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Convergence of MUSCL Relaxing Schemes to the Relaxed Schemes for Conservation Laws with Stiff Source Terms

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We consider the convergence and stability property of MUSCL relaxing schemes applied to conservation laws with stiff source terms. The maximum principle for the numerical schemes will be established. It will be also shown that the MUSCL relaxing schemes are uniformly l^1- and TV-stable in the sense that they are bounded by a constant independent of the relaxation parameter ε, the Lipschitz constant of the stiff source term and the time step Δt. The Lipschitz constant of the l^1 continuity in time for the MUSCL relaxing schemes is shown to be independent of ε and Δt. The convergence of the relaxing schemes to the corresponding MUSCL relaxed schemes is then established.

KEY WORDS: Relaxation scheme; nonlinear conservation laws; maximum principle; convergence.

1. INTRODUCTION

We consider the following Cauchy problem

$$\begin{cases} u_t + f(u)_x = q(u) & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.1}$$

where $f \in C^1(\mathbb{R})$, $f(0) = 0$ and $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. The nonlinear conservation law (1.1) is stiff if the time scale introduced by the source term q is small compared with the characteristic speed f' and some other appropriate length scale. It is observed that a realistic assumption on the source term

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is $q'(u) \leqslant 0$ for all $u \in \mathbb{R}$. It is indeed the case for many practical problems, e.g., in the model of combustion [4, 11], water waves in presence of the friction force in the bottom [29]. This assumption is also used by several authors in their theoretical and numerical analysis for (1.1), see, e.g., Chalabi [2], Chen *et al.*[3], Schroll and Winther [19], and Tang [25]. In the sense of Chen, Levermore and Liu [3], $q' < 0$ means the dissipativity of the source term. Furthermore, as usual, we assume that $u = 0$ is an equilibrium solution. Hence, throughout this paper we assume that

$$q(0) = 0, \quad -K \leqslant q'(u) \leqslant 0, \qquad \text{for some constant} \quad K \gg 1 \tag{1.2}$$

We want to approximate the global weak entropy solution of the Cauchy problem (1.1) by relaxation schemes. The problem (1.1) can be related to a singular perturbation problem:

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = q(u^\varepsilon) \\ v_t^\varepsilon + \mathsf{a} u_x^\varepsilon = -\dfrac{1}{\varepsilon}(v^\varepsilon - f(u^\varepsilon)), \qquad \varepsilon > 0 \end{cases} \tag{1.3}$$

where a is some positive constant. The corresponding relaxation system for the homogeneous conservation laws is

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = 0 \\ v_t^\varepsilon + \mathsf{a} u_x^\varepsilon = -\dfrac{1}{\varepsilon}(v^\varepsilon - f(u^\varepsilon)), \qquad \varepsilon > 0 \end{cases} \tag{1.4}$$

The relaxation limit for the 2×2 relaxation system without source term was first studied by Liu [9], who justified some nonlinear stability criteria for diffusion waves, expansion waves and traveling waves. A general mathematical framework was analyzed by Chen *et al.* [3] for the nonlinear systems (1.4). The presence of relaxation mechanisms is widespread in both the continuum mechanics as well as the kinetic theory contexts. Relaxation is known to provide a subtle dissipative mechanism for discontinuities against the destabilizing effect of nonlinear response [9]. The relaxation models can be loosely interpreted as discrete velocity kinetic equations. The relaxation parameter, ε, plays the role of the mean free path and the system models the macroscopic conservation law. In that sense they are a discrete velocity analogue of the kinetic equations introduced by Perthame and Tadmor [18] and Lions *et al.* [12].

On the numerical side, relaxation schemes proposed by Jin and Xin [5] are a class of nonoscillatory numerical schemes for systems of conservation laws. They provide a new way of approximating the solutions of the nonlinear conservation laws. The computational results, see, e.g., Jin and Xin [5] and Aregba-Driollet and Natalini [1], indicate that the relaxation methods obtained in the limit $\varepsilon \to 0$ provide a promising class of schemes. The main advantages of these schemes are that they neither require the computation of the Jacobians of fluxes for the conservation laws nor the use of Riemann-solvers. This important property is shared by other schemes such as the high resolution central schemes introduced by Nessyahu and Tadmor [15].

For homogeneous conservation laws, there have been many recent studies concerning the asymptotic convergence of the relaxation systems to the corresponding equilibrium conservation laws as the rate of the relaxation tends to zero. Katsoulakis and Tzavaras [6] introduced a class of relaxation systems, namely the contractive relaxation systems, and established an $\mathcal{O}(\sqrt{\varepsilon})$ error bound in the case that the equilibrium equation is a scalar multidimensional one. Kurganov and Tadmor [7] studied the convergence and error estimates for a class of relaxation systems, including (1.4) as a special case, and concluded an $\mathcal{O}(\varepsilon)$ order of convergence for scalar convex conversation laws. The novelty of their approach is the use of a weak *Lip*′-measure of the error, which allow them to obtain the sharp error bounds. For the relaxation system (1.4), Natalini [13] proved that the solutions to the relaxation system converges strongly to the unique entropy solution of the corresponding conservation laws as $\varepsilon \to 0$. Based on a general framework developed in [26], the $\mathcal{O}(\varepsilon)$ rate of convergence in L^1 is established by Teng [27] in the case that the equilibrium solutions are piecewisely smooth. In the same case, Tadmor and Tang [23] obtained the optimal $\mathcal{O}(\varepsilon)$-pointwise error estimates away from the shock discontinuities, which is based on a general framework of [22].

In this paper, we wish to analyze a class of fully-discretized MUSCL schemes for approximating the relaxation system (1.3). The schemes are extension of a prototype of MUSCL relaxing schemes introduced by Jin and Xin [5] for approximating systems of conservation laws. In the semi-implicit scheme (2.1) below, we treat the stiff source term $q(u)$ and the relaxation term $(v - f(u))/\varepsilon$ implicitly. It is the semi-implicit treatment that makes the CFL condition *independent* of the Lipschitz constant of the stiff source term. The convergence theory for the relaxing scheme (2.1) and the relaxed scheme (2.5) with $q \equiv 0$ can be found in [1, 28, 30]. Consult Natalini [14] for an overview of the recent developments for hyperbolic relaxation problems.

This paper is organized as follows. In Sec. 2 we introduce the MUSCL relaxing schemes for the Cauchy problem (1.1) and give some properties of the schemes. Section 3 is devoted to show the discrete maximum principle for the solutions of the relaxing scheme (2.1). In Sec. 4, we show that the solutions of the scheme (2.1) are l^1- and TV-bounded by a constant independent of the relaxation parameter ε, the Lipschitz constant of the stiff source term and the time step Δt. In Sec. 5 we show that the Lipschitz constant of the l^1-continuity in time for the numerical solutions of (2.1) is independent of ε and Δt. This property, together with the uniform TV-boundedness, leads to the convergence of the MUSCL relaxing schemes.

2. NUMERICAL SCHEMES

We choose a time step Δt, a spatial mesh size Δx, a parameter a which will be related to the characteristic speed of the conservation law and a small relaxation parameter $\varepsilon > 0$. For these we define the mesh ratio $\lambda = \Delta t / \Delta x$, the CFL parameter $\mu = \sqrt{\mathsf{a}}\,\lambda \in (0, 1)$ and the scale parameter $\mathsf{k} = \Delta t / \varepsilon$. The mesh is given by the points $(x_j, t_n) = (j\,\Delta x, n\,\Delta t)$ for $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$. The approximate solution takes the discrete values $u_j^{n,\varepsilon}$ at the grid points. The relaxing schemes involve the discrete relaxation fluxes $v_j^{n,\varepsilon}$.

The numerical schemes studied in this work, i.e., (2.1) and (2.5) below, are all natural extension of Jin and Xin's schemes for homogeneous conservation laws, see (4.8) and (4.11) in [5]. In this sense, the MUSCL relaxing scheme for Cauchy problem (1.1) is of the form

$$\frac{u_j^{n+1,\varepsilon} - u_j^{n,\varepsilon}}{\Delta t} + \frac{1}{2\,\Delta x}(v_{j+1}^{n,\varepsilon} - v_{j-1}^{n,\varepsilon}) - \frac{\sqrt{\mathsf{a}}}{2\,\Delta x}(u_{j+1}^{n,\varepsilon} - 2u_j^{n,\varepsilon} + u_{j-1}^{n,\varepsilon})$$

$$+ \frac{1-\mu}{4}[(\sigma_j^{+,\varepsilon} - \sigma_{j-1}^{+\varepsilon}) - (\sigma_{j+1}^{-,\varepsilon} - \sigma_j^{-\varepsilon})] = q(u_j^{n+1,\varepsilon}) \tag{2.1a}$$

$$\frac{v_j^{n+1,\varepsilon} - v_j^{n,\varepsilon}}{\Delta t} + \frac{\mathsf{a}}{2\,\Delta x}(u_{j+1}^{n,\varepsilon} - u_{j-1}^{n,\varepsilon}) - \frac{\sqrt{\mathsf{a}}}{2\,\Delta x}(v_{j+1}^{n,\varepsilon} - 2v_j^{n,\varepsilon} + v_{j-1}^{n,\varepsilon})$$

$$+ \frac{\sqrt{\mathsf{a}}(1-\mu)}{4}[(\sigma_j^{+,\varepsilon} - \sigma_{j-1}^{+,\varepsilon}) + (\sigma_{j+1}^{-,\varepsilon} - \sigma_j^{-,\varepsilon})]$$

$$= -\frac{1}{\varepsilon}(v_j^{n+1} - f(u_j^{n+1,\varepsilon})) \tag{2.1b}$$

where $\sigma_j^{\pm,\varepsilon}$ and $\theta_j^{\pm,\varepsilon}$ are defined by

$$\sigma_j^{\pm,\varepsilon} = \frac{1}{\Delta x}\Delta_+(v_j^{n,\varepsilon} \pm \sqrt{\mathsf{a}}\,u_j^{n,\varepsilon})\,\phi(\theta_j^{\pm,\varepsilon})$$

$$\theta_j^{\pm,\varepsilon} = \frac{\Delta_-(v_j^{n,\varepsilon} \pm \sqrt{\mathsf{a}}\,u_j^{n,\varepsilon}}{\Delta_+(v_j^{n,\varepsilon} \pm \sqrt{\mathsf{a}}\,u_j^{n,\varepsilon})} \tag{2.2}$$

In the above formulas, $\Delta_{\pm} u_j = \mp (u_j - u_{j\pm 1})$, and $\phi(\theta)$ is a limiter function satisfying

$$0 \leqslant \frac{\phi(\theta)}{\theta} \leqslant 2, \qquad 0 \leqslant \phi(\theta) \leqslant 2 \tag{2.3}$$

The discrete initial data are given by averaging the initial data $u_0(x)$ over mesh cells $I_j = ((j - \frac{1}{2})\,\Delta x, (j + \frac{1}{2})\,\Delta x)$, i.e., taking

$$u_j^{0,\varepsilon} = \frac{1}{\Delta x} \int_{I_j} u_0(x)\,dx, \qquad \text{and} \qquad v_j^{0,\varepsilon} = f(u_j^{0,\varepsilon}) \tag{2.4}$$

The corresponding relaxed scheme as $\varepsilon \to 0$ limit of (2.1) is as follows:

$$\begin{aligned} &v_j^n = f(u_j^n), \\ &\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2\,\Delta x}(v_{j+1}^n - v_{j-1}^n) - \frac{\sqrt{a}}{2\,\Delta x}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \\ &\qquad + \frac{1-\mu}{4}[(\sigma_j^+ - \sigma_{j-1}^+) - (\sigma_{j+1}^- - \sigma_j^-)] = q(u_j^{n+1}) \end{aligned} \tag{2.5}$$

To guarantee the entropy consistency of the relaxed scheme (2.5), the following slightly stronger conditions are proposed in Tang *et al.* [24]:

$$\sup_u |f'(u)| \leqslant \frac{1}{\beta}\sqrt{a} \qquad \text{subcharacteristic condition} \tag{2.6}$$

$$0 \leqslant \frac{\phi(\theta)}{\theta} \leqslant X, \quad 0 \leqslant \phi(\theta) \leqslant X \qquad \text{limiter function condition} \tag{2.7}$$

The parameters β and X in the condition (2.6) and (2.7) satisfy

$$\beta > 1, \qquad 0 < X < 2, \qquad 1 - \frac{1}{\beta} \geqslant X(1-\mu) \tag{2.8}$$

It is shown by Tang *et al.* [24] that under the assumptions (2.6)–(2.8) the MUSCL relaxed scheme (2.5) with $q \equiv 0$ satisfy the cell entropy inequalities.

We will introduce some notations useful in the following sections. We take the Riemann invariants

$$\begin{pmatrix} \mathbf{R}_{1,j}^{n,\varepsilon} \\ \mathbf{R}_{2,j}^{n,\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(u_j^{n,\varepsilon} - \frac{v_j^{n,\varepsilon}}{\sqrt{\mathsf{a}}}\right) \\ \frac{1}{2}\left(u_j^{n,\varepsilon} + \frac{v_j^{n,\varepsilon}}{\sqrt{\mathsf{a}}}\right) \end{pmatrix} \tag{2.9}$$

and define as usual the Maxwellians

$$\begin{pmatrix} \mathbf{M}_1(u_j^{n,\varepsilon}) \\ \mathbf{M}_2(u_j^{n,\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(u_j^{n,\varepsilon} - \frac{f(u_j^{n,\varepsilon})}{\sqrt{\mathsf{a}}}\right) \\ \frac{1}{2}\left(u_j^{n,\varepsilon} + \frac{f(u_j^{n,\varepsilon})}{\sqrt{\mathsf{a}}}\right) \end{pmatrix} \tag{2.10}$$

Then the relaxing scheme (2.1) can be rewritten as

$$\begin{aligned} \mathbf{R}_{1,j}^{n+1,\varepsilon} &= \mathbf{R}_{1,j}^{n+(1/2),\varepsilon} + \mathsf{k}(\mathbf{M}_1(u_j^{n+1,\varepsilon}) - \mathbf{R}_{1,j}^{n+1,\varepsilon})) + q(u_j^{n+1,\varepsilon})\,\Delta t/2 \\ \mathbf{R}_{2,j}^{n+1,\varepsilon} &= \mathbf{R}_{2,j}^{n+(1/2),\varepsilon} + \mathsf{k}(\mathbf{M}_2(u_j^{n+1,\varepsilon}) - \mathbf{R}_{2,j}^{n+1,\varepsilon})) + q(u_j^{n+1,\varepsilon})\,\Delta t/2 \end{aligned} \tag{2.11}$$

with

$$\begin{aligned} \mathbf{R}_{1,j}^{n+(1/2),\varepsilon} &:= (1-\mu)\,\mathbf{R}_{1,j}^{n,\varepsilon} + \mu\mathbf{R}_{1,j+1}^{n,\varepsilon} + \frac{\Delta t(1-\mu)}{4}\,(\sigma_{j+1}^{-,\varepsilon} - \sigma_j^{-,\varepsilon}) \\ \mathbf{R}_{2,j}^{n+(1/2),\varepsilon} &:= (1-\mu)\,\mathbf{R}_{2,j}^{n,\varepsilon} + \mu\mathbf{R}_{2,j-1}^{n,\varepsilon} + \frac{\Delta t(1-\mu)}{4}\,(\sigma_j^{+,\varepsilon} - \sigma_{j-1}^{+,\varepsilon}) \end{aligned} \tag{2.12}$$

It can be verified that

$$\begin{aligned} \mathbf{R}_{1,j}^{n+(1/2),\varepsilon} &= (1 - c_{1,j}^{n,\varepsilon})\,\mathbf{R}_{1,j}^{n,\varepsilon} + c_{1,j}^{n,\varepsilon}\mathbf{R}_{1,j+1}^{n,\varepsilon} \\ \mathbf{R}_{2,j}^{n+(1/2),\varepsilon} &= (1 - d_{-1,j}^{n,\varepsilon})\,\mathbf{R}_{2,j}^{n,\varepsilon} + d_{-1,j}^{n,\varepsilon}\mathbf{R}_{2,j-1}^{n,\varepsilon} \end{aligned}$$

where

$$\begin{aligned} c_{1,j}^{n,\varepsilon} &= \mu - \frac{(1-\mu)\,\mu}{2}\left[\frac{\phi(\theta_{j+1}^{-,\varepsilon})}{\theta_{j+1}^{-,\varepsilon}} - \phi(\theta_j^{-,\varepsilon})\right] \\ d_{-1,j}^{n,\varepsilon} &= \mu + \frac{(1-\mu)\,\mu}{2}\left[\frac{\phi(\theta_j^{+,\varepsilon})}{\theta_j^{+}} - \phi(\theta_{j-1}^{+,\varepsilon})\right] \end{aligned} \tag{2.13}$$

It is easily seen that condition (2.3) implies

$$\left[\frac{\phi(\theta_j^{\pm\varepsilon})}{\theta_j^{\pm\varepsilon}} - \phi(\theta_{j-1}^{\pm\varepsilon}) \right] \leqslant 2 \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\begin{aligned} 1 > 1-(1-\mu)^2 \geqslant c_{1,j}^{n,\varepsilon} \geqslant \mu^2 > 0 \\ 1 > 1-(1-\mu)^2 \geqslant d_{-1,j}^{n,\varepsilon} \geqslant \mu^2 > 0 \end{aligned} \tag{2.15}$$

The above results indicate that the coefficients $c_{1,j}^{n,\varepsilon}$ and $d_{-1,j}^{n,\varepsilon}$ take values in the interval $[\mu^2, 1)$.

3. THE DISCRETE MAXIMUM PRINCIPLE

The maximum principle is an important tool for analyzing and understanding the theoretical properties of numerical schemes. For the numerical scheme (2.1), it is not easy to establish such a principle due to the stiffness of the source term. We will devote this whole section to the following theorem. The key requirement is that the bounds for the numerical solution $u_j^{n,\varepsilon}$ should be independent of ε, Δt and the Lipschitz constant of the stiff source term q.

Theorem 3.1 (Maximum Principle). Let b be the L^∞ bound of initial data $u_0(x)$, i.e.,

$$\|u_0(\bullet)\|_{L^\infty} \leqslant \mathsf{b} \tag{3.1}$$

We assume that the subcharacteristic condition (2.6) is satisfied, namely for some constant $\beta > 1$

$$\sup_{|u| \leqslant \mathsf{b}} |f'(u)| \leqslant \frac{1}{\beta} \sqrt{\mathsf{a}} \tag{3.2}$$

Then any solution $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})_{j\in\mathbb{Z}, n\in\mathbb{N}_0}$ of the scheme (2.1) with initial data (2.4) satisfies

$$\mathbf{M}_i(-\mathsf{b}) \leqslant \mathbf{R}_{i,j}^{n,\varepsilon} \leqslant \mathbf{M}_i(\mathsf{b}), \qquad i=1,2 \tag{3.3}$$

$$|u_j^{n,\varepsilon}| \leqslant \mathsf{b} \tag{3.4}$$

provided that ε is sufficiently small.

Proof. We prove the theorem by induction. It is obvious that the averaging (2.4) maintains the bound

$$|u_j^{0,\varepsilon}| \leqslant \mathsf{b} \tag{3.5}$$

The Maxwellians $\mathbf{M}_i(u)$ are nondecreasing functions for $|u| \leqslant \mathsf{b}$ under the sub characteristic condition (3.2). This can be verified by differentiating (2.10). Then with $v_j^0 = f(u_j^0)$ we have for $i = 1, 2$, $j \in \mathbb{Z}$ that

$$\mathbf{M}_i(-\mathsf{b}) \leqslant R_{i,j}^{0,\varepsilon} = \frac{1}{2}\left(u_j^{0,\varepsilon} + (-1)^i \frac{v_j^{0,\varepsilon}}{\sqrt{\mathsf{a}}}\right) = \mathbf{M}_i(u_j^0) \leqslant \mathbf{M}_i(\mathsf{b}) \tag{3.6}$$

For the induction, we assume that the estimates (3.3)–(3.4) hold for $n = m$, namely,

$$\mathbf{M}_i(-\mathsf{b}) \leqslant \mathbf{R}_{i,j}^{m,\varepsilon} \leqslant \mathbf{M}_i(\mathsf{b}), \qquad i = 1, 2 \tag{3.7}$$

$$|u_j^{m,\varepsilon}| \leqslant \mathsf{b} \tag{3.8}$$

We will then show that (3.3)–(3.4) hold for $n = m+1$. First we prove $|u_j^{m+1,\varepsilon}| \leqslant \mathsf{b}$. Adding both sides of the two equations in (2.11) with (2.12) for $n = m$ gives

$$\begin{aligned} u_j^{m+1,\varepsilon} = {} & (1 - c_{1,j}^{m,\varepsilon})\, R_{1,j}^{m,\varepsilon} + c_{1,j}^{m,\varepsilon} \mathbf{R}_{1,j+1}^{m,\varepsilon} + (1 - d_{-1,j}^{m,\varepsilon})\, \mathbf{R}_{2,j}^{m,\varepsilon} \\ & + d_{-1,\varepsilon}^{m,\varepsilon} \mathbf{R}_{2,j-1}^{m,\varepsilon} + q(u_j^{m+1,\varepsilon})\, \Delta t \end{aligned}$$

The above result, together with $q(0) = 0$, leads to

$$\begin{aligned} u_j^{m+1,\varepsilon} = {} & (1 - q'(\xi_j^{m+1,\varepsilon})\, \Delta t)^{-1} \,((1 - c_{1,j}^{m,\varepsilon})\, \mathbf{R}_{1,j}^{m,\varepsilon} + c_{1,j}^{m,\varepsilon} \mathbf{R}_{1,j+1}^{m,\varepsilon} \\ & + (1 - d_{-1,j}^{m,\varepsilon})\, \mathbf{R}_{2,j}^{m,\varepsilon} + d_{-1,j}^{m,\varepsilon} \mathbf{R}_{2,j-1}^{m,\varepsilon}) \end{aligned} \tag{3.9}$$

where $\xi_j^{m+1,\varepsilon}$ is an intermediate value between $u_j^{m+1,\varepsilon}$ and 0. Observe that

$$\begin{aligned} &(1 - q'(\xi_j^{m+1,\varepsilon})\, \Delta t) \geqslant 1 && \text{since} \quad q'(u) \leqslant 0 \\ &0 < c_{1,j}^{m,\varepsilon} < 1, \quad 0 < d_{-1,j}^{m,\varepsilon} < 1 && \text{from} \quad (2.15) \\ &\mathbf{M}_1(u) + \mathbf{M}_2(u) = u && \text{for all} \quad u \in \mathbb{R} \end{aligned}$$

These observations, together with the induction assumption (3.7) and the equation (3.9), yield

$$|u_j^{m+1,\varepsilon}| \leqslant \mathsf{b} \tag{3.10}$$

Now we will show that (3.3) holds for $n = m+1$. Set

$$r_{i,j}^{m+1,\varepsilon} = \mathbf{R}_{i,j}^{m+1,\varepsilon} - \mathbf{M}_i(\mathsf{b}), \qquad r_{i,j}^{m+(1/2),\varepsilon} = \mathbf{R}_{i,j}^{m+(1/2),\varepsilon} - \mathbf{M}_i(\mathsf{b}), \qquad i = 1, 2$$

Then the scheme (2.11) can be rewritten as

$$\begin{aligned} r_{1,j}^{m+1,\varepsilon} &= r_{1,j}^{m+(1/2),\varepsilon} + \mathsf{k}(\mathbf{M}_1(u_j^{m+1,\varepsilon}) - \mathbf{M}_1(\mathsf{b}) - r_{1,j}^{m+1,\varepsilon}) \\ &\quad + \frac{\Delta t}{2}(q(u_j^{m+1,\varepsilon}) - q(\mathsf{b})) + \frac{\Delta t}{2} q(\mathsf{b}) \\ r_{2,j}^{m+1,\varepsilon} &= r_{2,j}^{m+(1/2),\varepsilon} + \mathsf{k}(\mathbf{M}_2(u_j^{m+1,\varepsilon}) - \mathbf{M}_2(\mathsf{b}) - r_{2,j}^{m+1,\varepsilon}) \\ &\quad + \frac{\Delta t}{2}(q(u_j^{m+1,\varepsilon}) - q(\mathsf{b})) + \frac{\Delta t}{2} q(\mathsf{b}) \end{aligned} \tag{3.11}$$

Applying the Mean-Value Theorem to the terms $\mathbf{M}_1$, $\mathbf{M}_2$ and q in (3.11) and noting that $u_j^{m+1,\varepsilon} - \mathsf{b} = r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}$ yield

$$\begin{aligned} r_{1,j}^{m+1,\varepsilon} &= r_{1,j}^{m+(1/2),\varepsilon} + \mathsf{k}\left(\frac{1}{2}(r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}) \right. \\ &\quad \left. - \frac{f'(\xi)}{2\sqrt{\mathsf{a}}}(r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}) - r_{1,j}^{m+1,\varepsilon}\right) \\ &\quad + \frac{\Delta t}{2} q'(\eta)(r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}) + \frac{\Delta t}{2} q(\mathsf{b}) \\ r_{2,j}^{m+1,\varepsilon} &= r_{2,j}^{m+(1/2),\varepsilon} + \mathsf{k}\left(\frac{1}{2}(r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}) \right. \\ &\quad \left. + \frac{f'(\xi)}{2\sqrt{\mathsf{a}}}(r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}) - r_{2,j}^{m+1,\varepsilon}\right) \\ &\quad + \frac{\Delta t}{2} q'(\eta)(r_{1,j}^{m+1,\varepsilon} + r_{2,j}^{m+1,\varepsilon}) + \frac{\Delta t}{2} q(\mathsf{b}) \end{aligned} \tag{3.12}$$

where ξ and η are intermediate values between $u_j^{m+1,\varepsilon}$ and b. It follows from (3.10) that $|\xi| \leqslant \mathsf{b}$. Solving the scheme (3.12) for $r_{1,j}^{m+1,\varepsilon}$ and $r_{2,j}^{m+1,\varepsilon}$ gives

$$r_{1,j}^{m+1,\varepsilon} = \frac{1}{\Delta}\left[(1+\alpha - q'(\eta)\,\Delta t/2)\, r_{1,j}^{m+(1/2),\varepsilon} + (\alpha + q'(\eta)\,\Delta t/2)\, r_{2,j}^{m+(1/2),\varepsilon}\right] + \frac{1}{\Delta}(1+2\alpha)\, q(\mathsf{b})\,\Delta t/2$$

$$r_{2,j}^{m+1,\varepsilon} = \frac{1}{\Delta}\left[(\gamma + q'(\eta)\,\Delta t/2)\, r_{1,j}^{m+(1/2),\varepsilon} + (1+\gamma - q'(\eta)\,\Delta t/2)\, r_{2,j}^{m+(1/2),\varepsilon}\right] + \frac{1}{\Delta}(1+2\gamma)\, q(\mathsf{b})\,\Delta t/2 \tag{3.13}$$

where

$$\alpha = \frac{\mathsf{k}}{2}\left(1 - \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right), \qquad \gamma = \frac{\mathsf{k}}{2}\left(1 + \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right)$$

$$\Delta = (1+\alpha+\gamma)(1 - q'(\eta)\,\Delta t)$$

It follows from the fact $|\xi| \leqslant \mathsf{b}$ and the subcharacteristic condition (3.2) that

$$\alpha \geqslant \frac{\mathsf{k}}{2}(1 - 1/\beta) > 0, \qquad \gamma \geqslant \frac{\mathsf{k}}{2}(1 - 1/\beta) > 0 \tag{3.14}$$

Using the above results we obtain

$$\Delta > 0, \qquad 1+\alpha - q'(\eta)\,\Delta t/2 > 0, \qquad 1+\gamma - q'(\eta)\,\Delta t/2 > 0 \tag{3.15}$$

Furthermore, using the first inequality of (3.14) and the subcharacteristic condition (3.2) we obtain

$$\alpha + q'(\eta)\,\Delta t/2 \geqslant \frac{\Delta t}{2}\left[\frac{\beta-1}{\varepsilon\beta} + q'(\eta)\right] \geqslant \frac{\Delta t}{2}\left[\frac{\beta-1}{\varepsilon\beta} - K\right] \geqslant 0 \tag{3.16}$$

provided that $\varepsilon \leqslant (\beta-1)/K\beta$. In (3.16), we used the definition for k, namely $\mathsf{k} = \Delta t/\varepsilon$. Similarly, the following estimate holds

$$\gamma + q'(\eta)\,\Delta t/2 \geqslant 0 \tag{3.17}$$

provided that $\varepsilon \leqslant (\beta - 1)/K\beta$. It follows from (2.12) and the induction assumption (3.7) that

$$\begin{aligned} r_{1,j}^{m+(1/2),\varepsilon} &= \mathbf{R}_{1,j}^{m+(1/2),\varepsilon} - \mathbf{M}_1(\mathsf{b}) \\ &= (1 - c_{1,j}^{m,\varepsilon})\, \mathbf{R}_{1,j}^{m,\varepsilon} + c_{1,j}^{m,\varepsilon} \mathbf{R}_{1,j+1}^{m,\varepsilon} - \mathbf{M}_1(\mathsf{b}) \leqslant 0 \end{aligned} \tag{3.18}$$

In a similar manner, we can show that

$$r_{2,j}^{m+(1/2),\varepsilon} \leqslant 0 \tag{3.19}$$

By noting that $q(0) = 0$, we have

$$q(\mathsf{b}) = q'(\xi)\, \mathsf{b} \leqslant 0 \tag{3.20}$$

where ξ is an intermediate value between b and zero. Applying the inequalities (3.15)–(3.20) to (3.13) gives

$$r_{i,j}^{m+1,\varepsilon} \leqslant 0, \qquad i = 1, 2$$

By using the definition for $r_{i,j}^{m+1,\varepsilon}$, the above estimate is equivalent to

$$\mathbf{R}_{i,j}^{m+1,\varepsilon} \leqslant \mathbf{M}_i(\mathsf{b}), \qquad i = 1, 2 \tag{3.21}$$

Similarly, by considering $\bar{r}_{i,j}^{m+1,\varepsilon} = \mathbf{R}_{i,j}^{m+1,\varepsilon} - \mathbf{M}_i(-\mathsf{b})$ we can obtain

$$\mathbf{M}_i(-\mathsf{b}) \leqslant \mathbf{R}_{i,j}^{m+1,\varepsilon}, \qquad i = 1, 2 \tag{3.22}$$

The inequalities (3.10), together with (3.21) and (3.22), imply that the estimates (3.3) and (3.4) hold for $n = m + 1$. This completes the proof of the induction. □

4. STABILITY

This section is devoted to establishing the l^1- and TV-stability for the numerical solution of MUSCL relaxing scheme (2.1) with initial data (2.4). Since $u_0 \in BV(\mathbf{R})$, there exists a constant $\mathbf{M}$ such that

$$\|u_0(\bullet)\|_{BV} \leqslant \mathbf{M} \tag{4.1}$$

By the definition of the initial data $u_j^{0,\varepsilon}$ given in (2.4), we have

$$TV(u^{0,\varepsilon}) := \sum_j |u_{j+1}^{0,\varepsilon} - u_j^{0,\varepsilon}| \leqslant \mathbf{M} \tag{4.2}$$

Furthermore, it follows from the L^∞-boundedness of u_0 in (3.1) and the subcharacteristic condition (3.2) that

$$\begin{aligned} TV(\mathbf{R}_1^{0,\varepsilon}, \mathbf{R}_2^{0,\varepsilon}) &:= \sum_j (|\mathbf{R}_{1,j+1}^{0,\varepsilon} - \mathbf{R}_{1,j}^{0,\varepsilon}| + |\mathbf{R}_{2,j+1}^{0,\varepsilon} - \mathbf{R}_{2,j}^{0,\varepsilon}|) \\ &\leqslant \sum_j \left(|u_{j+1}^{0,\varepsilon} - u_j^{0,\varepsilon}| + \frac{1}{\sqrt{\mathsf{a}}} \sup_{|\xi| \leqslant \mathsf{b}} |f'(\xi)| \, |u_{j+1}^{0,\varepsilon} - u_j^{0,\varepsilon}| \right) \\ &\leqslant 2\mathbf{M} \end{aligned} \tag{4.3}$$

In the following lemma, we show that the result in (4.3) holds for all time levels.

Lemma 4.1. Assume $u_0 \in BV \cap L^\infty$. If the subcharacteristic condition (3.2) is satisfied, then the relaxing scheme (2.11) is *TVD* (total variation diminishing), i.e.,

$$\begin{aligned} TV(R_1^{n+1,\varepsilon}, R_2^{n+1,\varepsilon}) &:= \sum_j (|R_{1,j+1}^{n+1,\varepsilon} - R_{1,j}^{n+1,\varepsilon}| + |R_{2,j+1}^{n+1,\varepsilon} - R_{2,j}^{n+1,\varepsilon}|) \\ &\leqslant TV(R_1^{n,\varepsilon}, R_2^{n,\varepsilon}) \\ &\leqslant 2\mathbf{M} \end{aligned} \tag{4.4}$$

Proof. We begin by setting

$$\bar{\mathbf{R}}_{i,j}^{n,\varepsilon} = \mathbf{R}_{i,j+1}^{n,\varepsilon} - \mathbf{R}_{i,j}^{n,\varepsilon}, \qquad \bar{\mathbf{R}}_{i,j}^{n+(1/2),\varepsilon} = \mathbf{R}_{i,j+1}^{n+(1/2),\varepsilon} - \mathbf{R}_{i,j}^{n+(1/2),\varepsilon}, \qquad i = 1, 2$$

Subtracting (2.12) with j from (2.12) with $j+1$ gives

$$\begin{aligned} (1+\gamma - q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{1,j}^{n+1,\varepsilon} - (\alpha + q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{2,j}^{n+1,\varepsilon} &= \bar{\mathbf{R}}_{1,j}^{n+(1/2),\varepsilon} \\ -(\gamma + q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{1,j}^{n+1,\varepsilon} + (1+\alpha - q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{2,j}^{n+1,\varepsilon} &= \bar{\mathbf{R}}_{2,j}^{n+(1/2),\varepsilon} \end{aligned} \tag{4.5}$$

where

$$\alpha = \frac{\mathsf{k}}{2}\left(1 - \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right), \qquad \gamma = \frac{\mathsf{k}}{2}\left(1 + \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right)$$

and ξ in the above equations and η in (4.5) are some intermediate values between $u_{j+1}^{n+1,\varepsilon}$ and $u_j^{n+1,\varepsilon}$. It follows from Theorem 3.1 that $|\xi| \leqslant \mathsf{b}$. Then from the subcharacteristic condition (3.2) we have $\alpha \geqslant 0$ and $\gamma \geqslant 0$. Solving the equation (4.5) gives

$$\bar{R}_{1,j}^{n+1,\varepsilon} = \frac{1}{\Delta} \left[(1+\alpha - q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{1,j}^{n+(1/2),\varepsilon} + (\alpha + q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{2,j}^{n+(1/2),\varepsilon} \right]$$

$$\bar{R}_{2,j}^{n+1,\varepsilon} = \frac{1}{\Delta} \left[(\gamma + q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{1,j}^{n+(1/2),\varepsilon} + (1+\gamma - q'(\eta)\,\Delta t/2)\,\bar{\mathbf{R}}_{2,j}^{n+(1/2),\varepsilon} \right]$$

where

$$\Delta = (1+\mathsf{k})(1 - q'(\eta)\,\Delta t/2)$$

It follows from the above two equations that

$$\begin{aligned} &|\bar{\mathbf{R}}_{1,j}^{n+1,\varepsilon}| + |\bar{\mathbf{R}}_{2,j}^{n+1,\varepsilon}| \\ &\quad \leqslant \frac{1}{\Delta} \{ [1+\alpha - q'(\eta)\,\Delta t/2 + |\gamma + q'(\eta)\,\Delta t/2|]\,|\bar{\mathbf{R}}_{1,j}^{n+(1/2),\varepsilon}| \\ &\qquad + [1+\gamma - q'(\eta)\,\Delta t/2 + |\gamma + q'(\eta)\,\Delta t/2|]\,|\mathbf{R}_{2,j}^{n+(1/2),\varepsilon}| \} \end{aligned} \tag{4.6}$$

where the facts $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $q'(\eta) \leqslant 0$ are used. Observe that

$$\begin{aligned} &1+\alpha - q'(\eta)\,\Delta t/2 + |\gamma + q'(\eta)\,\Delta t/2| \\ &\quad \leqslant \max(1+\mathsf{k},\ 1+\alpha - \gamma - q'(\eta)\,\Delta t) \\ &\quad \leqslant (1+\mathsf{k})(1 - q'(\eta)\,\Delta t) = \Delta \end{aligned}$$

and similarly

$$\begin{aligned} &1+\gamma - q'(\eta)\,\Delta t/2 + |\alpha + q'(\xi)\,\Delta t/2| \\ &\quad \leqslant \max(1+\mathsf{k},\ 1+\gamma - \alpha - q'(\eta)\,\Delta t) \leqslant \Delta \end{aligned}$$

where we have used the fact $|\alpha - \gamma| \leqslant \mathsf{k}$. The above results, together with (4.6), yield

$$|\bar{\mathbf{R}}_{1,j}^{n+1,\varepsilon}| + |\bar{\mathbf{R}}_{2,j}^{n+1,\varepsilon}| \leqslant |\bar{\mathbf{R}}_{1,j}^{n+(1/2),\varepsilon}| + |\bar{\mathbf{R}}_{2,j}^{n+(1/2),\varepsilon}| \tag{4.7}$$

It follows from (2.12) that

$$\bar{\mathbf{R}}_{1,j}^{n+(1/2),\varepsilon} = c_{1,j+1}^{n,\varepsilon}(\mathbf{R}_{1,j+2}^{n,\varepsilon} - \mathbf{R}_{1,j+1}^{n,\varepsilon}) + (1 - c_{1,j}^{n,\varepsilon})(\mathbf{R}_{1,j+1}^{n,\varepsilon} - \mathbf{R}_{1,j}^{n,\varepsilon})$$

$$\bar{\mathbf{R}}_{2,j}^{n+(1/2),\varepsilon} = (1 - d_{-1,j+1}^{n,\varepsilon})(\mathbf{R}_{2,j+2}^{n,\varepsilon} - \mathbf{R}_{2,j+1}^{n,\varepsilon}) + d_{-1,j}^{n,\varepsilon}(\mathbf{R}_{2,j+1}^{n,\varepsilon} - \mathbf{R}_{2,j}^{n,\varepsilon})$$

The above results, together with the facts $0 < c^{n,\varepsilon}_{1,j}, d^{n,\varepsilon}_{-1,j} < 1$, yield

$$\sum_j (|\bar{\mathbf{R}}^{n+(1/2),\varepsilon}_{1,j}| + |\bar{\mathbf{R}}^{n+(1/2),\varepsilon}_{2,j}|) \leqslant \sum_j (|\bar{\mathbf{R}}^{n,\varepsilon}_{1,j}| + |\bar{\mathbf{R}}^{n,\varepsilon}_{2,j}|) \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\sum_j (|\bar{\mathbf{R}}^{n+1,\varepsilon}_{1,j}| + |\bar{\mathbf{R}}^{n+1,\varepsilon}_{2,j}|) \leqslant \sum_j (|\bar{\mathbf{R}}^{n,\varepsilon}_{1,j}| + |\bar{\mathbf{R}}^{n,\varepsilon}_{2,j}|) \tag{4.9}$$

The inequalities (4.3) and (4.9) lead to the total variation estimate (4.4). □

Having the above lemma, we are now ready to state and prove the following stability results on the relaxing solutions $(u^{n,\varepsilon}_j, v^{n,\varepsilon}_j)$.

Theorem 4.1. Under the same assumptions as in Lemma 4.1, the solutions $(u^{n,\varepsilon}_j, v^{n,\varepsilon}_j)$ of the relaxing scheme (2.1) satisfy the following estimates:

- TV-stability

$$\begin{aligned} TV(u^{n,\varepsilon}) &= \sum_j |u^{n,\varepsilon}_{j+1} - u^{n,\varepsilon}_j| \leqslant 2\mathbf{M} \\ TV(v^{n,\varepsilon}) &= \sum_j |v^{n,\varepsilon}_{j+1} - v^{n,\varepsilon}_j| \leqslant 2\sqrt{\mathsf{a}}\,\mathbf{M} \end{aligned} \tag{4.10}$$

- l^1-stability

$$\begin{aligned} \sum_j |u^{n,\varepsilon}_j|\, \Delta x &\leqslant 2\, \|u_0(\bullet)\|_{L^1} \\ \sum_j |v^{n,\varepsilon}_j|\, \Delta x &\leqslant 2\sqrt{\mathsf{a}}\, \|u_0(\bullet)\|_{L^1} \end{aligned} \tag{4.11}$$

Proof. The TV-stability (4.10) follows from Lemma 4.1 and the one to one mapping between $(\mathbf{R}^{n,\varepsilon}_{1,j}, \mathbf{R}^{n,\varepsilon}_{2,j})$ and $(u^{n,\varepsilon}_j, v^{n,\varepsilon}_j)$ given by (2.9). We only need to prove the l^1-stability (4.11). Using the assumptions $f(0)=0$ and $q(0)=0$, we rewrite scheme (2.11) as

$$\begin{aligned} \mathbf{R}^{n+1,\varepsilon}_{1,j} &= \mathbf{R}^{n+(1/2),\varepsilon}_{1,j} + \mathsf{k}(\mathbf{M}_1(u^{n+1,\varepsilon}_j) - \mathbf{M}_1(0) - \mathbf{R}^{n+1,\varepsilon}_{1,j}) \\ &\quad + (q(u^{n+1,\varepsilon}_j) - q(0))\, \Delta t/2 \\ \mathbf{R}^{n+1,\varepsilon}_{2,j} &= \mathbf{R}^{n+(1/2),\varepsilon}_{2,j} + \mathsf{k}(\mathbf{M}_2(u^{n+1,\varepsilon}_j) - \mathbf{M}_2(0) - \mathbf{R}^{n+1,\varepsilon}_{2,j}) \\ &\quad + (q(u^{n+1,\varepsilon}_j) - q(0))\, \Delta t/2 \end{aligned}$$

Applying the Mean-Value Theorem to $\mathbf{M}_1$, $\mathbf{M}_2$ and q in the above equations gives

$$
\begin{aligned}
(1+\gamma-q'(\eta)\,\Delta t/2)\,\mathbf{R}_{1,j}^{n+1,\varepsilon}-(\alpha+q'(\eta)\,\Delta t/2)\,\mathbf{R}_{2,j}^{n+1,\varepsilon}&=\mathbf{R}_{1,j}^{n+(1/2),\varepsilon}\\
-(\gamma+q'(\eta)\,\Delta t/2)\,\mathbf{R}_{1,j}^{n+1,\varepsilon}+(1+\alpha-q'(\eta)\,\Delta t/2)\,\mathbf{R}_{2,j}^{n+1,\varepsilon}&=\mathbf{R}_{2,j}^{n+(1/2),\varepsilon}
\end{aligned}
\tag{4.12}
$$

where again we have

$$\alpha=\frac{\mathsf{k}}{2}\left(1-\frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right)\geqslant 0,\qquad \gamma=\frac{\mathsf{k}}{2}\left(1+\frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right)\geqslant 0$$

Similar to the proof of Lemma 4.1, we can obtain

$$|\mathbf{R}_{1,j}^{n+1,\varepsilon}|+|\mathbf{R}_{2,j}^{n+1,\varepsilon}|\leqslant|\mathbf{R}_{1,j}^{n+(1/2),\varepsilon}|+|\mathbf{R}_{2,j}^{n+(1/2),\varepsilon}| \tag{4.13}$$

Replacing the right hand side of the inequality (4.13) by (2.12) and noting that $0<c_{1,j}^{n,\varepsilon}, d_{-1,j}^{n,\varepsilon}<1$ yield

$$\sum_j(|\mathbf{R}_{1,j}^{n+1,\varepsilon}|+|\mathbf{R}_{2,j}^{n+1,\varepsilon}|)\leqslant\sum_j(|\mathbf{R}_{1,j}^{n,\varepsilon}|+|\mathbf{R}_{2,j}^{n,\varepsilon}|) \tag{4.14}$$

Then it follows from the fact $\mathbf{R}_{1,j}^{n,\varepsilon}+\mathbf{R}_{2,j}^{n,\varepsilon}=u_j^{n,\varepsilon}$ and the above inequality that

$$
\begin{aligned}
\sum_j|u_j^{n,\varepsilon}|\,\Delta x&\leqslant\sum_j(|\mathbf{R}_{1,j}^{n,\varepsilon}|+|\mathbf{R}_{2,j}^{n,\varepsilon}|)\,\Delta x\\
&\leqslant\sum_j(|\mathbf{R}_{1,j}^{0,\varepsilon}|+|\mathbf{R}_{2,j}^{0,\varepsilon}|)\,\Delta x\\
&\leqslant\sum_j\left(|u_j^0|+\frac{|f(u_j^{0,\varepsilon})|}{\sqrt{\mathsf{a}}}\right)\Delta x\\
&=\sum_j\left(1+\frac{|f'(\xi_j^{0,\varepsilon})|}{\sqrt{\mathsf{a}}}\right)|u_j^{0,\varepsilon}|\,\Delta x\\
&\leqslant 2\,\|u_0(\bullet)\|_{L^1}
\end{aligned}
\tag{4.15}
$$

where $\xi_j^{0,\varepsilon}$ is some intermediate value and also in the last step the sub-characteristic condition (3.2) is used. Similarly, we can show that

$$\sum_j|v_j^{n,\varepsilon}|\,\Delta x\leqslant\sqrt{\mathsf{a}}\sum_j(|\mathbf{R}_{1,j}^{n,\varepsilon}|+|\mathbf{R}_{2,j}^{n,\varepsilon}|)\leqslant 2\sqrt{\mathsf{a}}\,\|u_0(\bullet)\|_{L^1} \tag{4.16}$$

This completes the proof of Theorem 4.1. □

5. CONVERGENCE

In this section, we will discuss the convergence of the MUSCL relaxing schemes. In order to carry out the convergence analysis, it is necessary to investigate the Lipschitz continuity of the numerical solution in time and the difference between $v_j^{n,\varepsilon}$ and $f(u^{n,\varepsilon})$ in the l^1-norm. To this end, we first provide two lemmas.

Lemma 5.1. Under the same assumptions as in Lemma 4.1, the solutions of the relaxing scheme (2.1) with initial data (2.4) satisfy

$$\|v^{n,\varepsilon} - f(u^{n,\varepsilon})\|_{l^1} := \sum_j |v_j^{n,\varepsilon} - f(u_j^{n,\varepsilon})| \, \Delta x$$
$$\leqslant 4\sqrt{\mathsf{a}}(\mathbf{M}\lambda^{-1} + \tfrac{1}{2}K \, \|u_0(\bullet)\|_{L^1})\, \varepsilon \tag{5.1}$$

Proof. Set

$$\mathbf{G}_j^{n,\varepsilon} = \mathbf{M}_1(u_j^{n,\varepsilon}) - \mathbf{R}_{1,j}^{n,\varepsilon}$$

It follows from the definition for $\mathbf{R}_{1,j}^{n,\varepsilon}$ and $\mathbf{R}_{2,j}^{n,\varepsilon}$ in (2.9) that

$$\mathbf{G}_j^{n,\varepsilon} = \frac{1}{2}(\mathbf{R}_{2,j}^{n,\varepsilon} - \mathbf{R}_{1,j}^{n,\varepsilon}) - \frac{1}{2\sqrt{\mathsf{a}}} f(u_j^{n,\varepsilon}) \tag{5.2}$$

Subtracting (5.2) with n from (5.2) with $n+1$ gives

$$\mathsf{k}(\mathbf{G}_j^{n+1,\varepsilon} - \mathbf{G}_j^{n,\varepsilon}) = -\gamma(\mathbf{R}_{1,j}^{n+1,\varepsilon} - \mathbf{R}_{1,j}^{n,\varepsilon}) + \alpha(\mathbf{R}_{2,j}^{n+1,\varepsilon} - \mathbf{R}_{2,j}^{n,\varepsilon}) \tag{5.3}$$

with

$$\alpha = \frac{\mathsf{k}}{2}\left(1 - \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right) \geqslant 0, \qquad \gamma = \frac{\mathsf{k}}{2}\left(1 + \frac{f'(\xi)}{\sqrt{\mathsf{a}}}\right) \geqslant 0$$

where ξ is some intermediate value between $u_j^{n,\varepsilon}$ and $u_j^{n+1,\varepsilon}$. In obtaining (5.3) we have used the identity $u_j^{n,\varepsilon} = \mathbf{R}_{2,j}^{n,\varepsilon} + \mathbf{R}_{1,j}^{n,\varepsilon}$ and the Mean-Value Theorem. Using the scheme (2.11) with (2.12) in the right hand side of (5.3) yields

$$\mathsf{k}(\mathbf{G}_j^{n+1,\varepsilon} - \mathbf{G}_j^{n,\varepsilon}) = -(\gamma c_{1,j}^{n,\varepsilon}(\mathbf{R}_{1,j+1}^{n,\varepsilon} - \mathbf{R}_{1,j}^{n,\varepsilon}) + \alpha d_{-1,j}^{n,\varepsilon}(\mathbf{R}_{2,j}^{n,\varepsilon} - \mathbf{R}_{2,j-1}^{n,\varepsilon}))$$
$$- \mathsf{k}^2 \mathbf{G}_j^{n+1,\varepsilon} + (\alpha - \gamma)\, q(u_j^{n+1,\varepsilon})\, \Delta t/2 \tag{5.4}$$

The above equation, together with the facts $0<\alpha,\ \gamma<\mathsf{k}$, $0<c^{n,\varepsilon}_{1,j},\ d^{n,\varepsilon}_{-1,j}<1$ and $|\alpha-\gamma|\leqslant \mathsf{k}$, yield

$$|\mathbf{G}^{n+1,\varepsilon}_j|\leqslant\frac{1}{1+\mathsf{k}}\left(|\mathbf{R}^{n,\varepsilon}_{1,j+1}-\mathbf{R}^{n,\varepsilon}_{1,j}|+|\mathbf{R}^{n,\varepsilon}_{2,j}-\mathbf{R}^{n,\varepsilon}_{2,j-1}|\right)$$
$$+\frac{1}{2(1+\mathsf{k})}\,|q(u^{n+1}_j)\,\Delta t|+\frac{1}{1+\mathsf{k}}\,|\mathbf{G}^{n,\varepsilon}_j| \tag{5.5}$$

It follows from the above inequality and Lemma 4.1 that

$$\|\mathbf{G}^{n+1,\varepsilon}_j\|_{l^1}\leqslant(1+\mathsf{k})^{-1}\left(\|\mathbf{G}^{n,\varepsilon}_j\|_{l^1}+2\mathbf{M}\,\Delta x+\Delta t\sum_j|q(u^{n+1,\varepsilon}_j)|\,\Delta x\right)$$

The above inequality, together with $-K\leqslant q'(u)\leqslant 0$ in (1.2) and the l^1-bound of $u^{n,\varepsilon}$ in (4.11), lead to

$$\|\mathbf{G}^{n+1,\varepsilon}_j\|_{l^1}\leqslant(1+\mathsf{k})^{-1}\left(\|\mathbf{G}^{n,\varepsilon}_j\|_{l^1}+2\mathbf{M}\,\Delta x+K\,\|u_0(\bullet)\|_{L^1}\,\Delta t\right) \tag{5.6}$$

Iterating the inequality (5.6) gives

$$\|\mathbf{G}^{n+1,\varepsilon}_j\|_{l^1}\leqslant(1+\mathsf{k})^{-(n+1)}\,\|\mathbf{G}^{0,\varepsilon}_j\|_{l^1}$$
$$+\frac{1}{\mathsf{k}}\left(1-(1+\mathsf{k})^{-(n+1)}\right)\left(2\mathbf{M}\lambda^{-1}+K\,\|u_0(\bullet)\|_{L^1}\right)\Delta t \tag{5.7}$$

Noting that $v^{0,\varepsilon}_j=f(u^{0,\varepsilon}_j)$ in (2.4), we have $\|\mathbf{G}^{0,\varepsilon}_j\|_{l^1}=0$. Thus the inequality (5.7), together with the definition $\mathsf{k}=\Delta t/\varepsilon$, gives

$$\|\mathbf{G}^{n+1,\varepsilon}_j\|_{l^1}\leqslant\left(2\mathbf{M}\lambda^{-1}+K\,\|u_0(\bullet)\|_{L^1}\right)\varepsilon$$

By the definition of $\mathbf{G}$, we have $\mathbf{G}^{n,\varepsilon}_j=(v^{n,\varepsilon}_j-f(u^{n,\varepsilon}_j)/2\sqrt{\mathsf{a}}$. This fact, together with the above inequality, completes the proof of this lemma.

Remark 5.1. We have from (5.7) that

$$\|v^{n,\varepsilon}-f(u^{n,\varepsilon})\|_{l^1}\leqslant 2\sqrt{\mathsf{a}}(1+\mathsf{k})^{-n}\,\|v^{0,\varepsilon}_j-f(u^{0,\varepsilon}_j)\|_{l^1}+\mathcal{O}(\varepsilon) \tag{5.8}$$

In (5.8), the term $\|v^{n,\varepsilon}-f(u^{n,\varepsilon})\|_{l^1}$ serves as a measure of the derivation of solution of the relaxing scheme (2.1) from the solution of the relaxed scheme (2.5) which is $\varepsilon\to 0$ limit of (2.1). It can be seen from (5.8) that the derivation measured in l^1-norm is controlled by the relaxation error of order ε and the derivation of the initial data. It is seen that the first term on the right hand side of (5.8) is proportional to

$$(1+\mathsf{k})^{-n} \|v_j^{0,\varepsilon} - f(u_j^{0,\varepsilon})\|_{l^1} = \left(\frac{\varepsilon}{\Delta t + \varepsilon}\right)^n \|v_j^{0,\varepsilon} - f(u_j^{0,\varepsilon})\|_{l^1} \tag{5.9}$$

It is observed from (5.9) that in the case that $\varepsilon \ll \Delta t$, which is always the most interesting case, it is not necessary to require the initial consistency $\|v_j^{0,\varepsilon} - f(u_j^{0,\varepsilon})\|_{l^1} = \mathcal{O}(\varepsilon)$. If we set $v_j^{0,\varepsilon}$ in the way such that $\|v_j^{0,\varepsilon} - f(u_j^{0,\varepsilon})\|_{l^1}$ is *bounded* as $\varepsilon \to 0$, then the theoretical estimates in this section still hold. In the case that $\varepsilon \ll \Delta t$, the inequality (5.9) shows that the right hand side of (5.9) is of order ε as long as the initial error $\|v^{0,\varepsilon} - f(u^{0,\varepsilon})\|_{l^1} = \mathcal{O}(1)$. We point out that this fact does not contradict the result of Kurganov and Tadmor [7] who require that $\|v^{0,\varepsilon} - f(u^{0,\varepsilon})\|_{l^1} = \mathcal{O}(\varepsilon)$, since the relaxing scheme (2.1) with $\varepsilon \ll \Delta t$ can not catch the initial layer.

Lemma 5.2. Under the same assumptions as in Lemma 4.1, the solutions of the relaxing scheme (2.1) with the initial data (2.4) satisfy

$$\begin{aligned}\sum_j (|\mathbf{R}_{1,j}^{n+1,\varepsilon} - \mathbf{R}_{1,j}^{n,\varepsilon}| + |\mathbf{R}_{2,j}^{n+1,\varepsilon} - \mathbf{R}_{2,j}^{n,\varepsilon}|)\, \Delta x \\ \leqslant (6\mathbf{M}\lambda^{-1} + 4K \|u_0(\bullet)\|_{L^1})\, \Delta t\end{aligned} \tag{5.10}$$

Proof. Set

$$\tilde{\mathbf{R}}_{i,j}^{n+1,\varepsilon} = \mathbf{R}_{1,j}^{n+1,\varepsilon} - \mathbf{R}_{i,j}^{n,\varepsilon}, \qquad i = 1, 2$$

It follows from (2.11) and (2.12) that

$$\begin{aligned}\tilde{\mathbf{R}}_{1,j}^{n+1,\varepsilon} &= c_{1,j}^{n,\varepsilon}(\mathbf{R}_{1,j+1}^{n,\varepsilon} - \mathbf{R}_{1,j}^{n,\varepsilon}) \\ &\quad + \frac{\mathsf{k}}{2\sqrt{\mathsf{a}}}(v_j^{n+1,\varepsilon} - f(u_j^{n+1,\varepsilon})) + q(u_j^{n+1})\, \Delta t/2 \\ \tilde{\mathbf{R}}_{2,j}^{n+1,\varepsilon} &= -d_{-1,j}^{n,\varepsilon}(\mathbf{R}_{2,j}^{n,\varepsilon} - \mathbf{R}_{2,j-1}^{n,\varepsilon}) \\ &\quad - \frac{\mathsf{k}}{2\sqrt{\mathsf{a}}}(v_j^{n+1,\varepsilon} - f(u_j^{n+1,\varepsilon})) + q(u_j^{n+1})\, \Delta t/2\end{aligned}$$

Multiplying the first equation above by the sign of $\tilde{\mathbf{R}}_{1,j}^{n+1,\varepsilon}$ and the second equation above by the sign of $\tilde{\mathbf{R}}_{2,j}^{n+1,\varepsilon}$, adding up and summing the resulting equations over j yield

$$\sum_j (|\tilde{\mathbf{R}}_{1,j}^{n+1,\varepsilon}| + |\tilde{\mathbf{R}}_{2,j}^{n+1,\varepsilon}|)\,\Delta x$$
$$\leqslant \Delta x\, TV(\mathbf{R}_1^{n,\varepsilon}, \mathbf{R}_2^{n,\varepsilon}) + \frac{\mathsf{k}}{\sqrt{\mathsf{a}}}\,\|v^{n+1,\varepsilon} - f(u_j^{n+1,\varepsilon})\|_{l^1} + \Delta t \sum_j q(u_j^{n+1})\,\Delta x$$

Applying (4.4) in Lemma 4.1, (4.11) in Theorem 4.1 and (5.1) in Lemma 5.1 to the right hand side of the above inequality gives

$$\sum_j (|\tilde{\mathbf{R}}_{1,j}^{n+1,\varepsilon}| + |\tilde{\mathbf{R}}_{2,j}^{n+1,\varepsilon}|)\,\Delta x$$
$$\leqslant 2\mathbf{M}\,\Delta x + 4(\mathbf{M}\lambda^{-1} + \tfrac{1}{2}K\,\|u_0(\bullet)\|_{L^1})\,\Delta t + 2K\,\|u_0(\bullet)\|_{L^1}\,\Delta t \qquad (5.11)$$

This completes the proof of this lemma. □

The following results are immediate consequences of the definition (2.9) and Lemma 5.2.

Lemma 5.3. Under the same assumptions as in Lemma 4.1, the solutions of the relaxing scheme (2.1) with the initial data (2.4) satisfy

$$\sum_j |u_j^{n+1,\varepsilon} - u_j^{n,\varepsilon}|\,\Delta x \leqslant (6\mathbf{M}\lambda^{-1} + 4K\,\|u_0(\bullet)\|_{L^1})\,\Delta t \qquad (5.12)$$

$$\sum_j |v_j^{n+1,\varepsilon} - v_j^{n,\varepsilon}|\,\Delta x \leqslant \sqrt{\mathsf{a}}\,(6\mathbf{M}\lambda^{-1} + 4K\,\|u_0(\bullet)\|_{L^1})\,\Delta t \qquad (5.13)$$

We are now ready to state and to prove the main theorem of this section.

Theorem 5.1. Assume $u_0 \in BV \cap L^\infty$. If the subcharacteristic condition (3.2) is satisfied, then the solutions of the MUSCL relaxing scheme (2.1) converge to the solutions of the corresponding MUSCL relaxed scheme (2.5) as ε tends to zero for fixed Δt. Furthermore, the solutions of the relaxed scheme (2.5) satisfy the following estimates:

$$\|u^n\|_{l^\infty} \leqslant \mathsf{b}, \qquad \|v^n\|_{l^\infty} \leqslant \sqrt{\mathsf{a}}\,\mathsf{b} \qquad (5.14)$$

$$TV(u^n) \leqslant 2\mathbf{M}, \qquad TV(v^n) = 2\sqrt{\mathsf{a}}\,\mathbf{M} \qquad (5.15)$$

$$\sum_j |u_j^{n+1} - u_j^n|\,\Delta x \leqslant (6\mathbf{M}\lambda^{-1} + 4K\,\|u_0(\bullet)\|_{L^1})\,\Delta t \qquad (5.16)$$

$$\sum_j |v_j^{n+1} - v_j^n|\,\Delta x \leqslant \sqrt{\mathsf{a}}\,(6\mathbf{M}\lambda^{-1} + 4K\,\|u_0(\bullet)\|_{L^1})\,\Delta t \qquad (5.17)$$

for all nonnegative integer n.

Proof. Define the linear interpolant for the solutions of the relaxing scheme (2.1) as follows

$$(u^{n,\varepsilon}(x), v^{n,\varepsilon}(x)) = \sum_j (u_j^{n,\varepsilon}, v_j^{n,\varepsilon})\, \chi_{[x_j - \Delta x/2,\, x_j + \Delta x/2)}(x)$$

where $\chi_{[a,b)}(x)$ is the characteristic function on the intervat $[a, b)$. It follows from Theorems 3.1 and 4.1 that $(u^{n,\varepsilon}(x), v^{n,\varepsilon}(x))$, $n \in \mathbb{N}_0$ are bounded piecewise constant functions of bounded variation with respect to n and ε. By Helly's Theorem, for each fixed n there exists a subsequence $(u^{n,\varepsilon_i}(x), v^{n,\varepsilon_i}(x))$ converging to a piecewise constant function

$$(u^n(x), v^n(x)) = \sum_j (u_j^n, v_j^n)\, \chi_{[x_j - \Delta x/2,\, x_j + \Delta x/2)}(x)$$

pointwisely for $n \in \mathbb{N}_0$ as $i \to \infty$. Therefore $(u_j^{n,\varepsilon_i}, v_j^{n,\varepsilon_i})$ converges to (u_j^n, v_j^n) as $i \to \infty$ for $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$. Furthermore, it follows from Lemma 5.1 that

$$\sum_j |v_j^{n,\varepsilon_i} - f(u_j^{n,\varepsilon_i})|\, \Delta x \leqslant 4\sqrt{\mathsf{a}}\, (\mathbf{M}\lambda^{-1} + \tfrac{1}{2} K\, \|u_0(\bullet)\|_{L^1})\, \varepsilon_i$$

By letting $i \to \infty$, we obtain $\sum_j |v_j^n - f(u_j^n)|\, \Delta x = 0$, which implies that

$$v_j^n = f(u_j^n) \qquad \text{for} \quad j \in \mathbb{Z}, \quad n \in \mathbb{N}_0 \tag{5.18}$$

Taking the limit $\varepsilon_i \to 0$ in the first equation of the relaxing scheme (2.1) gives

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2\,\Delta x}(v_{j-1}^n - v_{j+1}^n) - \frac{\sqrt{\mathsf{a}}}{2\,\Delta x}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$
$$+ \frac{1-\mu}{4}\left[(\sigma_j^+ - \sigma_{j-1}^+) - (\sigma_{j+1}^- - \sigma_j^-)\right] = q(u_j^{n+1}) \tag{5.19}$$

The above two equations, (5.18)–(5.19), are exactly the relaxed scheme (2.5). The estimates (5.14)–(5.17) follow from Theorem 3.1, Theorem 4.1 and Lemma 5.3. □

Remark 5.2. Note that (u_j^n, v_j^n) is uniquely determined by the relaxed scheme (2.5) with a given limiter function. Hence the whole sequence of the solutions $(u_j^{n,\varepsilon}, v_j^{n,\varepsilon})$ of the corresponding relaxing scheme (2.1) converges to (u_j^n, v_j^n) as $\varepsilon \to 0$.

6. CONCLUDING REMARKS

In this work, we investigated the convergence and stability property of a class of MUSCL relaxing schemes applied to conservation laws with stiff source terms. The main difficulty is the involvement of the stiff source term in the numerical schemes. The maximum principle and the l^1- and TV-stability are established. The $l^\infty \cap l^1 \cap BV$ bounds for the numerical solutions are shown to be independent of the relaxation parameter ε and the Lipschitz constant K of the stiff source term. However, it is found that the Lipschitz constant of the l^1 continuity in time for the MUSCL relaxing schemes depends on the large constant K, see Theorem 5.1.

Finally, it is important to make a number of comments on the results obtained in this work.

1. It would be ideal to establish convergence and stability theories for the MUSCL relaxing schemes with $\varepsilon \to 0$ and $\Delta x \to 0$ (or $\varepsilon \sim \mathcal{O}(\Delta x)$). However, it seems difficult to do that directly. An alternative approach to study the convergence is the following: from relaxing scheme to relaxed scheme (as $\varepsilon \to 0$); and then from relaxed scheme to the conservation laws (as mesh sizes tend to zero). The present work is to study the convergence and stability properties of the first step. For the second step, it can be shown that the MUSCL relaxed scheme (2.5) is a consistent, stable and *entropy consistent* discretization of the conservation law (1.1), by using the same arguments as used in [24]. The importance of studying the stability and convergence for the first step was emphasized in the work of Jin and Xin [5]. They pointed out that "A good numerical discretization should possess the correct zero relaxation limit, in the sence that the zero relaxation limit ($\varepsilon \to 0$ for a *fixed mesh*) is a consistent and stable discretization." Our results demonstrate that the MUSCL relaxing scheme indeed possesses correct relaxation limit.
2. It will be interesting to provide a rigorous analysis for the convergence *rate* for the MUSCL relaxing scheme (2.1). Ideally, the error bounds should be independent of the relaxation parameter ε and the Lipschitz constant K of the stiff source term. However, it seems very difficult to avoid the Lipschitz constant K, particularly if the framework of Kuznetsov [8] is used. Even in the continuous situation, a natural question is to establish the following estimate:

$$\|u^\varepsilon - u\|_{L^1(\mathbb{R})} \leqslant C\varepsilon^\gamma$$

for some $0 < \gamma \leqslant 1$, where u^ε is the solution of the relaxation equation (1.3), u the solution of the corresponding conservation law (1.1), and C a constant *independent* of ε and K (here K is the measure for the stiffness of the source term, see (1.2) for its definition). Although the corresponding result for homogeneous conservation laws has been established, see, e.g., [6, 10, 27], it is still an open problem for conservation laws with stiff source terms.

3. For the historical reason, the MUSCL relaxed scheme studied in this work is called second-order scheme (see Theorem 4.2. of [5]). However, it seems that the relaxed scheme (2.5) is of *first-order* only. In fact, it was shown in [24] that the solution of (2.5) with $q \equiv 0$ satisfy the cell-entropy inequality. Using almost exactly the same arguments, we can show that the solution of (2.5) also satisfy the cell-entropy inequality. It is noted that Osher and Tadmor [17] has established that for general flux and general entropy, cell entropy inequality means *first-order* accuracy, see also [16, 21]. By standard truncation error analysis, it can be verified that the relaxed scheme (2.5) (for both $q \not\equiv 0$ and $q \equiv 0$) is indeed a first order scheme.

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