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Hierarchical construction of bounded solutions of $div U = F$ in critical regularity spaces

Eitan Tadmor and Changhui Tan

Abstract We implement the hierarchical decomposition introduced in [7], to construct uniformly bounded solutions of the problem $div U = F$, where the two-dimensional data is in the critical regularity space, $F \in L^2_\#(\mathbb{T}^2)$. Criticality in this context, manifests itself by the lack of linear mapping, $F \in L^2_\#(\mathbb{T}^2) \mapsto U \in L^\infty(\mathbb{T}^2)$, [1]. Thus, the intriguing aspect here is that although the problem is linear, the construction of its uniformly bounded solutions is not.

1 Introduction

We are concerned with the construction of uniformly bounded solutions, $U \in L^\infty(\mathbb{T}^2, \mathbb{R}^2)$ of the equation

$$div U = F, \quad F \in L^2_\#(\mathbb{T}^2), \tag{1}$$

where $L^2_\#(\mathbb{T}^2)$ is the space of L^2 integrable functions over the 2-dimensional torus $\mathbb{T}^2$ with zero mean.

The existence of uniformly bounded solutions of (1) follows from the closed range theorem together with Gagliardo-Nirenberg inequality, [1]. Moreover, Bourgain and Brezis [1] proved that any mapping, $F \in L^2_\# \mapsto U \in L^\infty(\mathbb{T}^2)$, must be *nonlinear*: thus, the intriguing aspect here is that although (1) is linear, the construction of its uniformly bounded solutions for $L^2_\#$-data is not.

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It follows, in particular, that the classical Helmholtz solution of (1), $U_{\text{Hel}} = \nabla\Delta^{-1}F$, cannot be a uniformly bounded solution for *all* $F \in L^2_\#$. Indeed, $F \in L^2_\#$ implies that $U_{\text{Hel}} \in H^1(\mathbb{T}^2)$, but since H^1 is not a subset of L^∞, Helmholtz solution need not be uniformly bounded. The following concrete counterexample due to L. Nirenberg, [1, Remark 7], demonstrates this type of unboundedness: fix $\theta \in (0,1/2)$, let $\zeta(r)$ be a smooth cut-off function supported near the origin, and set

$$F = \Delta v, \qquad v(x,y) := x|\log r|^\theta \zeta(r), \quad r = \sqrt{x^2+y^2}. \tag{2}$$

In this case, $F \in L^2_\#(\mathbb{T}^2)$, but the Helmholtz solution, $U_{\text{Hel}} = \nabla\Delta^{-1}F = \nabla v$, has a fractional logarithmic growth at the origin.

Inspired by the hierarchical decompositions which were introduced in [8, 9] in the context of image processing, Tadmor [7] utilized such decompositions as a constructive procedure to solve (1): the solution is given in terms of *hierarchical decomposition*, $U_{\text{Bdd}} = \sum \mathbf{u}_j$, where the $\{\mathbf{u}_j\}$'s can be computed recursively as the following minimizers,

$$\mathbf{u}_{j+1} = \underset{\mathbf{u}}{arg\,min}\left\{\|\mathbf{u}\|_{L^\infty} + \lambda_1 2^j \|F - div\,(\sum_{k=1}^{j}\mathbf{u}_k) - div\,\mathbf{u}\|^2_{L^2}\right\}, \quad j = 0,1,\cdots. \tag{3}$$

Here, λ_1 is any sufficiently large parameter, $\lambda_1 > 1/(2\|F\|_{BV})$, which guarantees that the hierarchical decomposition starts with a non-trivial solution of (3), consult (20) below.

In this paper, we propose a numerical approach to solve the minimization problem (3), which in turn generates the uniformly bounded hierarchical solution of problem (1).

We begin, in section 2, by quoting the hierarchical construction proposed in [7]. In section 3 we analyze the minimization problem (3) in terms of its corresponding dual problem. This dual problem amounts to a nonlinear PDE which governs the residual $r := f - div\,\mathbf{u}$, where f stands for $F - div\,(\sum \mathbf{u}_k)$. As a final step, we introduce a procedure to recover the desired minimizer $\mathbf{u}$ from its residual r. In section 4 we discuss the numerical solution of the governing PDE: it is solved by an iterative procedure which avoids significantly large errors in the recovering stage. In section 5, we report on our computations which compare the bounded hierarchical solution, U_{Bdd}, vs. the unbounded Helmholtz solution, U_{Hel}. Finally, in section 6, we introduce a new construction of bounded solutions for (1), based on two-step solution of the form,

$$U_{\text{2step}} = \mathbf{u}_1 + \nabla\Delta^{-1}r_1, \qquad [\mathbf{u}_1, r_1] = \underset{div\,\mathbf{u}+r=F}{arg\,min}\left\{\|\mathbf{u}\|_{L^\infty} + \lambda_1\|r\|^2_{L^2}\right\}. \tag{4}$$

This two-step solution consists of one hierarchical decomposition step, $\mathbf{u}_1$ followed by one Helmholtz step, which are shown to yield a uniformly bounded solution of (1).

2 Hierarchical solution of $div\, U = F \in L^2_{\#}(\mathbb{T}^2)$

Our starting point for the construction of a uniformly bounded solution of (1), $U \in L^\infty(\mathbb{T}^2, \mathbb{R}^2)$, is a decomposition of F,

$$F = div\,\mathbf{u}_1 + r_1, \qquad F \in L^2_{\#}(\mathbb{T}^2) := \left\{ g \in L^2(\mathbb{T}^2) \;\Big|\; \int_{\mathbb{T}^2} g(x)dx = 0 \right\}, \tag{5a}$$

where $[\mathbf{u}_1, r_1]$ is a minimizing pair of the functional,

$$[\mathbf{u}_1, r_1] = \underset{div\,\mathbf{u}+r=F}{arg\,min} \left\{ \|\mathbf{u}\|_{L^\infty} + \lambda_1 \|r\|^2_{L^2} \right\}. \tag{5b}$$

Here, λ_1 is a fixed parameter at our disposal where we distinguish between two cases, consult (20) below. If $\lambda_1 \le \dfrac{1}{2\|F\|_{BV}}$, then the minimizer of (5b) is the trivial one, $\mathbf{u}_1 \equiv 0, r_1 = F$; otherwise, by choosing λ_1 large enough, $\lambda_1 > \dfrac{1}{2\|F\|_{BV}}$, then (5b) admits a non-trivial minimizer, $[\mathbf{u}_1, r_1]$, which is characterized by a residual satisfying $\|r_1\|_{BV} = \dfrac{1}{2\lambda_1}$. By Gagliardo-Nirenberg isoperimetric inequality, e.g., [11, §2.7], there exists $\beta > 0$ such that

$$\|g\|_{L^2} \le \beta \|g\|_{BV}, \qquad \int_{\mathbb{T}^2} g(x)dx = 0. \tag{6}$$

It follows that r_1 is L^2-bounded:

$$\|r_1\|_{L^2} \le \beta \|r_1\|_{BV} = \frac{\beta}{2\lambda_1}. \tag{7}$$

Moreover, since F has a zero mean so does the residual r_1. We conclude that the residual $r_1 \in L^2_{\#}(\mathbb{T}^2)$, and we can therefore implement the same variational decomposition of F in (5), and use it to decompose r_1. To this end, we use the same variational statement, $\left\{ \|\mathbf{u}\|_{L^\infty} + \lambda_2 \|r\|^2_{L^2} \right\}$, with a new parameter, $\lambda = \lambda_2 > \lambda_1$,

$$r_1 = div\,\mathbf{u}_2 + r_2, \qquad [\mathbf{u}_2, r_2] = \underset{div\,\mathbf{u}+r=r_1}{arg\,min} \left\{ \|\mathbf{u}\|_{L^\infty} + \lambda_2 \|r\|^2_{L^2} \right\}. \tag{8}$$

Borrowing the terminology from our earlier work on image processing [8, 9], the decomposition (8) has the effect of "zooming" on the residual r_1, and it is here that we use the refined scale $\lambda_2 > \lambda_1$. Combining (8) with (5a) we obtain $F = div\,U_2 + r_2$ with $U_2 := \mathbf{u}_1 + \mathbf{u}_2$, which is viewed as an improved *approximate solution* of (1). Indeed, the "zooming" effect $\lambda_2 > \lambda_1$ implies that U_2 has a smaller residual $\|r_2\|_{BV} = 1/(2\lambda_2)$ compared with $\|r_1\|_{BV} = 1/(2\lambda_1)$ in (7). In particular,

$$\|r_2\|_{L^2} \le \beta \|r_2\|_{BV} = \frac{\beta}{2\lambda_2}.$$

This process can be repeated: if $r_j \in L^2_\#(\mathbb{T}^2)$ is the residual at step j, then we decompose it

$$r_j = div\,\mathbf{u}_{j+1} + r_{j+1}, \tag{9a}$$

where $[\mathbf{u}_{j+1}, r_{j+1}]$ is a minimizing pair of

$$[\mathbf{u}_{j+1}, r_{j+1}] = \underset{div\,\mathbf{u}+r=r_j}{arg\,min} \left\{ \|\mathbf{u}\|_{L^\infty} + \lambda_{j+1}\|r\|^2_{L^2} \right\}, \qquad j = 0,1,\ldots. \tag{9b}$$

For $j = 0$, the decomposition (9) is interpreted as (5a) by setting $r_0 := F$. Note that the recursive decomposition (9a) depends on the invariance that the residuals $r_j \in L^2_\#(\mathbb{T}^2)$: indeed, if r_j has a zero mean then so does r_{j+1}, and since by (20) the minimizer r_{j+1} has a bounded variation, $r_{j+1} \in L^2_\#(\mathbb{T}^2)$. The iterative process depends on a sequence of increasing scales, $\lambda_1 < \lambda_2 < \ldots \lambda_{j+1}$, which are yet to be determined.

The telescoping sum of the first k steps in (9a) yields an improved approximate solution, $U_k := \sum_{j=1}^k \mathbf{u}_j$:

$$F = div\,U_k + r_k, \qquad \|r_k\|_{L^2} \le \beta\|r_k\|_{BV} = \frac{\beta}{2\lambda_k} \downarrow 0, \qquad k = 1,2,\ldots. \tag{10}$$

The key question is whether the U_k's remain uniformly bounded, and it is here that we use the freedom in choosing the scaling parameters λ_k: comparing the minimizing pair $[\mathbf{u}_{j+1}, r_{j+1}]$ of (9b) with the trivial pair $[\mathbf{u} \equiv 0, r_j]$, we find

$$\|\mathbf{u}_{j+1}\|_{L^\infty} + \lambda_{j+1}\|r_{j+1}\|^2_{L^2} \le \|0\|_{L^\infty} + \lambda_{j+1}\|r_j\|^2_{L^2},$$

$$r_j = div\,\mathbf{u}_{j+1} + r_{j+1} = div\,(0) + r_j.$$

It remains to upper-bound the energy norm of the r_j's: for $j = 0$ we have $r_0 = F$; for $j > 0$, (10) implies that $\|r_j\|_{L^2} \le \beta/(2\lambda_j)$. We end up with

$$\|\mathbf{u}_{j+1}\|_{L^\infty} + \lambda_{j+1}\|r_{j+1}\|^2_{L^2} \le \lambda_{j+1}\|r_j\|^2_{L^2} \le \begin{cases} \lambda_1\|F\|^2_{L^2}, & j = 0, \\ \dfrac{\beta^2\lambda_{j+1}}{4\lambda_j^2}, & j = 1,2,\ldots. \end{cases} \tag{11}$$

We conclude that by choosing a sufficiently fast increasing λ_j's such that $\sum_j \lambda_{j+1}\lambda_j^{-2} < \infty$, then the approximate solutions $U_k = \sum_1^k \mathbf{u}_j$ form a Cauchy sequence in L^∞ whose limit, $U = \sum_1^\infty \mathbf{u}_j$, satisfies the following.

Theorem 2.1 ([7]) *Fix β such that* (6) *holds. Then, for any given $F \in L^2_\#(\mathbb{T}^2)$, there exists a uniformly bounded solution of* (1),

$$div\,U = F, \qquad \|U\|_{L^\infty} \le 2\beta\|F\|_{L^2}.$$

The solution U is given by $U = \sum_{j=1}^{\infty} \mathbf{u}_j$, where the $\{\mathbf{u}_j\}$'s are constructed recursively as minimizers of

$$[\mathbf{u}_{j+1}, r_{j+1}] = \underset{div\,\mathbf{u}+r=r_j}{arg\,min} \left\{ \|\mathbf{u}\|_{L^\infty} + \lambda_1 2^j \|r\|_{L^2}^2 \right\}, \qquad r_0 := F, \quad \lambda_1 = \frac{\beta}{\|F\|_{L^2}}. \quad (12)$$

Proof. Set $\lambda_j = \lambda_1 2^{j-1}$, $j = 1, 2, \ldots$, then, $\|U_k - U_\ell\|_{L^\infty} \lesssim 2^{-k}$, $k > \ell \gg 1$. Let U be the limit of the Cauchy sequence $\{U_k\}$ then $\|U_j - U\|_{L^\infty} + \|div\,U_j - F\|_{L^2} \lesssim 2^{-j} \to 0$, and since div has a closed graph on its domain $D := \{\mathbf{u} \in L^\infty :\ div\,\mathbf{u} \in L^2(\mathbb{T}^2)\}$, it follows that $div\,U = F$. By (11) we have

$$\|U\|_{L^\infty} \leq \sum_{j=1}^{\infty} \|\mathbf{u}_j\|_{L^\infty} \leq \lambda_1 \|F\|_{L^2}^2 + \frac{\beta^2}{4\lambda_1} \sum_{j=2}^{\infty} \frac{1}{2^{j-3}} = \lambda_1 \|F\|_{L^2}^2 + \frac{\beta^2}{\lambda_1}.$$

Here $\lambda_1 > \dfrac{1}{2\|F\|_{BV}}$ is a free parameter at our disposal: we choose $\lambda_1 := \beta/\|F\|_{L^2}$ which by (6) is admissible, $\lambda_1 = \dfrac{\beta}{\|F\|_{L^2}} > \dfrac{1}{2\|F\|_{BV}}$, and the result follows.

Remark 2.1 *[Energy decomposition] By squaring the refinement step (5a), $r_j = r_{j+1} + div\,\mathbf{u}_{j+1}$, and using the characterization of $[\mathbf{u}_{j+1}, r_{j+1}]$ as an* extremal pair *(consult remark 3.2 below), we find*

$$\|r_j\|_{L^2}^2 - \|r_{j+1}\|_{L^2}^2 = 2(r_{j+1}, div\,\mathbf{u}_{j+1}) + \|div\,\mathbf{u}_{j+1}\|_{L^2}^2 = \frac{1}{\lambda_{j+1}} \|\mathbf{u}_{j+1}\|_{L^\infty} + \|div\,\mathbf{u}_{j+1}\|_{L^2}^2.$$

A telescoping sum of the last equality yields the "energy decomposition"

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} \|\mathbf{u}_j\|_{L^\infty} + \sum_{j=1}^{\infty} \|div\,\mathbf{u}_j\|_{L^2(\mathbb{T}^2)}^2 = \|F\|_{L^2(\mathbb{T}^2)}^2 \quad (13)$$

Remark 2.2 *We note that the constructive proof of theorem 2.1 does not assume the existence of bounded solution for (14): it is deduced from the Gagliardo-Nirenberg inequality (6). The hierarchical construction of solutions for $\mathscr{L}U = F$, in the general setup of linear closed operators, $\mathscr{L} : \mathbb{B} \mapsto L^p_{\#}$, $1 < p < \infty$, with boundedly invertible duals $\mathscr{L}^*$, was proved in [7].*
In [2], Bourgain and Brezis proved that (1) admits a bounded solution in the smaller *space, $\mathbb{B} = L^\infty \cap H^1$. This requires a considerably more delicate argument, which could be justified by the refined dual estimate (compared with (6)), $\|g\|_{L^2(\mathbb{T}^2)} \lesssim \|\nabla g\|_{L^1 + H^{-1}(\mathbb{T}^2)}$. The proof of [2] is* constructive*: it is based on an intricate Littlewood-Paley decomposition, which cannot be readily implemented in actual computations.*

3 Construction of hierarchical minimizers

3.1 The minimization problem

We rewrite each minimization step of the hierarchical decompositions (3) in the following form,

$$\bar{\mathbf{u}} = \underset{\mathbf{u}:\mathbb{T}^2\to\mathbb{R}^2}{arg\,min}\left\{\|\mathbf{u}\|_{L^\infty} + \lambda\|f - div\,\mathbf{u}\|_{L^2}^2\right\}, \qquad \|\mathbf{u}\|_{L^\infty} := \operatorname*{ess\,sup}_{x,y}\sqrt{u_1^2+u_2^2}. \tag{14}$$

Here, f is an L^2 function with zero mean which stands for $F - div\,(\sum_{k=1}^{j}\mathbf{u}_k)$ in (3), and λ stands for the dyadic scales, $\lambda_1 2^j, j = 0, 1, \cdots$.

3.2 The dual problem

To circumvent the difficulty of handling the L^∞ norm in (14), we concentrate on the dual problem associated with (14). We let $\mathscr{N}(\mathbf{u}) = \|\mathbf{u}\|_{L^\infty} : V \mapsto \bar{\mathbb{R}}$, $\mathscr{E}(p) = \|f-p\|_{L^2}^2 : Y \mapsto \bar{\mathbb{R}}$, and $\Lambda = div : V \mapsto Y$ with $V = L^\infty(\mathbb{T}^2)$ and $Y = L^2(\mathbb{T}^2)$. By duality theorem, [4, §3,Remark 4.2], the variational problem (14),

$$(\mathscr{P}): \qquad \inf_{u\in V}[\mathscr{N}(u) + \mathscr{E}(\Lambda u)]$$

is equivalent to its dual problem

$$(\mathscr{P}^*): \qquad \sup_{p^*\in Y^*}[-\mathscr{N}^*(\Lambda^* p^*) - \mathscr{E}^*(-p^*)];$$

moreover, if $\bar{u}$ and $\bar{p}^*$ are solutions of $(\mathscr{P})$ and $(\mathscr{P}^*)$ respectively, then $\Lambda^*\bar{p}^* \in \partial\mathscr{N}(\bar{u})$, and $-\bar{p}^* \in \partial\mathscr{E}(\Lambda\bar{u})$. Here, $\mathscr{N}^*, \mathscr{E}^*$ are conjugate functions of $\mathscr{N}, \mathscr{E}$, expressed in terms of the usual L^2 pairing $\langle \mathbf{w}_1, \mathbf{w}_2\rangle := \int_{\mathbb{T}^2}\mathbf{w}_1\cdot\mathbf{w}_2\,dx$,

$$\begin{aligned}
\mathscr{N}^*(\mathbf{u}^*) &= \sup_{\mathbf{u}}\{\langle\mathbf{u},\mathbf{u}^*\rangle - \|\mathbf{u}\|_{L^\infty}\} \\
&= \sup_{\mathbf{u}}\{\|\mathbf{u}\|_{L^\infty}\|\mathbf{u}^*\|_{L^1} - \|\mathbf{u}\|_{L^\infty}\} = \chi_{\{\|\mathbf{u}^*\|_{L^1}\le 1\}} = \begin{cases} 0, & \text{if } \|\mathbf{u}^*\|_{L^1} \le 1 \\ +\infty, & \text{otherwise}\end{cases}; \\
\mathscr{E}^*(p^*) &= \sup_{p}\{\langle p, p^*\rangle - \lambda\|f-p\|_{L^2}^2\} \\
&= \sup_{p}\{-\lambda\langle p,p\rangle + \langle p^* + 2\lambda f, p\rangle - \lambda\langle f,f\rangle\} = \left\langle f + \frac{1}{4\lambda}p^*, p^*\right\rangle,
\end{aligned}$$

and $\Lambda^* = -\nabla$ is the dual operator of Λ.

We end up with the dual $(\mathscr{P}^*)$ problem

$$\inf_{\{p^*:\|\nabla p^*\|_{L^1}\leq 1\}} \left\langle \frac{1}{4\lambda}p^* - f, p^* \right\rangle$$

or

$$\inf_{p^*}\sup_{\mu\geq 0}\left[\left\langle \frac{1}{4\lambda}p^* - f, p^* \right\rangle + \mu(\|\nabla p^*\|_{L^1} - 1)\right]. \tag{15}$$

Moreover, $-\bar{p}^* \in \partial\mathscr{E}(\Lambda\bar{u})$, meaning that $p^* = 2\lambda r$, where r is the residual, $r = f - div\,\mathbf{u}$. So, we can express the dual problem (15) in terms of r,

$$\bar{r} = \underset{r}{arg\,min}\sup_{\mu\geq 0} L(r,\mu), \qquad L(r,\mu) := \lambda\left\langle r - 2f, r\right\rangle + \mu\left(\|\nabla r\|_{L^1} - \frac{1}{2\lambda}\right), \tag{16}$$

where $\bar{r} := f - div\,\bar{\mathbf{u}}$, is the residual corresponding to the optimal minimizer $\bar{u}$.

Since $L(\cdot,\mu)$ is convex and $L(r,\cdot)$ is concave and, for $r \in BV$ continuous, we can apply the minimax theorem, e.g., [4, §6], which allows us to interchange the infimum and supremum in (16), yielding

$$\sup_{\mu\geq 0}\min_{r}\left[\lambda\left\langle r - 2f, r\right\rangle + \mu\left(\|\nabla r\|_{L^1} - \frac{1}{2\lambda}\right)\right]. \tag{17}$$

The dual problem, (17), can be solved in two steps. An inner minimization problem

$$r_\mu = \underset{r}{arg\,min}\left[\lambda\left\langle r - 2f, r\right\rangle + \mu\left(\|\nabla r\|_{L^1} - \frac{1}{2\lambda}\right)\right]. \tag{18a}$$

Here, for any given $\mu \geq 0$, there exists a unique $r = r_\mu$ such that (μ, r_μ) is a saddle point of L. The optimal $\mu = \mu^*$ is determined by an outer maximization problem,

$$\mu^* = \underset{\mu\geq 0}{arg\,max}\left[P(\mu) + \mu Q(\mu)\right],$$
$$P(\mu) := \lambda\left\langle r_\mu - 2f, r_\mu\right\rangle, \;\; Q(\mu) := \|\nabla r_\mu\|_{L^1} - \frac{1}{2\lambda}. \tag{18b}$$

Once μ^* is found, then $\bar{r} = r_{\mu^*}$ is the optimal residual which is sought as the solution of (16) .

3.3 The outer maximization problem

We begin by characterizing the maximizer, $\mu = \mu^*$, of the outer problem (18b). Fix μ: since r_μ minimizes $L(r,\mu)$ we have

$$P(\mu) + \mu Q(\mu) \leq P(\nu) + \mu Q(\nu).$$

Similarly, $P(\nu)+\nu Q(\nu) \leq P(\mu)+\nu Q(\mu)$. Sum the last two inequalities to get, $(\mu-\nu)[Q(\mu)-Q(\nu)] \leq 0$, which yields that $Q(\cdot)$ is non-increasing.

Let μ^* be a maximizer of (18b). Then $\forall \mu \geq 0$,

$$P(\mu)+\mu Q(\mu) \leq P(\mu^*)+\mu^* Q(\mu^*) \leq P(\mu)+\mu^* Q(\mu),$$

which implies $(\mu^*-\mu)Q(\mu) \geq 0$. We distinguish between two cases.

Case #1: $\mu^*>0$. We have $Q(\mu) \leq 0$ if $\mu>\mu^*$ and $Q(\mu) \geq 0$ if $0 \leq \mu<\mu^*$. We conclude that μ^* is determined as a root of $Q(\cdot)$,

$$Q(\mu^*)=0, \quad \text{i.e.} \quad \|\nabla r_{\mu^*}\|_{L^1}=\frac{1}{2\lambda}. \tag{19}$$

Case #2: $\mu^*=0$. In this case, r_0 minimizes $\langle r-2f, r\rangle$, namely, $r_0=f$. This corresponds to the trivial minimizer of (14), $\bar{\mathbf{u}} \equiv 0$, which is the case we want to avoid. Case #2 happens when $Q(0) \leq 0$, i.e.

$$\mu^* \ \leftrightarrow \ \|\nabla r_0\|_{L^1}-\frac{1}{2\lambda} \leq 0 \ \leftrightarrow \ \|\nabla f\|_{L^1} \leq \frac{1}{2\lambda}.$$

So, to make sure that we pick a non-trivial minimizer, $\bar{\mathbf{u}} \not\equiv 0$, we must pick a sufficiently large λ such that

$$\lambda>\frac{1}{2\|f\|_{BV}} \ \leftrightarrow \ \bar{\mathbf{u}} \not\equiv 0, \ \|\bar{r}\|_{BV}=\frac{1}{2\lambda}. \tag{20}$$

This coincides with the same lower bound on λ's which yield non-trivial minimizers, asserted in [7, Lemma 5.3].

3.4 The inner minimization problem

We return to the inner minimization problem (18a). Fix $\mu=\mu^*$. The Euler-Lagrange equations characterizing minimizers of (18a) are

$$2\lambda(r_{\mu^*}-f)-\mu^* div\left(\frac{\nabla r_{\mu^*}}{|\nabla r_{\mu^*}|}\right)=0. \tag{21}$$

Take the L^2-inner product of (21) with r_{μ^*} to get

$$2\lambda\left\langle r_{\mu^*}-f, r_{\mu^*}\right\rangle-\mu^*\left\langle div\left(\frac{\nabla r_{\mu^*}}{|\nabla r_{\mu^*}|}\right), r_{\mu^*}\right\rangle=0.$$

Using (19) (and in the non-periodic case, the Neumann boundary condition $\nabla r_{\mu^*} \cdot \mathbf{n}=0$), we find

$$\left\langle div\left(\frac{\nabla r_{\mu^*}}{|\nabla r_{\mu^*}|}\right), r_{\mu^*}\right\rangle = -\left\langle \frac{\nabla r_{\mu^*}}{|\nabla r_{\mu^*}|}, \nabla r_{\mu^*}\right\rangle = -\int_{\mathbb{T}^2} |\nabla r_{\mu^*}| dx = -\frac{1}{2\lambda}.$$

This yields, $\mu^* = 4\lambda^2 \langle f - r_{\mu^*}, r_{\mu^*}\rangle$, and the governing equation (21) for the optimal residual, $\bar{r} = r_{\mu^*}$, amounts to

$$(\bar{r} - f) - 2\lambda \langle f - \bar{r}, \bar{r}\rangle\, div\left(\frac{\nabla \bar{r}}{|\nabla \bar{r}|}\right) = 0. \tag{22}$$

Remark 3.1 *This system has two solutions: one solution, $\bar{r} = f$, corresponds to the trivial case, $\bar{\mathbf{u}} \equiv 0$. The other is the target solution, i.e., the optimal residual $\bar{r}$ for (16). We will discuss numerical algorithms to solve system (22) in section 4.*

3.5 From r to $\mathbf{u}$: recovering the uniformly bounded solution

So far, we identified the residual, $\bar{r} = f - div\,\bar{\mathbf{u}}$, corresponding to the uniformly bounded solution $\bar{\mathbf{u}}$ of (14). To recover $\bar{\mathbf{u}}$ itself, we substitute $\bar{r} - f = -div\,\bar{\mathbf{u}}$ as the first term of (22), and get

$$div\left(\bar{\mathbf{u}} - 2\lambda \langle \bar{r} - f, \bar{r}\rangle \frac{\nabla \bar{r}}{|\nabla \bar{r}|}\right) = 0. \tag{23}$$

Therefore, we can recover a solution $\bar{\mathbf{u}}$ of (14),

$$\bar{\mathbf{u}} = 2\lambda \langle \bar{r} - f, \bar{r}\rangle \frac{\nabla \bar{r}}{|\nabla \bar{r}|}. \tag{24}$$

Observe that this $\bar{\mathbf{u}}$ is indeed uniformly bounded:

$$\|\bar{\mathbf{u}}\|_{L^\infty} = 2\lambda |\langle \bar{r} - f, \bar{r}\rangle| < \infty. \tag{25}$$

Remark 3.2 *The explicit expression of $\bar{\mathbf{u}}$ in (24) shows that $[\bar{\mathbf{u}}, \bar{r}]$ forms an extremal pair, [5, Theorem 4],[9, Theorem 2.3],[7, Theorem 5.1], in the sense of achieving an equality in the duality inequality of pairing $div\,\bar{\mathbf{u}}$ and $\bar{r}$:*

$$|\langle div\,\bar{\mathbf{u}}, \bar{r}\rangle| = \|\bar{\mathbf{u}}\|_{L^\infty} \frac{1}{2\lambda} = \|\bar{\mathbf{u}}\|_{L^\infty} \|\nabla \bar{r}\|_{L^1}.$$

4 Numerical algorithms for the hierarchical solution

We solve problem (1) using its hierarchical decomposition. In each iteration, we solve the minimization problem (14). Each iteration consists of three stages:

Stage 1. Find the non-trivial solution, r_j, of Euler-Lagrange equations (22) with $\lambda = \lambda_j$ and $f = f_j$;
Stage 2. Recover $\mathbf{u}_j$ from r_j using equation (24);
Stage 3. Update $\lambda_{j+1} \leftarrow 2\lambda_j$, $f_{j+1} \leftarrow r_j$.

Initially, we set λ_1 sufficiently large so that $\lambda_1 > (2\|F\|_{BV})^{-1}$, and $f_1 := f$. The iterations terminate when $\|f_j\|_{L^2}$ is sufficiently small. The final solution U for (1) is given by the sum of all $\mathbf{u}_j$'s.

4.1 Numerical discretization for the PDE system

We begin with *regularization*: to avoid the singularity in (18a) when $|\nabla r| = 0$, a standard approach is to regularize the problem using a small parameter $\varepsilon > 0$,

$$r_{\mu,\varepsilon} = \underset{r}{arg\,min}\left\{\lambda\,\langle r-2f,r\rangle + \mu\left(\int_{\mathbb{T}^2}\sqrt{\varepsilon^2+|\nabla r|^2}dxdy - \frac{1}{2\lambda}\right)\right\}. \tag{26}$$

At stage 1 of each regularized iteration, we find the minimizer $r = r_{\mu^*,\varepsilon}$. The corresponding Euler-Lagrange equations of the regularized problem read,

$$(r-f) - 2\lambda\,\langle f-r,r\rangle \cdot div\left(\frac{\nabla r}{\sqrt{\varepsilon^2+|\nabla r|^2}}\right) = 0. \tag{27}$$

In the non-periodic case, these equations are augmented with Neumann boundary condition, $\nabla r \cdot \mathbf{n} = 0$.

To solve (27), we cover $\mathbb{T}^2$ with a computational grid with cell size h. Let D_{+x}, D_{-x} and D_{0x} be the usual forward, backward and centered divided difference operator on x, namely, $D_{\pm x} r_{i,j} = \pm(r_{i\pm1,j} - r_{i,j})/h$, $D_{0x} r_{i,j} = (r_{i+1,j} - r_{i-1,j})/2h$. Similarly, we can define $D_{\pm y}$ and D_{0y}. A straightforward discretization of (27) yields,

$$\begin{aligned}
r_{i,j} = f_{i,j} &- K(r)\cdot D_{-x}\left[\frac{1}{\sqrt{\varepsilon^2+(D_{+x}r_{i,j})^2+(D_{0y}r_{i,j})^2}}D_{+x}r_{i,j}\right]\\
&- K(r)\cdot D_{-y}\left[\frac{1}{\sqrt{\varepsilon^2+(D_{0x}r_{i,j})^2+(D_{+y}r_{i,j})^2}}D_{+y}r_{i,j}\right]\\
= f_{i,j} &- \frac{K(r)}{h^2}\left[\frac{r_{i+1,j}-r_{i,j}}{\sqrt{\varepsilon^2+(D_{+x}r_{i,j})^2+(D_{0y}r_{i,j})^2}} - \frac{r_{i,j}-r_{i-1,j}}{\sqrt{\varepsilon^2+(D_{+x}r_{i-1,j})^2+(D_{0y}r_{i-1,j})^2}}\right]\\
&- \frac{K(r)}{h^2}\left[\frac{r_{i,j+1}-r_{i,j}}{\sqrt{\varepsilon^2+(D_{0x}r_{i,j})^2+(D_{+y}r_{i,j})^2}} - \frac{r_{i,j}-r_{i,j-1}}{\sqrt{\varepsilon^2+(D_{0x}r_{i,j-1})^2+(D_{+y}r_{i,j-1})^2}}\right].
\end{aligned} \tag{28}$$

Here, $K(r) := 2\lambda \langle r - f, r\rangle$, which is approximated using any appropriate numerical quadrature.

4.2 Computing the residuals r by implicit iterations

We use implicit iteration method to solve the nonlinear system (28),

$$
\begin{aligned}
r_{i,j}^{(n+1)} &= f_{i,j} \\
&- \frac{K(r^{(n)})}{h^2}\left[\frac{r_{i+1,j}^{(n+1)} - r_{i,j}^{(n+1)}}{\sqrt{\varepsilon^2 + (D_{+x}r_{i,j}^{(n)})^2 + (D_{0y}r_{i,j}^{(n)})^2}} - \frac{r_{i,j}^{(n+1)} - r_{i-1,j}^{(n+1)}}{\sqrt{\varepsilon^2 + (D_{+x}r_{i-1,j}^{(n)})^2 + (D_{0y}r_{i-1,j}^{(n)})^2}}\right] \\
&- \frac{K(r^{(n)})}{h^2}\left[\frac{r_{i,j+1}^{(n+1)} - r_{i,j}^{(n+1)}}{\sqrt{\varepsilon^2 + (D_{0x}r_{i,j}^{(n)})^2 + (D_{+y}r_{i,j}^{(n)})^2}} - \frac{r_{i,j}^{(n+1)} - r_{i,j-1}^{(n+1)}}{\sqrt{\varepsilon^2 + (D_{0x}r_{i,j-1}^{(n)})^2 + (D_{+y}r_{i,j-1}^{(n)})^2}}\right],
\end{aligned}
\tag{29}
$$

subject to initial condition which we set to be $r^{(0)} = f/2$.

Remark 4.1 *Recall that $K(r)$ is continuous, and $K(\bar{r}) < 0$ while $K(f) = 0$. To avoid the convergence of $r^{(n)}$ to the trivial solution, $\bar{r} = f$ (mentioned in remark (3.1)), we set $r^{(0)}$ small enough, $K(r^{(0)}) < K(\bar{r}) < K(f)$, so that $r^{(n)}$ is expected to reach the non-trivial solution $\bar{r}$, rather than f. As $\arg\min_r K(r) = f/2$, a good choice of the initial condition of the iteration is $r^{(0)} = f/2$.*

In the non-periodic case, we also need to apply Neumann boundary condition $\nabla r \cdot \mathbf{n} = 0$. To this end, we mirror r at the boundary, meaning $r_{0,j} = r_{2,j}, r_{N+1,j} = r_{N-1,j}$, etc, where the size of the grid is $N \times N$. So we only need to add the weight of the outer points to their corresponding inner points.

In summary, at the nth iteration amounts to an $N \times N$ linear system, $A(r^{(n)})\tilde{r}^{(n+1)} = \tilde{f}$, for the discretized nodes, $\{r^{(n+1)}\}$. Here, A is a sparse matrix with at most 5 non-zero entries every row or column, whose values depend on $r^{(n)}$.

4.3 Recovering $\mathbf{u}$ from r and control of errors

After we get a non-trivial solution r at stage 1, we move to stage 2 to recover $\mathbf{u}$ by (24). Normally, we apply centered divided difference operator on r to compute the discrete gradient, ∇r. However, this will cause a significant error of the solution $\mathbf{u}$.

For example, consider $u_{i,j}^1 = K \cdot \dfrac{r_{i+1,j} - r_{i-1,j}}{2h\sqrt{\varepsilon^2 + |\nabla r_{i,j}|^2}}$. Suppose the error for r in stage 1 is $e(r)$. Then, at points (x,y) such that $|\nabla r(x,y)| \approx 0$, the error for u^1 is of order $Ke(r)/(h\varepsilon)$. Therefore, dividing by $h\varepsilon$ with $\varepsilon \approx 0$, the error bound of u^1 can be significantly amplified at stage 2 of recovering $\mathbf{u}$, even if we obtain a sufficiently small $e(r)$ at stage 1. This amplification will get worse as we refine the mesh and h becomes smaller.

In order to get a reliable solution for $\mathbf{u}$, we cannot carry out stage 2 independent of the discretization stencil of stage 1. To this end, let

$$u^{1,(n+1)}_{i+1/2,j} = \frac{K^{(n)}}{h} \cdot \frac{r^{(n+1)}_{i+1,j} - r^{(n+1)}_{i,j}}{\sqrt{\varepsilon^2 + (D_{+x} r^{(n)}_{i,j})^2 + (D_{0y} r^{(n)}_{i,j})^2}}, \tag{30a}$$

$$u^{2,(n+1)}_{i,j+1/2} = \frac{K^{(n)}}{h} \cdot \frac{r^{(n+1)}_{i,j+1} - r^{(n+1)}_{i,j}}{\sqrt{\varepsilon^2 + (D_{0x} r^{(n)}_{i,j})^2 + (D_{+y} r^{(n)}_{i,j})^2}}. \tag{30b}$$

We then have

$$r_{i,j} = f_{i,j} - \frac{u^1_{i+1/2,j} - u^1_{i-1/2,j}}{h} - \frac{u^2_{i,j+1/2} - u^2_{i,j-1/2}}{h}.$$

The last two terms represent a numerical discretization of $div\,\mathbf{u}$. Therefore, we use (30) to recover $\mathbf{u}$ from the residual $r = f - div\,\mathbf{u}$ calculated at (29).

5 Hierarchical solution vs. Helmholtz solution

We apply our algorithm for the hierarchically constructed uniformly bounded solution for the example of $F \in L^2_\#$ defined at (2) with

$$\mathbb{T}^2 = [-1,1]\times[-1,1], \quad \theta = 1/3, \quad \zeta(r) \begin{cases} = e^{-\frac{1}{1-r^2}}, & |r|<1, \\ \equiv 0, & |r| \geq 1. \end{cases} \tag{31}$$

We concentrate on the first component of the solution U, denoted by U^1. Figure 1 shows Helmholtz solution, U^1_{Hel}, which slowly diverges at the origin. Figure 2 provides the hierarchical solution U^1_{Bdd} which remains uniformly bounded.

The computed hierarchical solution $\|U^{1,N}_{\text{Bdd}}\|_{L^\infty}/\|F^N\|_{L^2}$ remains uniformly bounded when N increases ($U^{1,N}_{\text{Bdd}}$ stands for the first component of hierarchical solution with grid size $N \times N$.) In contrast, table 1 illustrates the (slow) growth of the ratio $\|U^{1,N}_{\text{Hel}}\|_{L^\infty}/\|F^N\|_{L^2}$.

6 Hierarchical solution meets Helmholtz solution

The hierarchical solution is uniformly bounded. However, as observed in figure 2, the hierarchical solution U^1_{Bdd} is oscillatory outside the support of F. As each step

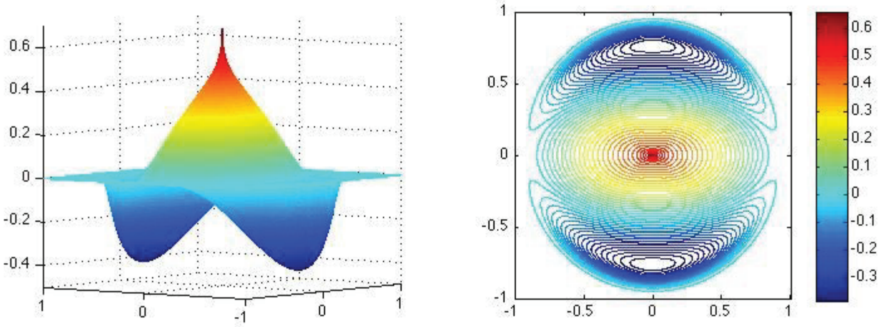

Fig. 1 Helmholtz solution U^1_{Hel} of example (2),(31).

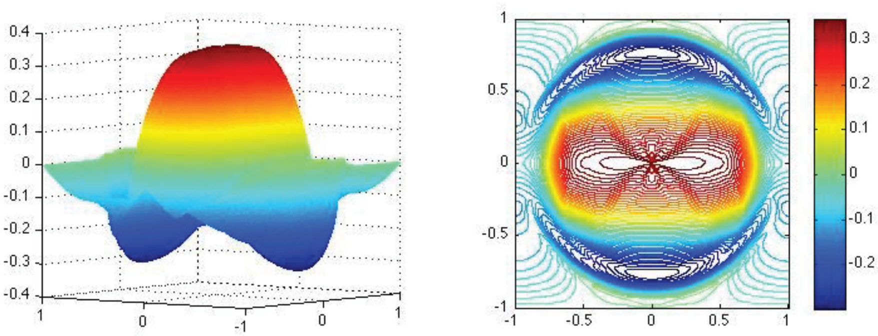

Fig. 2 Hierarchical solution U^1_{Bdd} of (2),(31).

The $N \times N$ grid	50×50	100×100	200×200	400×400	800×800
$\dfrac{\Vert U^{1,N}_{\mathrm{Hel}} \Vert_{L^\infty}}{\Vert F^N \Vert_{L^2}}$	0.2295	0.2422	0.2540	0.2650	0.2752
$\dfrac{\Vert U^{1,N}_{\mathrm{Bdd}} \Vert_{L^\infty}}{\Vert F^N \Vert_{L^2}}$	0.1454	0.1451	0.1455	0.1458	0.1451

Table 1 L^∞ norm of numerical solutions for different grids: Helmholtz vs. hierarchical construction

of the hierarchical decomposition relies on the previous steps, these oscillations will grow throughout the iterations. To limit their effect, we introduce a new, two-step method to construct bounded solutions of (1). It consists of one hierarchical decomposition step, whose residual is treated using Helmholtz decomposition:

Step 1. Solve minimization problem

$$\mathbf{u}_1 := argmin\mathbf{u}\{\|\mathbf{u}\|_{L^\infty} + \lambda_1 \|F - div\,\mathbf{u}\|_{L^2}^2\}. \tag{32a}$$

Step 2. Find the Helmholtz solution for $div\,\mathbf{u}_r = r_1$, i.e.

$$\mathbf{u}_r := \nabla\Delta^{-1} r_1, \quad r_1 = F - div\,\mathbf{u}_1. \tag{32b}$$

Clearly, the two-step solution, $U_{2\text{step}} = \mathbf{u}_1 + \mathbf{u}_r$, satisfies $div\,U = F$. Furthermore, it is uniformly bounded.

Proposition 6.1 *The two-step solution, $U_{2step} = \mathbf{u}_1 + \mathbf{u}_r$ given in* (32)*, is a uniformly bounded solution of* (1)*.*

Proof. Clearly, $\mathbf{u}_1$, as the first iteration of the hierarchical solution, is uniformly bounded. Next, $\mathbf{u}_r = \nabla\Delta^{-1} r_1 = \left(-\frac{1}{2\pi}\frac{\mathbf{x}}{|\mathbf{x}|^2}\right) \star r_1$. The Newtonian potential, $\left(-\frac{1}{2\pi}\frac{\mathbf{x}}{|\mathbf{x}|^2}\right)$, belongs to the Lorentz space $L^{2,\infty}$. The residual, r_1 is BV-bounded and hence, [10, 3], $r_1 \in BV \subset L^{2,1}$. By Hölder's inequality for Lorentz spaces, [6, 10], $\mathbf{u}_r$ and therefore $U_{2\text{step}}$, are uniformly bounded.

From Proposition 6.1, we know that $U_{2\text{step}}$ is also a solution of (1). As the minimization problem is solved only once, we expect fewer oscillations in $U_{2\text{step}}$ than U_{Bdd}.

Figure 3 shows the two-step solution of the example in Section 5. From the contour plot, we observe fewer oscillations than the hierarchical solution U_{Bdd}. Yet, the solution is not as smooth as U_{Bdd} at the origin. Table 2 reports that the ratio $\|U_{2\text{step}}^{1,N}\|_{L^\infty}/\|F^N\|_{L^2}$ is also stable when N is large. This verifies the uniformly boundedness of the two-step solution.

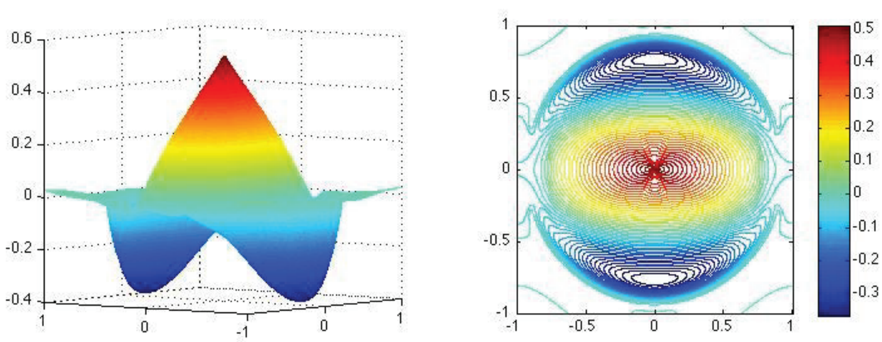

Fig. 3 Two-step solution, $U_{2\text{step}}^1$.

The $N \times N$ grid	50×50	100×100	200×200	400×400	800×800
$\dfrac{\|U^{1,N}_{2\text{step}}\|_{L^\infty}}{\|F^N\|_{L^2}}$	0.2096	0.2128	0.2144	0.2151	0.2154

Table 2 The two-step solution of (2),(31) for different grids.

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