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Admissible Concentration Factors for Edge Detection from Non-uniform Fourier Data

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Abstract
Edge detection from Fourier data has been emerging in many applications. The concentration factor method has been widely used in detecting edges from Fourier data. We present a theoretic analysis of the concentration factor method for non-uniform Fourier data in this paper. Specifically, we propose admissible conditions for the concentration factors such that the edge detector converges to a smoothed approximation of the jump function. Moreover, we also introduce some specific choices of admissible concentration factors and present estimates of convergence rates correspondingly.

1 Introduction

Detecting edges (discontinuities) of a piece-wise smooth functions plays an important role in many applications, such as detection of the bright band in radar data [11], brain tumor detection [12], lane detection in autonomous driving [18]. When the collected data are within the physical domain, many efficient numerical methods of edge detection have been developed, such as Canny edge detector [1] and Sobel operator [16]. On the other hand side, in applications such as magnetic resonance imaging (MRI) and synthetic aperture radar (SAR), the data are frequently collected in the frequency (Fourier) domain.

It will be much more challenging to detect edges from Fourier (spectral) data, since the Fourier data are essentially global information of the unknown function and the edges are local features. The concentration factor method has been introduced in [8] to detect edges from uniform Fourier data. In particular, a characterization of admissible concentration factors is

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presented in [8] and convergence analysis are also included there. Moreover, various refined concentration factor methods have been discussed in [5,9,10,15].

There have also been effort in developing concentration factor methods [3,6,7,14,17] for non-uniform Fourier data due to its emerging popularity in applications such as MRI and SAR. In particular, [6] obtain certain analytic forms of the concentration factors through utilization the Fourier frames, while [14,17] try to compute the concentration factors by solving discrete optimization problems. Moreover, [7] incorporates the Fourier frames approximation into an optimization model and solve the concentration factors according to a priori information on the discontinuous pattern of the underlying function. While the analysis of the concentration factor method for uniform Fourier data has been well studied, its analysis for nonuniform Fourier data has not been fully developed yet. In this paper, we will employ the tools in frame theory to analyze the concentration methods for non-uniform Fourier data. In particular, we will present *admissible conditions* on concentration factors for non-uniform Fourier data such that the edge detector converges to the edge function. Specifically, we will view the edge detection problem as a function approximation of a "smoothed" edge function. In this regard, we will employ the results of function approximation with Fourier frames in [13] to derive admissible conditions such that the function approximation converges. Moreover, we will also introduce some specific choices of admissible concentration factors and develop the convergence analysis for such choices correspondingly.

We will briefly introduce the mathematical formulation of detecting edges from non-uniform Fourier data and present the concentration factor method for non-uniform Fourier data in Sect. 2. We then propose in Sect. 3 admissible conditions for concentration factors through analyzing the error between the proposed edge detector and a "smoothed" edge function. Moreover, we provide some examples of admissible concentration factors in Sect. 4 and present the convergence analysis correspondingly there. Numerical examples will be shown in Sect. 5 to demonstrate the performance of our proposed methods. Finally, we make some concluding remarks in Sect. 6.

2 Edge Detection from Non-uniform Fourier Data

In this section, we will introduce the concentration factor method of detecting the edges of a piece-wise smooth function from its non-uniform Fourier data. Specifically, we consider an unknown function $f : \mathbb{R} \to \mathbb{R}$ that is supported on $[0, 1]$ and piece-wise smooth. We assume f has finitely many jump discontinuities in $(0, 1)$. For some non-uniform frequencies $\lambda_j \in \mathbb{R}$, $-m \le j \le m$, we let

$$\psi_j(x) = \mathrm{e}^{2\pi i \lambda_j x}, \quad x \in [0, 1] \tag{2.1}$$

and we are given the following non-uniform Fourier data

$$\hat{f}(\lambda_j) = \langle f, \psi_j \rangle, \quad -m \le j \le m, \tag{2.2}$$

where the above inner product is defined as $\langle g_1, g_2 \rangle = \int_0^1 g_1(x)\overline{g_2(x)}dx$ for $g_1, g_2 \in L^2[0, 1]$. We will try to locate all the jump discontinuities from such finite non-uniform Fourier data.

We shall next reformulate the edge detection problem as a function approximation problem. To this end, we will define the *jump function* by

$$[f](x) = f(x^+) - f(x^-), \quad x \in [0, 1].$$

We point out that detecting the jump discontinuities is equivalent to recover the above jump function. In simplicity of presentation, we will assume the underlying function f has only a single jump discontinuity $\xi \in (0, 1)$ in the following. We will see later that our method could be extended directly to the more general case of multiple jump discontinuities. It follows that

$$[f](x) = [f](\xi)\chi_\xi(x), \quad x \in [0, 1],$$

where the indicator function $\chi_\xi(x)$ is 1 if $x = \xi$ and 0 otherwise.

We note that the indicator function is nontrivial only in a single point with measure zero and it is not practical to obtain its approximation in $L^2[0, 1]$ directly. Instead, we will consider a smoothed version of the indicator function. That is, instead of recovering the indicator function, we will consider the following smoothed approximation

$$h_\epsilon(x) := h\left(\frac{x - \xi}{\epsilon}\right), \tag{2.3}$$

where h is a function centered around 0 taking a peak value 1 at 0 and ϵ is a positive constant. We will discuss more about the choices of h later. It is direct to observe that h_ϵ would be centered around ξ and would be close to the indicator function χ_ξ when the parameter ϵ is close to 0. For practical purpose, we often require h to be bell-shaped around 0 and ϵ to be small.

We shall next turn to approximate the smoothed jump function $[f](\xi)h_\epsilon$ from the given non-uniform Fourier data $\hat{f}(\lambda_j), -m \leq j \leq m$. We first give a conceptual description of our method of approximating $[f](\xi)h_\epsilon$. Specifically, we will derive an approximation by using its frame expansion. To this end, we review the definition of frames below.

Suppose $\mathcal{H}$ is a separable Hilbert space. We say $\{f_j : j \in \mathbb{N}\}$ is a *frame* [2] in $\mathcal{H}$ if there exist positive constants A and B such that for any $f \in \mathcal{H}$,

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle_{\mathcal{H}}|^2 \leq B\|f\|_{\mathcal{H}}^2.$$

We use S to denote the *frame operator* as

$$Sf = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle f_j, \quad f \in \mathcal{H},$$

and $\tilde{f}_j = S^{-1} f_j$, $j \in \mathbb{N}$, is the *canonical dual frame*. It is direct to observe that we have the following frame expansion for any $f \in \mathcal{H}$

$$f = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle \tilde{f}_j.$$

In this paper, we will consider the Hilbert space $\mathcal{H} = L^2[0, 1]$ and choose $\{\psi_j : j \in \mathbb{Z}\}$ to be a frame in $L^2[0, 1]$ with frame bounds A, B. It follows immediately that we could write the smoothed jump function $[f](\xi)h_\epsilon$ as follows

$$[f](\xi)h_\epsilon = \sum_{j\in\mathbb{Z}} \langle [f](\xi)h_\epsilon, \psi_j\rangle \widetilde{\psi}_j,$$

where $\{\widetilde{\psi}_j : j \in \mathbb{Z}\}$ is the canonical dual frame of $\{\psi_j : j \in \mathbb{Z}\}$. We will derive approximations of $\langle [f](\xi)h_\epsilon, \psi_j\rangle$ and $\widetilde{\psi}_j$ separately.

We start with an approximation of $\langle [f](\xi)h_\epsilon, \psi_j\rangle$. It follows from the definition of h_ϵ in (2.3) and the definition of ψ_j in (2.1) that

$$\langle [f](\xi)h_\epsilon, \psi_j\rangle = \int_0^1 [f](\xi)h\left(\frac{t-\xi}{\epsilon}\right)\mathrm{e}^{-2\pi i\lambda_j t}dt.$$

A direct calculation through a change of variable gives that

$$\langle [f](\xi)h_\epsilon, \psi_j\rangle = [f](\xi)\epsilon\mathrm{e}^{-2\pi i\lambda_j\xi}\int_{-\xi/\epsilon}^{(1-\xi)/\epsilon} h(u)\mathrm{e}^{-2\pi i\lambda_j u\epsilon}du.$$

We let

$$R_j(\epsilon) = [f](\xi)\epsilon\mathrm{e}^{-2\pi i\lambda_j\xi}\left(\int_{-\infty}^{-\xi/\epsilon} h(u)\mathrm{e}^{-2\pi i\lambda_j u\epsilon}du + \int_{(1-\xi)/\epsilon}^{\infty} h(u)\mathrm{e}^{-2\pi i\lambda_j u\epsilon}du\right). \tag{2.4}$$

It follows that

$$\langle [f](\xi)h_\epsilon, \psi_j\rangle = [f](\xi)\epsilon\mathrm{e}^{-2\pi i\lambda_j\xi}\hat{h}(\lambda_j\epsilon) - R_j(\epsilon). \tag{2.5}$$

We will further obtain an approximation of $\langle [f](\xi)h_\epsilon(t), \psi_j(t)\rangle$ from the given non-uniform Fourier data $\hat{f}(\lambda_j)$, $-m \le j \le m$. To this end, we first derive another equivalent form of $\hat{f}(\lambda_j)$. It follows from the definition of $\hat{f}_j$ in (2.2) that

$$\hat{f}_j = \langle f, \psi_j\rangle = \int_0^1 f(x)\mathrm{e}^{-2\pi i\lambda_j x}dx, \quad -m \le j \le m.$$

An integration by parts yields that for $\lambda_j \ne 0$

$$\hat{f}_j = \frac{1}{2\pi i\lambda_j}[f](\xi)\mathrm{e}^{-2\pi i\lambda_j\xi} + \frac{1}{2\pi i\lambda_j}\int_0^1 f'(x)\mathrm{e}^{-2\pi i\lambda_j x}dx, \quad -m \le j \le m.$$

Also note that $[f](\xi) = -\int_0^1 f'(x)dx$. It implies for all $-m \le j \le m$

$$[f](\xi)\mathrm{e}^{-2\pi i\lambda_j\xi} = 2\pi i\lambda_j\hat{f}_j - \int_0^1 f'(x)\mathrm{e}^{-2\pi i\lambda_j x}dx.$$

Combining the above equality with (2.5), we have

$$\langle [f](\xi)h_\epsilon, \psi_j\rangle = 2\pi i\lambda_j\hat{f}_j\epsilon\hat{h}(\lambda_j\epsilon) - T_j(\epsilon) - R_j(\epsilon). \tag{2.6}$$

where

$$T_j(\epsilon) = \epsilon\hat{h}(\lambda_j\epsilon)\int_0^1 f'(x)\mathrm{e}^{-2\pi i\lambda_j x}dx. \tag{2.7}$$

We point out that we will choose appropriate h and ϵ to make both $T_j(\epsilon)$ and $R_j(\epsilon)$ small. We will present more details about the error analysis in next section. In this regard, we will use $2\pi i\lambda_j\hat{f}_j\epsilon\hat{h}(\lambda_j\epsilon)$ as an approximation of $\langle [f](\xi)h_\epsilon(t), \psi_j(t)\rangle$.

We next present an approximation of $\widetilde{\psi}_j$. We remark that we do not have the closed form of the canonical dual frame in general. That is, we will need to derive a numerical approximation. We will employ the admissible frame approach introduced in [13]. In particular, we will use the following approximation of $\widetilde{\psi}_j$:

$$\widetilde{\psi}_j^\dagger = \sum_{l=-n}^{n} b_{l,j}\phi_l, \tag{2.8}$$

where $\phi_l(x) = e^{2\pi i l x}$, $\mathrm{B} = [b_{l,j}]_{l=-n,j=-m}^{n,m}$ is the Moore-Penrose pseudo-inverse of $\left[\langle \psi_j, \phi_l \rangle\right]_{j=-m,l=-n}^{m,n}$, and n has to satisfy certain sampling rate condition as shown in [13]. We note that the error analysis of this dual frame approximation is presented in [13]. We will discuss more details in next section.

We are now ready to present the edge detection method of approximating the smoothed jump function $[f](\xi)h_\epsilon(t)$ through combining the approximation of $\langle [f](\xi)h_\epsilon(t), \psi_j(t)\rangle$ in (2.6) and the approximation of $\widetilde{\psi}_j$ in (2.8):

$$E_m(\epsilon) = \sum_{|j|\le m} 2\pi i \lambda_j \hat{f}_j \epsilon \hat{h}(\lambda_j \epsilon)\, \widetilde{\psi}_j^\dagger. \tag{2.9}$$

We remark that we could view $\sigma_j = \epsilon \hat{h}(\lambda_j \epsilon)$ as the *concentration factors* for edge detection with non-uniform Fourier data. It shares a similar form as that in edge detection with uniform Fourier data. We will discuss in next section how to choose σ_j, in particularly, h and ϵ, to obtain a convergent approximation to the smoothed jump function $[f](\xi)h_\epsilon(t)$.

3 Error Analysis and Admissible Conditions

In this section, we will present the error analysis of the edge detection method (2.9) of approximating the smoothed jump function $[f](\xi)h_\epsilon(t)$. In particular, we will investigate the admissible conditions of h and ϵ such that the edge detector $E_m(\epsilon)$ converges to the smoothed jump function $[f](\xi)h_\epsilon(t)$. Furthermore, we will analyze the convergence rates for some specific choices of h and ϵ_n. We point out that we will use the uniform metric to measure the error. Since oscillations in the approximation might lead to false detections of the edges, we will try to suppress the oscillations by requiring the convergence in the uniform metric, which is used in [4] for detecting edges from uniform Fourier data.

We shall focus on estimating the approximation error $\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty$, where $\|\cdot\|_\infty$ denotes the supremum norm in $L^\infty[0, 1]$. We first decompose the error into a few terms. To this end, we let

$$h_{\epsilon,m} = \sum_{|j|\le m} \langle h_\epsilon, \psi_j\rangle \widetilde{\psi}_j^\dagger. \tag{3.1}$$

It is direct to observe that

$$\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty \le \|E_m(\epsilon) - [f](\xi)h_{\epsilon,m}\|_\infty + \|[f](\xi)h_{\epsilon,m} - [f](\xi)h_\epsilon\|_\infty$$

It follows from (2.6), (2.9), and (3.1) that

$$\|E_m(\epsilon) - [f](\xi)h_{\epsilon,m}\|_\infty \le \left\| \sum_{|j|\le m} T_j(\epsilon)\widetilde{\psi}_j^\dagger \right\|_\infty + \left\| \sum_{|j|\le m} R_j(\epsilon)\widetilde{\psi}_j^\dagger \right\|_\infty.$$

We denote

$$\kappa_m = \max_{|j|\le m} \|\widetilde{\psi}_j^\dagger\|_\infty. \tag{3.2}$$

A direct computation yields

$$\|E_m(\epsilon) - [f](\xi)h_{\epsilon,m}\|_\infty \le \kappa_m \sum_{|j|\le m} |T_j(\epsilon)| + \kappa_m \sum_{|j|\le m} |R_j(\epsilon)|.$$

Consequently, we have

$$\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty \le \kappa_m \sum_{|j|\le m} |T_j(\epsilon)| + \kappa_m \sum_{|j|\le m} |R_j(\epsilon)| + |[f](\xi)|\|h_{\epsilon,m} - h_\epsilon\|_\infty. \tag{3.3}$$

We will estimate the three terms on the right hand side of the above inequality separately.

We start with an estimate of $\sum_{|j|\le m} |T_j(\epsilon)|$.

Proposition 3.1 *If f is piece-wise smooth with a single jump at $\xi \in (0, 1)$, then*

$$\sum_{|j|\le m} |T_j(\epsilon)| \le \frac{4\|f'\|_\infty + \|f''\|_\infty}{2\pi} \epsilon \sum_{|j|\le m, \lambda_j \ne 0} \frac{|\hat{h}(\lambda_j \epsilon)|}{\lambda_j} + \epsilon|\hat{h}(0)|\|f'\|_\infty.$$

Proof It follows from an integration by parts that for $\lambda_j \ne 0$

$$\int_0^1 f'(x)\mathrm{e}^{-2\pi i\lambda_j x}dx = \frac{1}{-2\pi i\lambda_j}\Bigg[f'(x)\mathrm{e}^{-2\pi i\lambda_j x}\Big|_{x=0}^{x=\xi^-} + f'(x)\mathrm{e}^{-2\pi i\lambda_j x}\Big|_{x=\xi^+}^{x=1} - \int_0^1 f''(x)\mathrm{e}^{-2\pi i\lambda_j x}dx\Bigg].$$

A direct computation yields for $\lambda_j \ne 0$

$$\left|\int_0^1 f'(x)\mathrm{e}^{-2\pi i\lambda_j x}dx\right| \le \frac{4\|f'\|_\infty + \|f''\|_\infty}{2\pi\lambda_j}.$$

When $\lambda_j = 0$, we have

$$\left|\int_0^1 f'(x)\mathrm{e}^{-2\pi i\lambda_j x}dx\right| \le \|f'\|_\infty.$$

The desired result follows from substituting the above two inequalities into the definition of $T_j(\epsilon)$ in (2.7). □

We next estimate the second term $\sum_{|j|\le m} |R_j(\epsilon)|$ in (3.3).

Proposition 3.2 *For any $m \in \mathbb{N}$ and $\epsilon > 0$, there holds that*

$$\sum_{|j|\le m} |R_j(\epsilon)| \le (2m+1)|[f](\xi)|\epsilon\left(\int_{-\infty}^{-\xi/\epsilon} |h(u)|du + \int_{(1-\xi)/\epsilon}^{\infty} |h(u)|du\right)$$

Proof A direct computation from the definition of $R_j(\epsilon)$ in (2.4) yields that for any $j \in \mathbb{N}$,

$$|R_j(\epsilon)| \le |[f](\xi)|\epsilon\left(\int_{-\infty}^{-\xi/\epsilon} |h(u)|du + \int_{(1-\xi)/\epsilon}^{\infty} |h(u)|du\right),$$

which implies the desired result immediately. □

It remains to estimate the third term $\|h_{\epsilon,m} - h_\epsilon\|_\infty$ in (3.3). We point out that we always assume h is a bell-shaped function satisfying the following conditions:

$$\begin{cases} h(0) = \max_{x\in\mathbb{R}} h(x) = 1;\ h(x) \ge 0, \quad x \in \mathbb{R} \\ h \text{ is decreasing on } (0,\infty) \text{ and increasing on } (-\infty, 0). \end{cases} \tag{3.4}$$

In particular, we consider a specific Fourier frame given by the jittered non-uniform sampling frequencies:

$$\lambda_j = j + \delta_j, \quad -m \leq j \leq m, \tag{3.5}$$

where δ_j is uniformly distributed in $[-\frac{1}{4}, \frac{1}{4}]$. It is well known [19] that it constitutes a frame in $L^2[0, 1]$.

Proposition 3.3 *Suppose $\{\lambda_k\}$ is the jittered sampling with A, B as the corresponding frame bounds and $n = \frac{A\pi^2}{A\pi^2+128}m$. There holds that*

$$\begin{aligned}\|h_{\epsilon,m} - h_\epsilon\|_\infty \leq\ & \sqrt{2n+1}\frac{2B}{A}\epsilon\left(\sum_{|l|>n} \hat{h}^2(\epsilon l)\right)^{1/2} + \epsilon \sum_{|l|>n} \left|\hat{h}(\epsilon l)\right| \\ & + \left(1 + \frac{2B}{A}\sqrt{2n+1}\right)\left[\sum_{k\geq 1} h\left(\frac{k-\xi}{\epsilon}\right) + \sum_{k\leq -1} h\left(\frac{1+k-\xi}{\epsilon}\right)\right].\end{aligned}$$

Proof We will estimate the difference of h_ϵ and $h_{\epsilon,m}$ through their shifted versions in the following:

$$H_\epsilon(x) = \sum_{k\in\mathbb{Z}} h_\epsilon(x+k) \quad \text{and} \quad H_{\epsilon,m} = \sum_{k\in\mathbb{Z}} h_{\epsilon,m}(x+k). \tag{3.6}$$

That is, we shall estimate $\|h_{\epsilon,m} - h_\epsilon\|_\infty$ through the following decomposition:

$$\|h_{\epsilon,m} - h_\epsilon\|_\infty \leq \|H_{\epsilon,m} - H_\epsilon\|_\infty + \|H_\epsilon - h_\epsilon\|_\infty + \|H_{\epsilon,m} - h_{\epsilon,m}\|_\infty. \tag{3.7}$$

We first estimate $\|H_\epsilon - H_{\epsilon,m}\|_\infty$. To this end, we define $Q_n(g) = \sum_{|l|\leq n}\langle g, \phi_l\rangle\phi_l$ for $g \in L^2[0, 1]$, which is the standard Fourier partial sum. It is direct to observe that

$$\|H_{\epsilon,m} - H_\epsilon\|_\infty \leq \|H_{\epsilon,m} - Q_n H_\epsilon\|_\infty + \|Q_n H_\epsilon - H_\epsilon\|_\infty.$$

We start with an estimate of $\|H_{\epsilon,m} - Q_n H_\epsilon\|_\infty$. In particular, we will employ the L^2 estimate in [13] to derive the L^∞ estimate. Note that from (2.8) and (3.1), we know $H_{\epsilon,m} \in \operatorname{span}\{\phi_l : -n \leq l \leq n\}$. On the other hand, we also have $Q_n H_\epsilon \in \operatorname{span}\{\phi_l : -n \leq l \leq n\}$. It implies

$$\|Q_n H_\epsilon - H_{\epsilon,m}\|_\infty \leq \sqrt{2n+1}\|Q_n H_\epsilon - H_{\epsilon,m}\|_2,$$

where $\|\cdot\|_2$ denote the norm in $L^2[0, 1]$. We point out that the condition $n = \frac{A\pi^2}{A\pi^2+128}m$ would ensure the convergence of the admissible frame method in [13]. Specifically, by combining the results in Theorem 5.1, Lemma 5.4, and Equation (5.2) in [13], we obtain that

$$\|Q_n H_\epsilon - H_{\epsilon,m}\|_2 \leq \frac{2B}{A}\left(\sum_{|l|>n} |\langle H_\epsilon, \phi_l\rangle|^2\right)^{1/2},$$

which implies

$$\|Q_n H_\epsilon - H_{\epsilon,m}\|_\infty \leq \sqrt{2n+1}\frac{2B}{A}\left(\sum_{|l|>n} |\langle H_\epsilon, \phi_l\rangle|^2\right)^{1/2}. \tag{3.8}$$

It follows from the definition of H_ϵ in (3.6) and the definition of h_ϵ in (2.3) that

$$\langle H_\epsilon, \phi_l\rangle = \sum_{k\in\mathbb{Z}} \int_0^1 h\left(\frac{t+k-\xi}{\epsilon}\right) e^{-2\pi i l t} dt.$$

A simple substitution yields

$$\langle H_\epsilon, \phi_l\rangle = \epsilon \mathrm{e}^{-2\pi i l\xi}\int_{-\infty}^{\infty} h(u)\mathrm{e}^{-2\pi i\epsilon l u}du = \epsilon \mathrm{e}^{-2\pi i l\xi}\hat{h}(\epsilon l). \tag{3.9}$$

Substitute it into (3.8) and we have

$$\|Q_nH_\epsilon - H_{\epsilon,m}\|_\infty \le \sqrt{2n+1}\frac{2B}{A}\epsilon\left(\sum_{|l|>n}\hat{h}^2(\epsilon l)\right)^{1/2}. \tag{3.10}$$

We next estimate the second term $\|Q_nH_\epsilon - H_\epsilon\|_\infty$. It follows from a direct computation that

$$\|Q_nH_\epsilon - H_\epsilon\|_\infty = \left\|\sum_{|l|>n}\langle H_\epsilon, \phi_l\rangle\phi_l\right\|_\infty \le \sum_{|l|>n}\left|\langle H_\epsilon, \phi_l\rangle\right|.$$

Substituting (3.9) into the above inequality yields

$$\|Q_nH_\epsilon - H_\epsilon\|_\infty \le \epsilon\sum_{|l|>n}\left|\hat{h}(\epsilon l)\right|,$$

which combined with Eq. (3.10) implies

$$\|H_\epsilon - H_{\epsilon,m}\|_\infty \le \sqrt{2n+1}\frac{2B}{A}\epsilon\left(\sum_{|l|>n}\hat{h}^2(\epsilon l)\right)^{1/2} + \epsilon\sum_{|l|>n}\left|\hat{h}(\epsilon l)\right|. \tag{3.11}$$

It remains to estimate the second term $\|H_\epsilon - h_\epsilon\|_\infty$ and the third term $\|H_{\epsilon,m} - h_{\epsilon,m}\|_\infty$ in the error decomposition (3.7). It follows from a direct computation from the definition of H_ϵ in (3.6) that

$$\|H_\epsilon - h_\epsilon\|_\infty = \left\|\sum_{|k|\ge 1}h_\epsilon(x+k)\right\|_\infty = \left\|\sum_{|k|\ge 1}h\left(\frac{x+k-\xi}{\epsilon}\right)\right\|_\infty.$$

By the assumption of h in (3.4), we have

$$\|H_\epsilon - h_\epsilon\|_\infty \le \sum_{k\ge 1}h\left(\frac{k-\xi}{\epsilon}\right) + \sum_{k\le -1}h\left(\frac{1+k-\xi}{\epsilon}\right). \tag{3.12}$$

We next estimate $\|H_{\epsilon,m} - h_{\epsilon,m}\|_\infty$. It follows from Theorem 5.1 and Equation (5.1) in [13] that

$$\|H_{\epsilon,m} - h_{\epsilon,m}\|_2 \le \frac{2B}{A}\|H_\epsilon - h_\epsilon\|_2.$$

Note that $H_{\epsilon,m} - h_{\epsilon,m} \in \operatorname{span}\{\phi_l : -n \le l \le n\}$, which implies $\|H_{\epsilon,m} - h_{\epsilon,m}\|_\infty \le \sqrt{2n+1}\|H_{\epsilon,m} - h_{\epsilon,m}\|_2$. On the other hand, we have $\|H_\epsilon - h_\epsilon\|_2 \le \|H_\epsilon - h_\epsilon\|_\infty$. It follows that

$$\|H_{\epsilon,m} - h_{\epsilon,m}\|_\infty \le \frac{2B}{A}\sqrt{2n+1}\|H_\epsilon - h_\epsilon\|_\infty. \tag{3.13}$$

Consequently, substituting (3.11), (3.12), and (3.13) into (3.7) yields the desired result. □

We are now ready to combine these above estimates to obtain an estimate of the approximation error $\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty$.

Theorem 3.4 *Suppose f is piece-wise smooth with a single jump at $\xi \in (0,1)$, $\{\lambda_k\}$ is the jittered sampling with A, B as the corresponding frame bounds and $n = \frac{A\pi^2}{A\pi^2+128}m$. If the bell-shaped function h satisfies the assumption (3.4), then for any $m \in \mathbb{N}$ and $\epsilon > 0$*

$$\begin{aligned}\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty \le\ & \frac{2}{A}\frac{4\|f'\|_\infty + \|f''\|_\infty}{2\pi}\epsilon \sum_{|j|\le m, \lambda_j \ne 0} \frac{|\hat{h}(\lambda_j\epsilon)|}{\lambda_j} + \frac{2}{A}\epsilon|\hat{h}(0)|\|f'\|_\infty \\ & + \frac{2}{A}\sqrt{2n+1}(2m+1)|[f](\xi)|\epsilon\left(\int_{-\infty}^{-\xi/\epsilon}|h(u)|du + \int_{(1-\xi)/\epsilon}^{\infty}|h(u)|du\right) \\ & + |[f](\xi)|\sqrt{2n+1}\frac{2B}{A}\epsilon\left(\sum_{|l|>n}\hat{h}^2(\epsilon l)\right)^{1/2} + |[f](\xi)|\epsilon\sum_{|l|>n}\left|\hat{h}(\epsilon l)\right| \\ & + |[f](\xi)|\left(1 + \frac{2B}{A}\sqrt{2n+1}\right)\left[\sum_{k\ge 1} h\left(\frac{k-\xi}{\epsilon}\right) + \sum_{k\le -1} h\left(\frac{1+k-\xi}{\epsilon}\right)\right].\end{aligned}$$

Proof We first obtain an estimate of κ_m as defined in (3.2). It follows from Theorem 5.1 and Equation (4.3) in [13] that for $1 \le j \le m$,

$$\|\widetilde{\psi}_j^\dagger\|_2 \le \frac{2}{A}\|\psi_j\|_2 = \frac{2}{A}.$$

Moreover, we note that for $1 \le j \le m$, $\widetilde{\psi}_j^\dagger \in \text{span}\{\phi_l : -n \le l \le n\}$, which implies

$$\|\widetilde{\psi}_j^\dagger\|_\infty \le \sqrt{2n+1}\|\widetilde{\psi}_j^\dagger\|_2 \le \frac{2}{A}\sqrt{2n+1}.$$

It follows that

$$\kappa_m \le \frac{2}{A}\sqrt{2n+1}.$$

This combined with Propositions 3.1, 3.2, 3.3, and the error decomposition (3.3) yields the desired result. □

In the rest of this section, we will discuss the admissible conditions for the choices of h and ϵ such that $\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty$ converges to 0. In particular, we will choose ϵ_m depending on m and the bell-shaped function h according to the following *admissible conditions*: when $m \to \infty$

(i) $\sqrt{m}\epsilon_m \sum_{|j|\le m} \frac{\hat{h}(\lambda_j\epsilon_m)}{\lambda_j} \to 0$

(ii) $m^{3/2}\epsilon_m\left(\int_{-\infty}^{-\xi/\epsilon_m}|h(u)|du + \int_{(1-\xi)/\epsilon_m}^{\infty}|h(u)|du\right) \to 0$

(iii) $\sqrt{m}\epsilon_m\left(\sum_{|l|>n}\hat{h}^2(\epsilon_m l)\right)^{1/2} \to 0$

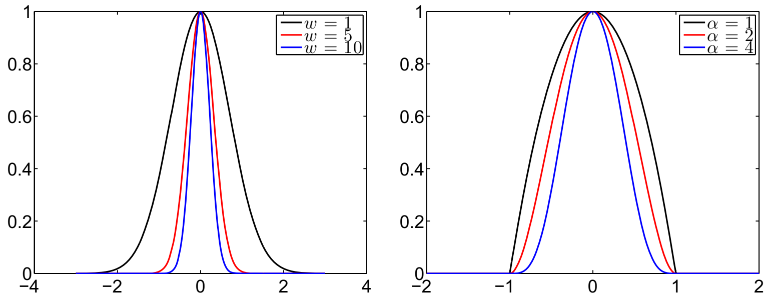
(iv) $\epsilon_m \sum_{|l|>n}\left|\hat{h}(\epsilon_m l)\right| \to 0$

(v) $\sqrt{m}\left[\sum_{k\ge 1} h\left(\frac{k-\xi}{\epsilon_m}\right) + \sum_{k\le -1} h\left(\frac{1+k-\xi}{\epsilon_m}\right)\right] \to 0$

Theorem 3.5 *Assume the same conditions in Theorem 3.4 hold. If we choose the bell-shaped function h and a sequence $\{\epsilon_m\}$ satisfying the above admissible conditions, then*

$$\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty \to 0, \quad \textit{as } m \to \infty.$$

Proof It follows from a direct computation by substituting the above admissible conditions into Theorem 3.4. □



(a) $h(x) = \mathrm{e}^{-wx^2}$ **(b)** $h(x) = (1-x^2)_+^\alpha$

Fig. 1 Examples of admissible smoothing functions

4 Examples of Admissible Concentration Factors and Convergence Rates

We point out the admissible conditions for h and ϵ are not straightforward to check in practical applications. We will present in this section a few examples of h and ϵ that are easier to use in practice. Moreover, we will also analyze the convergence rates for such specific choices of h and ϵ.

We first consider h satisfying the following condition: there exist positive constants c_0 and δ such that

$$|h(t)| + |\hat{h}(t)| \leq c_0(1+|t|)^{-1-\delta}, \quad t \in \mathbb{R}. \tag{4.1}$$

We point out that the above condition is also considered in [4], which requires both the funtion h and its Fourier transform have decay properties. We give a few examples of h satisfying the above condition below

(a) $h(x) = \mathrm{e}^{-wx^2}, \quad w > 0.$
(b) $h(x) = (1-x^2)_+^\alpha, \quad \alpha \geq 1.$

Figure 1 displays some specific examples of them.

We shall next derive a specific estimate of the approximation error $\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty$ for such choices of h.

Proposition 4.1 *Suppose the same conditions in Theorem 3.4 hold. If additionally the bell-shaped function h satisfies the above assumption (4.1), then there exists a positive constant c such that for any $m \in \mathbb{N}$, $\epsilon > 0$, and $1 \leq k \leq m$ with $\log k \leq (\epsilon k)^{-1-\delta}$,*

$$\|E_m(\epsilon) - [f](\xi)h_\epsilon\|_\infty \leq c[m^{1/2}\epsilon(\epsilon k)^{-1-\delta} + m^{3/2}\epsilon^{1+\delta} + (\epsilon m)^{-\delta}].$$

Proof We will estimate the four terms in the result of Theorem 3.4 separately by using the additional assumption (4.1) on h.

We first present an estimate of $\sum_{|j|\leq m} \frac{\hat{h}(\lambda_j\epsilon)}{\lambda_j}$. For any $1 \leq k \leq m$, we have

$$\sum_{|j|\leq m} \frac{\hat{h}(\lambda_j\epsilon)}{\lambda_j} = \sum_{|j|\leq k} \frac{\hat{h}(\lambda_j\epsilon)}{\lambda_j} + \sum_{k<|j|\leq m} \frac{\hat{h}(\lambda_j\epsilon)}{\lambda_j}$$

By the assumption (4.1), we will apply $|\hat{h}(t)| \leq c_0$ in the first term and apply $|\hat{h}(t)| \leq c_0|t|^{-1-\delta}$ in the second term. It follows that

$$\sum_{|j|\leq m} \frac{\hat{h}(\lambda_j \epsilon)}{\lambda_j} \leq c_0 \sum_{|j|\leq k} \frac{1}{\lambda_j} + c_0 \epsilon^{-1-\delta} \sum_{k<|j|\leq m} \lambda_j^{-2-\delta}$$

By the assumption of $\{\lambda_j\}$, there exists some positive constant c_1 such that

$$\sum_{|j|\leq m} \frac{\hat{h}(\lambda_j \epsilon)}{\lambda_j} \leq c_1 \log k + c_1 (\epsilon k)^{-1-\delta}$$

For $1 \leq k \leq m$ and $\log k \leq (\epsilon k)^{-1-\delta}$, we have

$$\sum_{|j|\leq m} \frac{\hat{h}(\lambda_j \epsilon)}{\lambda_j} \leq 2c_1 (\epsilon k)^{-1-\delta}$$

We next estimate the second term. It is enough to estimate the sum of the two integrals $\int_{-\infty}^{-\xi/\epsilon} |h(u)|du + \int_{(1-\xi)/\epsilon}^{\infty} |h(u)|du$. We will first estimate the second integral. By the assumption (4.1), we have $|h(t)| \leq c|t|^{-1-\delta}$. It implies

$$\int_{(1-\xi)/\epsilon}^{\infty} |h(u)|du \leq c_0 \int_{(1-\xi)/\epsilon}^{\infty} u^{-1-\delta} du = \frac{c_0}{\delta}(1-\xi)^{-\delta}\epsilon^{\delta}.$$

Similarly, we have

$$\int_{-\infty}^{-\xi/\epsilon} |h(u)|du \leq \frac{c_0}{\delta}\xi^{-\delta}\epsilon^{\delta}.$$

It follows

$$\int_{(1-\xi)/\epsilon}^{\infty} |h(u)|du + \int_{-\infty}^{-\xi/\epsilon} |h(u)|du \leq \frac{c_0}{\delta}[\xi^{-\delta} + (1-\xi)^{-\delta}]\epsilon^{\delta}.$$

We continue with estimates of $\left(\sum_{|l|>n} \hat{h}^2(\epsilon l)\right)^{1/2}$ and $\sum_{|l|>n} \left|\hat{h}(\epsilon l)\right|$ in the third term. It follows from a direct substitution of the assumption (4.1) that

$$\left(\sum_{|l|>n} \hat{h}^2(\epsilon l)\right)^{1/2} \leq \frac{c_0}{\sqrt{1+2\delta}} \epsilon^{-(1+\delta)} n^{-\delta-1/2}$$

and

$$\sum_{|l|>n} \left|\hat{h}(\epsilon l)\right| \leq \frac{c_0}{\delta}\epsilon^{-(1+\delta)} n^{-\delta}.$$

It remains to estimate $\sum_{k\geq 1} h\left(\frac{k-\xi}{\epsilon}\right) + \sum_{k\leq -1} h\left(\frac{1+k-\xi}{\epsilon}\right)$ in the fourth term. It follows from (4.1) that

$$\sum_{k\geq 1} h\left(\frac{k-\xi}{\epsilon}\right) \leq c_0 \epsilon^{1+\delta} \sum_{k\geq 1} (k-\xi)^{-1-\delta} \leq \frac{c_0}{\delta}(1-\xi)^{-\delta}\epsilon^{1+\delta}.$$

Similarly,

$$\sum_{k\le -1} h\left(\frac{1+k-\xi}{\epsilon}\right) \le \frac{c_0}{\delta}\xi^{-\delta}\epsilon^{1+\delta}.$$

The desired result follows directly from substituting the above four estimates into the result of Theorem 3.4. □

We will choose ϵ to achieve the optimal convergence rate in the above result.

Theorem 4.2 *Suppose the same conditions in Theorem 3.4 hold. If additionally the bell-shaped function h satisfies the above assumption (4.1) and we choose*

$$\epsilon_m = m^{-\frac{\delta+1/2}{1+\delta}} \log^{-\frac{1}{1+\delta}} m, \tag{4.2}$$

then there exists a positive constant c such that for any $m \in \mathbb{N}$

$$\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty \le c[m^{-\frac{\delta}{2(1+\delta)}} \log^{\frac{\delta}{1+\delta}} m + m^{-(\delta-3/2)} \log^{-1} m].$$

Proof It follows from substituting the above specific choice of ϵ_m (4.2) into Proposition 4.1 that there exists a positive constant c_1 such that for any $m \in \mathbb{N}$, $1 \le k \le m$ with $\log k \le k^{-1-\delta} m^{\delta-1/2} \log m$

$$\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty \le c_1[k^{-1-\delta} m^{\frac{\delta^2+\delta+1/2}{1+\delta}} \log^{\frac{\delta}{1+\delta}} m$$
$$+m^{-(\delta-3/2)} \log^{-1} m + m^{-\frac{\delta}{2(1+\delta)}} \log^{\frac{\delta}{1+\delta}} m].$$

The desired result follows from setting $k = m^{\frac{\delta+1/2}{1+\delta}}$ in the above inequality. □

We next consider another special class of functions: compactly supported functions. That is, in addition to the assumption (4.1), we further assume h is supported on $[-1, 1]$.

Corollary 4.3 *Suppose the same conditions in Theorem 3.4 hold. If additionally the bell-shaped function h satisfies the above assumption (4.1) and is supported in $[-1, 1]$ and we choose ϵ_m as in (4.2), then there exists a positive constant c such that for any $m \in \mathbb{N}$*

$$\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty \le c m^{-\frac{\delta}{2(1+\delta)}} \log^{\frac{\delta}{1+\delta}} m.$$

Proof It is direct to observe that $R_j(\epsilon) = 0$, $H_\epsilon = h_\epsilon$ when $\epsilon \le \min\{\xi, 1-\xi\}$ and h is supported on $[-1, 1]$. That is, the second term and the fourth term in the result of Theorem 3.4 would vanish. The desired result follows in a similar way to the proof of Theorem 4.2. □

5 Numerical Experiments

We will present in this section the numeric performance of the edge detector $E_m(\epsilon)$ in (2.9). In particular, we will choose admissible h and ϵ for various data size and test functions to show the numerical convergence of the edge detector $E_m(\epsilon)$.

In all the numerical experiments below, we will use the jittered non-uniform sampling frequencies as defined in Eq. (3.5).

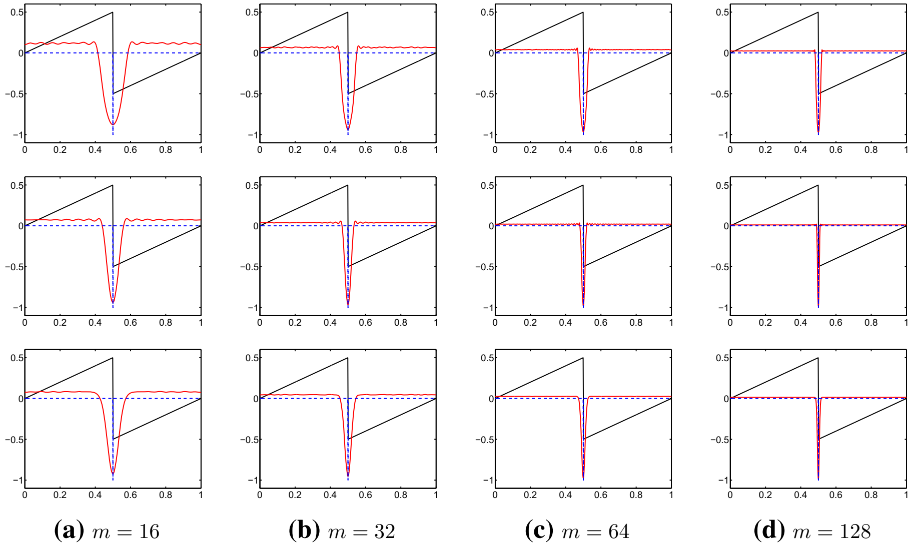


Fig. 2 Edge detectors with different bell functions h_1 (top row), h_2 (middle row), and h_3 (bottom row) for test function f_1

Table 1 Approximation errors $\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty$ for the test function f_1

m	h_1		h_2		h_3	
	Error	ϵ_m	Error	ϵ_m	Error	ϵ_m
16	1.4E−1	8.8E−2	9.1E−2	6.9E−2	8.7E−2	6.1E−2
32	9.2E−2	5.0E−2	5.9E−2	3.6E−2	4.9E−2	3.1E−2
64	5.9E−2	3.0E−2	3.7E−2	1.9E−2	2.7E−2	1.6E−2
128	4.2E−2	1.9E−2	2.2E−2	1.0E−2	1.4E−2	7.8E−3

Example 5.1 We consider the following test function with a a single jump discontinuity at $\xi = 0.5$:

$$f_1(x) = \begin{cases} x, & 0 \le x \le 0.5 \\ x - 1, & 0.5 < x \le 1. \end{cases}$$

For each m ranging in 16, 32, 64, 128, we take the following three pairs of admissible h and ϵ:

(a) $h_1(x) = (1 - x^2)_+^1$ and $\epsilon_m = m^{-1/2} \log^{-1} m$
(b) $h_2(x) = (1 - x^2)_+^2$ and $\epsilon_m = m^{-5/6} \log^{-1/3} m$
(c) $h_3(x) = \mathrm{e}^{-5x^2}$ and $\epsilon_m = m^{-1} \log^{1/2} m$

and compute the edge detectors in (2.9). We display them in Fig. 2.

Moreover, we compute the approximation error $\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty$ between the edge detector and the smoothed jump function for each m ranging in 16, 32, 64, 128 and each admissible choice of the bell-shaped function h and ϵ_m in the above three examples. We list the results in Table 1.

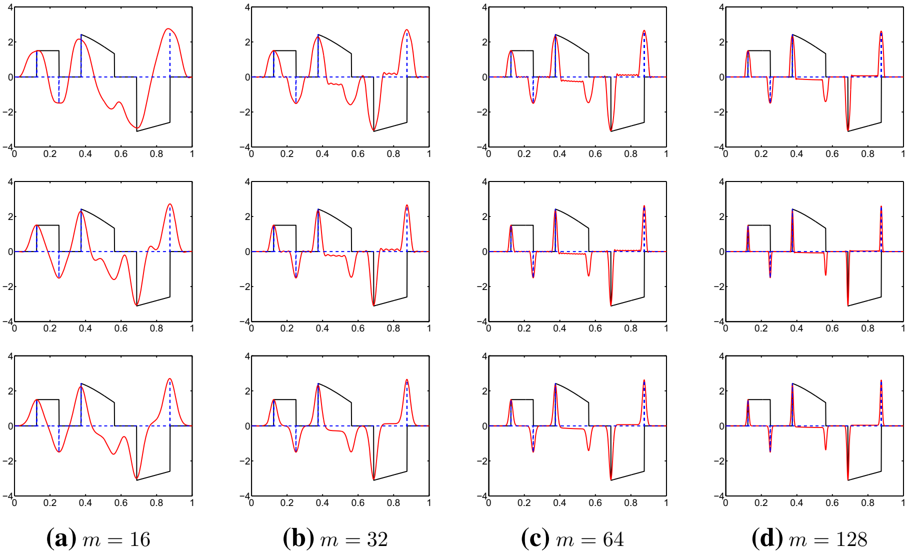


Fig. 3 Edge detectors with different bell functions h_1 (top row), h_2 (middle row), and h_3 (bottom row) for test function f_2

Table 2 Approximation errors $\|E_m(\epsilon_m) - [f](\xi)h_{\epsilon_m}\|_\infty$ for the test function f_2

m	h_1		h_2		h_3	
	Error	ϵ_m	Error	ϵ_m	Error	ϵ_m
16	8.4E−1	8.8E−2	5.3E−1	6.9E−2	5.1E−1	6.1E−2
32	5.1E−1	5.0E−2	2.9E−1	3.6E−2	3.0E−1	3.1E−2
64	3.4E−1	3.0E−2	1.6E−1	1.9E−2	1.7E−1	1.6E−2
128	2.1E−1	1.9E−2	8.8E−2	1.0E−2	9.2E−2	7.8E−3

Example 5.2 We consider another test function with multiple jump discontinuities

$$f_2(x) = \begin{cases} \frac{3}{2}, & 1/8 \leq x \leq 1/4 \\ \frac{7}{4} - \frac{x}{2} + \sin(2\pi x - 1/4), & 3/8 < x \leq 9/16 \\ \frac{11}{4}x - 5, & 11/16 \leq x \leq 7/8 \\ 0, & \text{otherwise.} \end{cases}$$

For each m ranging in 16, 32, 64, 128, we use the same three pairs of admissible h and ϵ as in Example 5.1 and display the the edge detectors in Fig. 3.

Similarly, we compute the approximation error $\|E_m(\epsilon_m) - [f](\xi)h_\epsilon\|_\infty$ and list the results in Table 2.

We remark that we focus on the approximation of a smoothed edge function in this paper and the smoothness parameter δ in Eq. (4.1) of the function h and its Fourier transform $\hat{h}$ would determine the approximation accuracy. Since h_3 has a greater δ than h_1 and h_2, it has a better numerical performance than the other two. Moreover, the functions in the Schwartz class would satisfy the condition in Eq. (4.1) and would provide a large class of admissible smoothing functions.

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

6 Conclusion

By converting the edge detection problem into the approximation of a smoothed edge function and employing the admissible frame technique, we are able to develop a theoretic analysis of the concentration factor method for non-uniform Fourier data. In particular, we derive admissible conditions on concentration factors for non-uniform Fourier data such that the edge detector with the concentration factor method converges to the smoothed edge function. We present some examples of concentration factors satisfying such admissible conditions and further obtain the convergence rates for such specific examples. Our numerical experiments with a few admissible concentration factors demonstrate the convergence results in detecting edges from non-uniform Fourier data for functions with a single jump discontinuity and with multiple jumps.

We point out that we only consider the case of jittered sampling that is well known as a Fourier frame in $L^2[0, 1]$. Our approach could also be generalized to other nonuniform schemes that constitute frames in $L^2[0, 1]$. The key point would rely on deriving the sampling rate in the admissible frame approach of [13]. Moreover, we focus on the 1D nonuniform sampling scheme in this paper. The 2D nonuniform sampling scheme needs a slightly different approach and would be investigated in a sequential paper.

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