout this section that

$$\lambda\left(M_f + (2+\mu)M_b\right) \le 1. \tag{4.2}$$

LEMMA 4.1. *Assume that the initial data* (σ_j^0, v_j^0) *of* (2.10) *satisfies* (4.1) *for sufficiently small* δ *and* Δt*. Then*

$$\sup_j \left\{(1-\mu)\alpha_j^n, \mu\beta_j^n, 0\right\} \le \sup_j \left\{(1-\mu)\alpha_j^0, \mu\beta_j^0, 0\right\}.$$

Proof. Define function $\alpha = \alpha(\bar\alpha, \bar\beta, \alpha_L)$ and $\beta = \beta(\bar\alpha, \bar\beta, \alpha_L)$ implicitly by

$$\begin{aligned} \alpha &= \bar\alpha - \lambda a\left(\bar\alpha - \alpha_L\right) - \lambda b\left(\alpha + \beta + \bar\alpha + \bar\beta\right)\bar\alpha - \frac{\Delta t}{\delta}((1-\mu)\alpha - \mu\beta), \\ \beta &= \bar\beta + \frac{\Delta t}{\delta}((1-\mu)\alpha - \mu\beta). \end{aligned} \tag{4.3}$$

Here a and b are positive constants, bounded by M_f and M_b, respectively.

Recall that it follows from (3.4) that if $a = a_j^n$ and $b = b_j^n$, then $\alpha_j^{n+1} = \alpha(\alpha_j^n, \beta_j^n, \alpha_{j-1}^n)$ and $\beta_j^{n+1} = \beta(\alpha_j^n, \beta_j^n \;, \alpha_{j-1}^n)$. Recall also that Lemma 3.2 implies that $|\alpha_j^n| \le \mu$ and $|\beta_j^n| \le 1-\mu$.

We will first show that, under the assumptions that $|\bar\alpha|, |\alpha_L| \le \mu$, $|\bar\beta| \le 1-\mu$ and

$$\bar\alpha \le \mu^2 \frac{\Delta t}{2\delta + \mu\Delta t}, \tag{4.4}$$

the functions α and β are monotonically increasing in all three arguments. Observe that the second equation of (4.3) implies that

$$\beta = \frac{\delta}{\delta + \mu\Delta t}\bar\beta + \frac{(1-\mu)\Delta t}{\delta + \mu\Delta t}\alpha. \tag{4.5}$$

Hence we can eliminate β from the first equation. We obtain the equation

$$c\alpha = r, \tag{4.6}$$

where

$$c = c(\bar{\alpha}) = 1 + \lambda b \frac{\delta + \Delta t}{\delta + \mu\Delta t}\bar{\alpha} + \frac{(1-\mu)\Delta t}{\delta + \mu\Delta t} = (1 + \lambda b\bar{\alpha})\,\frac{\delta + \Delta t}{\delta + \Delta t\mu}$$

and

$$r = r(\bar{\alpha}\ , \bar{\beta}\ , \alpha_L) = (1-\lambda a)\bar{\alpha} + \lambda a\alpha_L - \lambda b\bar{\alpha}^2 - \lambda b\frac{2\delta + \mu\Delta t}{\delta + \mu\Delta t}\bar{\alpha}\bar{\beta} + \frac{\mu\Delta t}{\delta + \mu\Delta t}\bar{\beta}.$$

Note that since $\bar{\alpha} \geq -\mu$, it follows that

$$c \geq c(-\mu) \geq \frac{\delta + \Delta t}{\delta + \mu\Delta t}\left(1 - \mu\lambda M_b\right),$$

and hence (4.2) implies that $c > 0$. Observe that

$$\frac{\partial r}{\partial \alpha_L} = \lambda a > 0,$$

which implies that $\frac{\partial\alpha}{\partial\alpha_L} > 0$.

Similarly, by (4.2) and (4.4), we get

$$\frac{\partial r}{\partial \bar{\beta}} = \frac{\mu\Delta t - \lambda b(2\delta + \mu\Delta t)\bar{\alpha}}{\delta + \mu\Delta t} \geq \frac{\mu\Delta t}{\delta + \mu\Delta t}(1 - \lambda b\mu) \geq 0,$$

which implies that

$$\frac{\partial \alpha}{\partial \bar{\beta}} \geq 0.$$

Finally, we observe that

$$\begin{aligned} c\frac{\partial\alpha}{\partial\bar{\alpha}} &= \frac{\partial r}{\partial\bar{\alpha}} - \alpha\frac{dc}{d\bar{\alpha}} \\ &= (1-\lambda a) - 2\lambda b\bar{\alpha} - \lambda b\frac{2\delta + \mu\Delta t}{\delta + \mu\Delta t}\bar{\beta} - \lambda b\frac{\delta + \Delta t}{\delta + \mu\Delta t}\alpha \\ &\geq (1-\lambda a) - 2\lambda b\mu - \lambda b\frac{2\delta + \mu\Delta t}{\delta + \mu\Delta t}(1-\mu) - \lambda b\frac{\delta + \Delta t}{\delta + \mu\Delta t}\mu. \end{aligned}$$

This implies that

$$c\frac{\partial\alpha}{\partial\bar{\alpha}} \geq 1 - \lambda(a + b(2+\mu)).$$

Hence, it follows from (4.2) that

$$\frac{\partial\alpha}{\partial\bar{\alpha}} \geq 0.$$

We have therefore established that the function α is an increasing function in all three of its arguments. Furthermore, from (4.5) we easily derive that β has the corresponding property. We now use induction to complete the proof. Assume that

$$z^n \equiv \sup_j \left\{(1-\mu)\alpha_j^n\ , \mu\beta_j^n\ , 0\right\} \leq z^0.$$

In particular, this implies that (cf. (4.4))

$$\alpha_j^n \le \mu^2 \frac{\Delta t}{2\delta + \mu \Delta t}.$$

Hence, the monotonicity property of α implies that

$$\alpha_j^{n+1} \le \alpha \left(\frac{z^n}{1-\mu}, \frac{z^n}{\mu}, \frac{z^n}{1-\mu} \right).$$

Furthermore, since $z^n \ge 0$,

$$c\left(\frac{z^n}{1-\mu} \right) \ge \frac{\delta + \Delta t}{\delta + \mu \Delta t}$$

and

$$r\left(\frac{z^n}{1-\mu}, \frac{z^n}{\mu}, \frac{z^n}{1-\mu} \right) \le \frac{z^n}{1-\mu} + \frac{\Delta t \, z^n}{\delta + \mu \Delta t} = \frac{z^n}{1-\mu} \left(\frac{\delta + \Delta t}{\delta + \mu \Delta t} \right).$$

We therefore obtain from (4.6) that

$$\alpha_j^{n+1} = \frac{r\left(\frac{z^n}{1-\mu}, \frac{z^n}{\mu}, \frac{z^n}{1-\mu} \right)}{c\left(\frac{z^n}{1-\mu} \right)} \le \frac{z^n}{1-\mu}.$$

Finally, from (4.5), we derive

$$\beta_j^{n+1} \le \beta \left(\frac{z^n}{1-\mu}, \frac{z^n}{\mu}, \frac{z^n}{1-\mu} \right) \le \frac{\delta}{\delta + \mu \Delta t} \frac{z^n}{\mu} + \frac{(1-\mu)\Delta t}{\delta + \mu \Delta t} \frac{z^n}{1-\mu} = \frac{z^n}{\mu}.$$

Hence, we conclude that $z^{n+1} \le z^n$. ◻

Next we will show that the finite difference solution satisfies a "discrete entropy inequality." Recall that the initial data (σ^0, v^0) satisfies a one-sided bound of the form (cf. (2.12v))

$$\sup_j \left\{ \sigma_j^0 - \sigma_{j-1}^0, v_j^0 - v_{j-1}^0 \right\} \le M \Delta t, \tag{4.7}$$

where $M > 0$ is a finite constant independent of δ and the mesh parameters. For $(\sigma, v), (k, q) \in \mathcal{S}$, we define

$$G(\sigma, v, k, q) = \frac{g(\sigma + v) - g(k + q)}{(\sigma + v) - (k + q)}.$$

Hence,

$$G(\sigma, v, k, q) \ge M_f^{-1} > 0.$$

For a fixed $(k, q) \in \mathcal{S}$, let

$$G_j^n = G\left(\sigma_j^n, v_j^n, k, q \right),$$

where $\{(\sigma_j^n, v_j^n)\}$ denotes the solution of the difference scheme (2.10). Observe that it follows from (2.10) that

$$G_j^{n+1} - G_j^n = -\lambda \frac{G_j^{n+1} - G_j^n}{\left(\sigma_j^{n+1} + v_j^{n+1}\right) - \left(\sigma_j^n + v_j^n\right)} \cdot \frac{f\left(u_j^{n+1}\right) - f\left(u_j^n\right)}{u_j^{n+1} - u_j^n}\left(\sigma_j^n - \sigma_{j-1}^n\right).$$

Therefore, since f is increasing and g is concave (because $g'' = -f''/(f')^3 \le 0$), it follows that there is a positive constant M, depending only on f (or g), such that

$$G_j^{n+1} - G_j^n \le M \max\left(0, \sigma_j^n - \sigma_{j-1}^n\right). \tag{4.8}$$

Hence, we obtain from (4.8), (4.7), and Lemma 4.1 that

$$G_j^{n+1} - G_j^n \le M\Delta t, \tag{4.9}$$

where $M > 0$ is independent of δ and the mesh parameters.

LEMMA 4.2. *There is a positive constant M, independent of δ and the mesh parameters such that for any $(k,q) \in \mathcal{S}$ the solution of* (2.10) *satisfies*

$$\begin{aligned} &G_j^{n+1}\left(\left|\sigma_j^{n+1} - k\right| + \left|v_j^{n+1} - q\right|\right) \\ &\quad \le G_j^n\left(\left|\sigma_j^n - k\right| + \left|v_j^n - q\right|\right) - \lambda\left(\left|\sigma_j^n - k\right| - \left|\sigma_{j-1}^n - k\right|\right) \\ &\quad -\frac{\Delta t}{\delta} G_j^n R\left(\sigma_j^{n+1}, v_j^{n+1}\right)\left[sgn\left(\sigma_j^{n+1} - k\right) - sgn\left(v_j^{n+1} - q\right)\right] \\ &\quad +M\Delta t\left[\left|v_j^{n+1} - q\right| - \left(v_j^{n+1} - q\right)\, sgn\, \left(\sigma_j^{n+1} - q\right)\right], \end{aligned}$$

where, as above, $G_j^n = G(\sigma_j^n, v_j^n, k, q)$.

Proof. Let $(k,q) \in \mathcal{S}$. From the first equation in (2.10) we directly obtain

$$\begin{aligned} G_j^{n+1}\left(\sigma_j^{n+1} - k\right) &= G_j^n\left(\sigma_j^n - k\right) - \lambda\left(\sigma_j^n - \sigma_{j-1}^n\right) \\ &\quad - \left(G_j^{n+1} - G_j^n\right)\left(v_j^{n+1} - q\right) - G_j^n\left(v_j^{n+1} - v_j^n\right). \end{aligned}$$

Hence, by using the second equation of (2.10), this can be written in the form

$$\begin{aligned} G_j^{n+1}\left(\sigma_j^{n+1} - k\right) &= G_j^n\left(\sigma_j^n - k\right) - \lambda\left[\left(\sigma_j^n - k\right) - \left(\sigma_{j-1}^n - k\right)\right] \\ &\quad - \left(G_j^{n+1} - G_j^n\right)\left(v_j^{n+1} - q\right) - \frac{\Delta t}{\delta} G_j^n R_j^{n+1}, \end{aligned} \tag{4.10}$$

where $R_j^{n+1} = R(\sigma_j^{n+1}, v_j^{n+1})$.

The next step in the derivation is to multiply (4.10) by sgn $(\sigma_j^{n+1} - k)$. Observe that since $0 < \lambda \le M_f^{-1} \le G_j^n$, the inequality

$$\begin{aligned} &\left\{G_j^n\left(\sigma_j^n - k\right) - \lambda\left[\left(\sigma_j^n - k\right) - \left(\sigma_{j-1}^n - k\right)\right]\right\} \operatorname{sgn}\left(\sigma_j^{n+1} - k\right) \\ &\quad \le G_j^n\left|\sigma_j^n - k\right| - \lambda\left(\left|\sigma_j^n - k\right| - \left|\sigma_{j-1}^n - k\right|\right) \end{aligned}$$

holds. Hence, from (4.10), we obtain

$$\begin{aligned} G_j^{n+1}\left|\sigma_j^{n+1} - k\right| &\le G_j^n\left|\sigma_j^n - k\right| - \lambda\left(\left|\sigma_j^n - k\right| - \left|\sigma_{j-1}^n - k\right|\right) \\ &\quad - \left[\left(G_j^{n+1} - G_j^n\right)\left(v_j^{n+1} - q\right) + \frac{\Delta t}{\delta} G_j^n R_j^{n+1}\right] \operatorname{sgn}\left(\sigma_j^{n+1} - k\right). \end{aligned} \tag{4.11}$$

Next, write the second equation of (2.10) in the form

$$G_j^{n+1}\left(v_j^{n+1}-q\right)=G_j^n\left(v_j^n-q\right)+\left(G_j^{n+1}-G_j^n\right)\left(v_j^{n+1}-q\right)$$
$$+\frac{\Delta t}{\delta}G_j^n R_j^{n+1}.$$

Hence, if we multiply this equation by sgn $(v_j^{n+1}-q)$ and add the result to (4.11) we obtain the inequality

$$\begin{aligned}G_j^{n+1}\left(\left|\sigma_j^{n+1}-k\right|+\left|v_j^{n+1}-q\right|\right)&\le G_j^n\left(\left|\sigma_j^n-k\right|+\left|v_j^n-q\right|\right)\\&-\lambda\left(\left|\sigma_j^n-k\right|-\left|\sigma_{j-1}^n-k\right|\right)\\&-\frac{\Delta t}{\delta}G_j^nR_j^{n+1}\left[\operatorname{sgn}\left(\sigma_j^{n+1}-k\right)-\operatorname{sgn}\left(v_j^{n+1}-q\right)\right]\\&+\left(G_j^{n+1}-G_j^n\right)\left(v_j^{n+1}-q\right)\left[\operatorname{sgn}\left(v_j^{n+1}-q\right)-\operatorname{sgn}\left(\sigma_j^{n+1}-k\right)\right].\end{aligned}\tag{4.12}$$

However, note that

$$0\le\left(v_j^{n+1}-q\right)\left[\operatorname{sgn}\left(v_j^{n+1}-q\right)-\operatorname{sgn}\left(\sigma_j^{n+1}-k\right)\right];$$

therefore, it follows from the one-sided bound (4.9) that

$$\begin{aligned}&\left(G_j^{n+1}-G_j^n\right)\left(v_j^{n+1}-q\right)\left[\operatorname{sgn}\left(v_j^{n+1}-q\right)-\operatorname{sgn}\left(\sigma_j^{n+1}-k\right)\right]\\&\qquad\le M\Delta t\left[\left|v_j^{n+1}-q\right|-\left(v_j^{n+1}-q\right)\operatorname{sgn}\left(\sigma_j^{n+1}-q\right)\right],\end{aligned}$$

and hence the desired inequality follows from (4.12). □

Consider a real valued function $\mathcal{E}:\mathcal{S}\mapsto\mathbb{R}$ of the form

$$\mathcal{E}(\sigma,v)=\mathcal{L}(g(\sigma+v))+\int_{\mathcal{S}}P(k,q)G(\sigma,v,k,q)(|\sigma-k|+|v-q|)\,dk\,dq.$$

Here, $\mathcal{L}$ is a linear function and $P:\mathcal{S}\mapsto\mathbb{R}$ is a smooth, nonnegative function. Define, correspondingly,

$$\begin{aligned}\mathcal{F}(\sigma)&=\mathcal{L}(\sigma)+\int_{\mathcal{S}}P(k,q)|\sigma-k|\,dk\,dq,\\\mathcal{G}(\bar\sigma,\bar v,\sigma,v)&=\int_{\mathcal{S}}P(k,q)G(\bar\sigma,\bar v,k,q)R(\sigma,v)[\operatorname{sgn}(\sigma-k)-\operatorname{sgn}(v-q)]\,dk\,dq,\\\mathcal{H}(v)&=\int_{\mathcal{S}}P(k,q)\left[|v-q|-(v-q)\operatorname{sgn}(\sigma-k)\right]\,dk\,dq.\end{aligned}$$

It follows from (2.10) and by integrating the inequality of Lemma 4.2 that the solution of (2.10) satisfies the discrete entropy inequality

$$\begin{aligned}\mathcal{E}\left(\sigma_j^{n+1},v_j^{n+1}\right)&\le\mathcal{E}\left(\sigma_j^n,v_j^n\right)-\lambda\left[\mathcal{F}\left(\sigma_j^n\right)-\mathcal{F}\left(\sigma_{j-1}^n\right)\right]\\&\quad-\frac{\Delta t}{\delta}\mathcal{G}\left(\sigma_j^n,v_j^n,\sigma_j^{n+1},v_j^{n+1}\right)+M\Delta t\mathcal{H}\left(v_j^{n+1}\right).\end{aligned}\tag{4.13}$$

The properties of the entropy solutions of the system (2.1) will be derived from the corresponding properties of the finite difference solutions generated by the scheme (2.10). The convergence of the finite difference solutions is first established by a proper application of Helly's theorem, cf., e.g., [16].

Lemma 4.3. *Suppose* (σ^0, v^0) *is the initial data which satisfies all the assumptions in* (2.12) *and let* $(\sigma^N, v^N)_\Delta$ *be the piecewise constant representation of the data generated by the scheme* (2.10). *Then, as the mesh parameters* Δx *and* Δt *tend to zero, there is a subsequence of* $(\sigma^N, v^N)_\Delta$*, which converges in* $(L^1_{loc}(\mathbb{R} \times \mathbb{R}))^2$ *to a pair of functions* (σ, v)*. Furthermore,* $\sigma(\cdot, t), v(\cdot, t) \in BV$*, for all* $t \geq 0$*, and* $(\sigma(x, y), v(x, t)) \in \mathcal{S}$ *for* $(x, t) \in \mathbb{R} \times \mathbb{R}_0^+$*, and the following estimates hold:*

1. $(\sigma(x, t), v(x, t)) \in \mathcal{S}, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}_0^+,$
2. $TV(\sigma(\cdot, t)) + TV(v(\cdot, t)) \leq TV(\sigma^0) + TV(v^0),$
3. $\|p(\cdot, t)\|_1 \leq M\delta,$
4. $\|\sigma(\cdot, t) - \sigma(\cdot, \tau)\|_1 + \|v(\cdot, t) - v(\cdot, \tau)\|_1 \leq M|t - \tau|,$
5. $Lip^+(\sigma(\cdot, t)) \leq M Lip^+(\sigma^0), \quad Lip^+(v(\cdot, t)) \leq M Lip^+(v^0), \quad \forall t \geq 0.$

Here, M *is a constant independent of* t *and* δ*.*

From the entropy inequality in (4.13), we derived that the limit solution is the entropy solution of (2.1).

Lemma 4.4. *The limit solution* (σ, v) *constructed in Lemma* 4.3 *is the entropy solution of the system* (2.1)*, which satisfies the following Kruzkov-type inequality:*

$$\begin{aligned}(4.14)\quad &\int_0^T \int_{\mathbb{R}} [G(\sigma, v, k, q)(|\sigma - k| + |v - q|)\phi_t + |\sigma - k|\phi_x]\, dx\, dt \\ &+ \int_{\mathbb{R}} G\left(\sigma^0, v^0, k, q\right) \left(\left|\sigma^0 - k\right| + \left|v^0 - q\right|\right) \phi(x, 0)\, dx \\ &- \int_{\mathbb{R}} G(\sigma(x, T), v(x, T), k, q)(|\sigma(x, T) - k| + |v(x, T) - q|)\phi(x, T)\, dx \\ &+ M \int_0^T \int_{\mathbb{R}} [|v - q| - (v - q) sgn\ (\sigma - k)]\, \phi\, dx\, dt \\ &\geq \frac{1}{\delta} \int_0^T \int_{\mathbb{R}} G(\sigma, v, k, q) R(\sigma, v)[sgn\ (\sigma - k) - sgn\ (v - q)]\phi\, dx\, dt.\end{aligned}$$

Here, $(k, q) \in \mathcal{S}$ *and* $\phi \in \mathcal{D}_+(T)$ *is any test function with compact support. We recall that the function* $G = G(\sigma, v, k, q)$ *is defined as*

$$G(\sigma, v, k, q) = \frac{g(\sigma + v) - g(k + q)}{(\sigma + v) - (k + q)}.$$

Proof. Let $\phi \in \mathcal{D}_+(T)$ be a test function with compact support. We multiply the inequality in (4.13) by $\phi(x_j, t_n)$, then sum over $0 \leq n \leq N - 1$ and $j \in \mathcal{Z}$, and apply summation by parts with respect to n and j, and we obtain the following:

$$\begin{aligned}\Delta t \sum_{n=0}^{N-1} \Delta x \sum_{j \in \mathcal{Z}} &\left[\mathcal{E}\left(\sigma_j^{n+1}, v_j^{n+1}\right) \frac{\phi\left(x_j, t_{n+1}\right) - \phi\left(x_j, t_n\right)}{\Delta t} \right. \\ &+ \left. \mathcal{F}\left(\sigma_j^n\right) \frac{\phi\left(x_{j+1}, t_n\right) - \phi\left(x_j, t_n\right)}{\Delta x} \right] \\ &+ \Delta x \sum_{j \in \mathcal{Z}} \mathcal{E}\left(\sigma_j^0, v_j^0\right) \phi\left(x_j, t^0\right) - \Delta x \sum_{j \in \mathcal{Z}} \mathcal{E}\left(\sigma_j^N, v_j^N\right) \phi\left(x_j, t^N\right)\end{aligned}$$

$$+ \Delta t \sum_{n=0}^{N-1} \Delta x \sum_{j\in\mathcal{Z}} M\mathcal{H}\left(v_j^{n+1}\right)\phi\left(x_j,t_n\right)$$
$$\geq \frac{1}{\delta}\Delta t \sum_{n=0}^{N-1} \Delta x \sum_{j\in\mathcal{Z}} \mathcal{G}\left(\sigma_j^n, v_j^n, \sigma_j^{n+1}, v_j^{n+1}\right)\phi\left(x_j,t_n\right).$$

Now, by letting $\Delta x, \Delta t \to 0$ in the previous inequality, we get

$$\int_0^T \int_{\mathbb{R}} [\mathcal{E}(\sigma,v)\phi_t + \mathcal{F}(\sigma)\phi_x + M\mathcal{H}(v)\phi]\, dx\, dt$$
$$+ \int_{\mathbb{R}} [\mathcal{E}(\sigma^0,v^0)\phi(x,0) - \mathcal{E}(\sigma(x,T),v(x,T))\phi(x,T)]\, dx$$
$$\geq \frac{1}{\delta}\int_0^T \int_{\mathbb{R}} \mathcal{G}(\sigma,v,\sigma,v)\phi\, dx\, dt.$$

Hence, by choosing a sequence of smooth function pairs $(\mathcal{E}_\theta, \mathcal{F}_\theta, \mathcal{G}_\theta, \mathcal{H}_\theta)$ such that, as $\theta \to 0$,

$$\mathcal{E}_\theta \to G(\sigma,v,k,q)(|\sigma-k|+|v-q|),$$
$$\mathcal{F}_\theta \to |\sigma-k|,$$
$$\mathcal{G}_\theta \to G(\sigma,v,k,q)R(\sigma,v)[\text{ sgn }(\sigma-k) - \text{ sgn }(v-q)],$$
$$\mathcal{H}_\theta \to |v-q| - (v-q)\text{ sgn }(\sigma-k),$$

uniformly, and we get the inequality (4.14) in Lemma 4.4 by the dominated convergence theorem. ◻

The uniqueness and continuous dependence with respect to the initial data in L^1 is then obtained by the Kruzkov-type argument.

LEMMA 4.5. *Let* (σ,v) *and* $(\bar\sigma,\bar v)$ *be two entropy solutions of the system* (2.1) *with initial data* (σ^0,v^0) *and* $(\bar\sigma^0,\bar v^0)$*, respectively. Then,*

$$\|\sigma(\cdot,t)-\bar\sigma(\cdot,t)\|_{L^1} + \|v(\cdot,t)-\bar v(\cdot,t)\|_{L^1} \leq \bar{M}e^{Mt}\left[\left\|\sigma^0-\bar\sigma^0\right\|_{L^1} + \left\|v^0-\bar v^0\right\|_{L^1}\right].$$

Proof. The uniqueness of the entropy solutions is proved by generalizing the arguments by Kruzkov [5] for scalar conservation laws. In this paper, only the sketch of the proof is given, and we refer to [14, 16] for the details in the proof.

For any $\theta \in (0,1]$, we introduce the mollifier function ω_θ on $\mathbb{R}$ as

$$\omega_\theta(x) = \frac{1}{\theta}\Omega\left(\frac{x}{\theta}\right),$$

where $\Omega : \mathbb{R} \to \mathbb{R}$ is a nonnegative, symmetric $\mathcal{C}^\infty$-function with support in $[-1,1]$ and satisfying

$$\int_{\mathbb{R}} \Omega(x)\, dx = 1.$$

Let $T > 0$. In (4.14), we choose $(k,q) = (\bar\sigma(y,\tau), \bar v(y,\tau))$ and $\phi(x,t) = \omega_\theta(x-y)\omega_\theta(t-\tau)$ for solution (σ,v), and integrate over $\mathbb{R}\times[0,T]$ with respect to y and τ, and we get an inequality. For the solution $(\bar\sigma,\bar v)$, we perform a similar operation, but where we reverse the role of the variable (x,t) and (y,τ), we get another inequality. Now, adding these two inequalities, we get

$$L(\theta) + \frac{1}{\delta} l(\theta) \leq R(\theta) + 2Mr(\theta),$$

where

$$\begin{aligned} L(\theta) = &\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} G(\sigma(x,T), v(x,T), \bar{\sigma}, \bar{v})(|\sigma(x,T) - \bar{\sigma}| + |v(x,T) - \bar{v}|) \\ &\qquad\qquad \omega_\theta(x-y)\omega_\theta(T-\tau)\, dx\, dy\, d\tau \\ &+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} G(\bar{\sigma}(y,T), \bar{v}(y,T), \sigma, v)(|\bar{\sigma}(y,T) - \sigma| + |\bar{v}(y,T) - v|) \\ &\qquad\qquad \omega_\theta(x-y)\omega_\theta(T-\tau)\, dx\, dy\, d\tau \end{aligned}$$

and

$$\begin{aligned} R(\theta) = &\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} G(\sigma(x,0), v(x,0), \bar{\sigma}, \bar{v})(|\sigma(x,0) - \bar{\sigma}| + |v(x,0) - \bar{v}|) \\ &\qquad\qquad \omega_\theta(x-y)\omega_\theta(\tau)\, dx\, dy\, d\tau \\ &+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} G(\bar{\sigma}(y,0), \bar{v}(y,0), \sigma, v)(|\bar{\sigma}(y,0) - \sigma| + |\bar{v}(y,0) - v|) \\ &\qquad\qquad \omega_\theta(x-y)\omega_\theta(\tau)\, dx\, dy\, d\tau \end{aligned}$$

$$\begin{aligned} l(\theta) = &\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} G(\sigma, v, \bar{\sigma}, \bar{v})[\text{ sgn } (\sigma - \bar{\sigma}) - \text{ sgn } (v - \bar{v})] \\ &\qquad [R(\sigma, v) - R(\bar{\sigma}, \bar{v})]\omega_\theta(x-y)\omega_\theta(t-\tau)\, dx\, dt\, dy\, d\tau \end{aligned}$$

and

$$r(\theta) = 2\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} |v - \bar{v}| \omega_\theta(x-y)\omega_\theta(t-\tau)\, dx\, dt\, dy\, d\tau.$$

First we note that $l(\theta)$ is non-negative. In order to estimate the turns $L(\theta)$ and $R(\theta)$, we introduce the function $\mathcal{N}(t)$ as

$$\mathcal{N}(t) = \int_{\mathbb{R}} G(\sigma(x,t), v(x,t), \bar{\sigma}(x,t), \bar{v}(x,t))(|\sigma(x,t) - \bar{\sigma}(x,t)| + |v(x,t) - \bar{v}(x,t)|)\, dx.$$

Note that the function $\mathcal{N}(t)$ is equivalent to

$$A(t) := \|\sigma(\cdot,t) - \bar{\sigma}(\cdot,t)\|_{L^1} + \|v(\cdot,t) - \bar{v}(\cdot,t)\|_{L^1}$$

in the sense that there exist two positive constants, M_1, M_2, such that

$$M_1 A(t) \leq \mathcal{N}(t) \leq M_2 A(t). \tag{4.15}$$

Then, as $\theta \to 0$, we get(cf., e.g., [16])

$$L(\theta) \to \mathcal{N}(T), \quad R(\theta) \to \mathcal{N}(0),$$

and

$$r(\theta) \to 2M \int_0^T \|v(\cdot,t) - \bar{v}(\cdot,t)\|_{L^1} \, dt.$$

Combining these estimates we conclude, in the limit case as $\theta \to 0$, that

$$\mathcal{N}(T) \le \mathcal{N}(0) + M \int_0^T \mathcal{N}(t)\, dt,$$

where M is a finite constant independent of δ. Thus, it follows that

$$\mathcal{N}(T) \le \mathcal{N}(0) e^{MT},$$

and again, using (4.15), we get

$$\|\sigma(\cdot,t) - \bar{\sigma}(\cdot,t)\|_{L^1} + \|v(\cdot,t) - \bar{v}(\cdot,t)\|_{L^1} \le \bar{M} e^{Mt} \left[\left\|\sigma^0 - \bar{\sigma}^0\right\|_{L^1} + \left\|v^0 - \bar{v}^0\right\|_{L^1} \right],$$

where $\bar{M}$ and M are finite constants independent of δ. This completes the proof of Theorem 2.2. ∎

5. Rate of convergence towards equilibrium: Proof of Theorem 2.3 and Corollary 2.4. We recall that Lemma 4.3 establishes bounds, uniformly with respect to δ, on the solutions $(\sigma_\delta, v_\delta)$ of the non-equilibrium model (1.1) or (2.1). By combining these estimates with standard compactness arguments we could have concluded, more or less directly, that these solutions converge to a solution of the equilibrium model (1.2) or (2.2) as the relaxation parameter δ tends to zero. However, we are not only interested in convergence, but also in a rate of convergence. Hence, in order to prove the error estimates in Theorem 2.3 and Corollary 2.4, we shall follow the work of Tadmor [15] and Kurganov and Tadmor [6]. First we observe that the entropy solutions of (1.1) are weak solutions of a scalar equation with an "error term."

LEMMA 5.1. *Let (u,σ) (resp., (σ,v)) be the entropy solutions of* (1.1) *(resp.,* (2.1)*). Then the solutions u are weak solutions of the following "error equation"*

$$u_t + \mu f(u)_x = -R(\sigma,v)_x$$

in the sense that the following integral equation holds for all test functions $\phi \in \mathcal{D}_+(T)$:

$$\int_0^T \int_{\mathbb{R}} \left(u\phi_t + \mu f(u)_x \phi_x\right)\, dx\, dt + \int_{\mathbb{R}} [u(x,0)\phi(x,0) - u(x,T)\phi(x,T)]\, dx$$
$$= -\int_0^T \int_{\mathbb{R}} R(\sigma,v)\phi_x \, dx\, dt.$$

In addition, u satisfies the Lip^+ bound

$$\mathrm{Lip}^+(u(\cdot,t)) \le M, \quad \forall t \ge 0.$$

Proof. Let (σ,v) be the entropy solutions of (2.1). Then they satisfy the Kruzkov-type inequality given in (2.6). Choosing $(k = \sigma_m, q = v_m)$, where $\sigma_m = \min(\sigma)$ and $v_m = \min(v)$, (one can use, e.g., $k = q = 0$), the last terms on the left-hand side and the right-hand side are 0. Using the definition of G, the relation $u = g(\sigma + v)$, and the fact that (k,q) are constants, we get

$$\int_0^T \int_{\mathbb{R}} [u\phi_t + \sigma\phi_x]\, dx\, dt + \int_{\mathbb{R}} (u(x,0)\phi(x,0) - u(x,T)\phi(x,T)) \ge 0.$$

Similarly, by choosing ($k = \sigma_M, q = v_M$), where $\sigma_M = \max(\sigma)$ and $v_M = \max(v)$ (e.g., $k = \mu, q = 1 - \mu$), we get

$$\int_0^T \int_{\mathbb{R}} [u\phi_t + \sigma\phi_x] \ dx \ dt + \int_{\mathbb{R}} (u(x,0)\phi(x,0) - u(x,T)\phi(x,T)) \leq 0.$$

These two inequalities lead to

$$\int_0^T \int_{\mathbb{R}} [u\phi_t + \sigma\phi_x] \ dx \ dt + \int_{\mathbb{R}} [u(x,0)\phi(x,0) - u(x,T)\phi(x,T)] = 0.$$

Furthermore, using the relation

$$\sigma - \mu f(u) = \sigma - \mu(\sigma + v) = (1-\mu)\sigma - \mu v = R(\sigma, v),$$

we get the weak formulation in Lemma 5.1, and thus u is a weak solution of the error equation. Finally, the Lip^+ bound follows from the monotonicity of the function g. ∎

Let $T > 0$ be given and define $E = -R_x = -p_x$. Hence, $u = u_\delta$ is a weak solution of the inhomogeneous equation

$$u_t + \mu f(u)_x = E,$$

and $\bar{u}$ is a solution of the corresponding homogeneous equation (1.2). Furthermore, these solutions satisfy an Oleinik condition of the form

$$\text{Lip}^+(u(\cdot,t)), \ \text{Lip}^+(\bar{u}(\cdot,t) \leq M, \quad \forall t \geq 0.$$

Since the flux function f is convex, we can therefore conclude from the arguments in Kurganov and Tadmor [6] that the following stability estimate holds:

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{\text{Lip}'} \leq M \sup_{0\leq\tau\leq t} \|E(\cdot,\tau)\|_{\text{Lip}'}, \quad 0 \leq t \leq T.$$

From Lemma 4.3 we obtain that

$$\|E(\cdot,t)\|_{\text{Lip}'} \leq \|p(\cdot,t)\|_{L^1} \leq M\delta.$$

This completes the proof of Theorem 2.3.

The L^p estimate in Corollary 2.4 can be proved by interpolation between the Lip′-error estimate in Theorem 2.3 and the BV-boundness of the error, exactly in the same way as is done in Nessyahu and Tadmor [11]. We therefore omit the details.

The L^1 estimate for $\sigma - \bar{\sigma}$ follows from the L^1 estimate for $u - \bar{u}$. To be precise, since $\bar{\sigma} = \mu f(\bar{u})$, we have

$$\begin{aligned}\|\sigma - \bar{\sigma}\|_{L^1} &= \|\sigma - \mu f(u) + \mu f(u) - \bar{\sigma}\|_{L^1} \leq \|\sigma - \mu f(u)\|_{L^1} + \|\mu(f(u) - f(\bar{u}))\|_{L^1} \\ &\leq \|p\|_{L^1} + M\|u(\cdot,t) - \bar{u}(\cdot,t)\|_{L^1} \\ &\leq M\sqrt{\delta},\end{aligned}$$

which gives the second estimate in Corollary 2.4.

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